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On the application of Turán's method to the theory of Dirichlet L-functions

by

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1. Consider $\Psi(x) = \sum_{n\leqslant x} \Lambda(n)$, where $\Lambda(n)$ is the Mangoldt function, and denote by $\Delta(x)$ the remainder-term of the prime-number formula

$$\Delta(x) = \Psi(x)-x.$$

P. Turán proved the following theorem ([4], p. 150):

If γ_1 is the supremum of the numbers γ for which

$$\Delta(x) = O(xe^{-c_1\log^\gamma x}), \tag{1.1}$$

where c_1 is a numerical constant, and γ_2 is the infimum of the numbers γ' for which the function of Riemann $\zeta(s) \neq 0$ in the region

$$\sigma > 1-\frac{c_2}{\log^{\gamma'}|t|}, \quad |t| \geqslant c_3, \tag{1.2}$$

where c_2, c_3 are constants, then

$$\gamma_1 = \frac{1}{1+\gamma_2} \quad \textit{or} \quad \gamma_2 = \frac{1}{\gamma_1}-1.$$

The subject of this note is to prove a similar theorem for the case of Dirichlet L-functions $L(s,\chi)$, $\chi \bmod k$, $k\geqslant 1$.

2. Let us introduce the following notations

$$\Psi(x,k,l) = \sum_{\substack{n\equiv l (\bmod k)\\ n\leqslant x}} \Lambda(n),$$

$$\Psi(x,\chi) = \sum_{n\leqslant x} \chi(n)\Lambda(n),$$

where k, l are integers, $k\geqslant 1$, $0<l\leqslant k$, $(l,k)=1$ and χ is a character $\bmod k$.

Let us denote

$$E_0 = E_0(\chi, k) = \begin{cases} 1 & \text{if} \quad \chi = \chi_0, \\ 0 & \text{if} \quad \chi \neq \chi_0; \end{cases}$$

$$E_1 = E_1(\chi, k) = \begin{cases} 1 & \text{if} \quad \chi = \chi_1, \\ 0 & \text{if} \quad \chi \neq \chi_1, \end{cases}$$

where χ_0 is the principal character and χ_1 is the exceptional character $\bmod k$.

Let us introduce further the following remainders

$$\Delta(x, k, l) = \Psi(x, k, l) - \frac{x}{\varphi(k)} + \frac{\chi_1(l)}{\varphi(k)} \cdot \frac{x^{\beta_1}}{\beta_1},$$

$$\Delta(x, \chi) = \Psi(x, \chi) - E_0 x + E_1 \frac{x^{\beta_1}}{\beta_1},$$

where β_1 denotes the exceptional real zero of $L(s, \chi_1)$ (if there exists such a zero; [1], p. 140).

3. The solution of the problem stated under 1 reduces essentially to the conversion of the following theorem of T. Tatuzawa (see [3] and [1], p. 319).

If $\prod_{\chi \bmod k} L(s, \chi) \neq 0$, $s \neq \beta_1$ *in the region*

$$\sigma \geqslant 1 - \frac{c_1}{\max\{\log k, \log^{\gamma}(|t|+3)\}}, \quad 0 < \gamma \leqslant 1, \tag{3.1}$$

then

$$|\Delta(x, k, l)| \leqslant c \frac{x}{\varphi(k)} \exp\left\{-c_2 \frac{\log x}{\max(\log k, \log^{\gamma/1+\gamma} x)}\right\} \tag{3.2}$$

for

$$1 \leqslant k \leqslant \exp\left(c_3 \frac{\log x}{\log\log x}\right),$$

where c, c_1, c_2, c_3 *are numerical constants.*

We will prove the following theorems.

THEOREM 1. *If* $0 < \gamma \leqslant 1$, $\omega(k)$ *is a positive valued function, and*

$$\Delta(x, k, l) < c_1 \frac{x}{\varphi(k)} \exp\left\{-c_2 \frac{\log x}{\max(\log k, \log^{\gamma/1+\gamma} x)}\right\} \tag{3.3}$$

for $x \geqslant \omega(k)$ *and any* l, $0 < l \leqslant k$, $(l, k) = 1$, *then*

$$\prod_{\chi \bmod k} L(s, \chi) \neq 0$$

in the region

$$\sigma > 1 - \frac{1}{30} \cdot \frac{(c_2/2)^{1+\gamma}}{\max\{\max(1, (c_2/2)^{1+\gamma}) \log k, \log^{\gamma}|t|\}}, \tag{3.4}$$

$$|t| \geqslant \max\left(c_3, k^{1/9}, \frac{1}{k} \exp \frac{\log \omega(k)}{\log k}\right),$$

where c_3 *is a constant depending only on* c_1, c_2 *and* γ. *In the case* $k = 1$ *the term* $\frac{1}{k} \exp \frac{\log \omega(k)}{\log k}$ *has to be omitted.*

From this theorem we will deduce

THEOREM 2. *If* γ_1 *is the supremum of the numbers* γ *for which*

$$|\Delta(x, k, l)| < c_1 \frac{x}{\varphi(k)} \exp\left\{-c_2 \frac{\log x}{\max(\log k, \log^{1-\gamma} x)}\right\} \tag{3.5}$$

with

$$x \geqslant \omega(k), \quad 0 < l \leqslant k, \quad (l, k) = 1,$$

where $\omega(k)$ *is any function satisfying the condition*

$$\exp \log^2 k \leqslant \omega(k) \leqslant \exp(A \log^2 k) \qquad (A \text{ constant} \geqslant 1)$$

and if γ_2 *is the infimum of* γ' *for which* $\prod_{\chi \bmod k} L(s, \chi) \neq 0$, $s \neq \beta_1$ *in the region*

$$\sigma > 1 - \frac{c_3}{\max\{\log k, \log^{\gamma'}(|t|+3)\}}, \tag{3.6}$$

c_2, c_3 *are constants, then*

$$\gamma_1 = \frac{1}{1+\gamma_2}.$$

Theorem 2 solves completely the problem stated above.

4. Proof of Theorem 1 will rest on four lemmas.

LEMMA 1. *Let* $z_1, z_2, \ldots, z_h$ *be complex numbers such that*

$$|z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_h|, \quad |z_1| \geqslant 1$$

and let $b_1, b_2, \ldots, b_h$ *be any complex numbers.*

Then, if m is positive and $N \geqslant h$, there exists an integer ν such that $m \leqslant \nu \leqslant m+N$,

$$|b_1 z_1^\nu + b_2 z_2^\nu + \ldots + b_h z_h^\nu| \geqslant \left(\frac{1}{48e^2} \cdot \frac{N}{2N+m}\right)^N \min_{1 \leqslant j \leqslant h} |b_1 + b_2 + \ldots + b_j|. \tag{4.1}$$

This lemma is Turán's second main theorem (see [4], p. 52).

LEMMA 2. *For $\sigma > 1$, $\xi > 1$ we have*

$$\left|(-1)^r \sum_{n \geqslant \xi} \frac{\Lambda(n)\chi(n)}{n^s} \cdot \frac{\log^{r+1}(n/\xi)}{(r+1)!} + \sum_{\varrho} \frac{\xi^{\varrho - s}}{(\varrho - s)^{r+2}} - E_0 \frac{\xi^{1-s}}{(1-s)^{r+2}}\right| < c_4 \frac{\log k(|t|+2)}{\xi^{3/2}}, \tag{4.2}$$

where r is a positive integer and ϱ runs through all zeros of $L(s, \chi)$ in the strip $0 \leqslant \sigma < 1$.

Proof of Lemma 2 does not essentially differ from that of lemma contained in [2], p. 315, and can be dropped.

LEMMA 3. *For $\sigma > 0$, $|t| \geqslant 2$ we have*

$$|L(s, \chi)| < c\{1 + (k|t|)^{1-\sigma}\} \log k|t| \tag{4.3}$$

and for $\sigma > 1$

$$\frac{1}{|L(s, \chi)|} \leqslant 1 + \frac{1}{\sigma - 1}. \tag{4.4}$$

It is easily seen that (4.3) can be deduced from [1], pp. 113–116. The proof of (4.4) is trivial.

LEMMA 4. *If $s_0 = 1 + \mu + it'$, $0 < \mu \leqslant 1/40$, $R = 16\mu$, $t' \geqslant 10$ and N_1 stands for the number of roots of $L(s, \chi)$, $\chi \bmod k$ in the circle $|s - s_0| \leqslant R/2$, then*

$$N_1 < 25\mu \log kt' + c_5 \log\log kt' + \log \frac{1+\mu}{\mu}. \tag{4.5}$$

This lemma follows from Lemma 3 by the use of Jensen inequality (compare [4], p. 187).

5. Proof of Theorem 1. It can easily be shown that

$$|\Delta(x, \chi)| \leqslant \sum_{\substack{1 \leqslant l \leqslant k \\ (l, k) = 1}} |\Delta(x, k, l)|.$$

Hence owing to (3.3) we have for $0 < \gamma \leqslant 1$, $x \geqslant \omega(k)$, the estimate

$$|\Delta(x, \chi)| < c_1 x \exp\left\{-c_2 \frac{\log x}{\max(\log k, \log^{\gamma/(1+\gamma)} x)}\right\} \tag{5.1}$$

for any $\chi \bmod k$.

Let t be positive and

$$f_{N_1 N_2}(\chi, t) = \sum_{N_1 \leqslant n \leqslant N_2} \chi(n)\Lambda(n) \exp(-it \log n). \tag{5.2}$$

From the definition of $\Delta(x, \chi)$ we get for $n > 1$

$$\chi(n)\Lambda(n) = \Delta(n, \chi) - \Delta(n-1, \chi) + E_0 - E_1 \frac{n^{\beta_1} - (n-1)^{\beta_1}}{\beta_1}. \tag{5.3}$$

Hence

$$\begin{aligned} |f_{N_1 N_2}(\chi, t)| \leqslant E_0 \Big|\sum_{N_1 \leqslant n \leqslant N_2} \exp(-it \log n)\Big| + \\ + \Big|\sum_{N_1 \leqslant n \leqslant N_2} [\Delta(n, \chi) - \Delta(n-1, \chi)] \exp(-it \log n)\Big| + \\ + E_1 \Big|\sum_{N_1 \leqslant n \leqslant N_2} \frac{n^{\beta_1} - (n-1)^{\beta_1}}{\beta_1} \exp(-it \log n)\Big| = E_0 J_1 + J_2 + E_1 J_3. \end{aligned} \tag{5.4}$$

By partial summation we have

$$J_2 \leqslant |\Delta(N_2, \chi)| + |\Delta(N_1 - 1, \chi)| + \sum_{N_1 \leqslant n \leqslant N_2} |\Delta(n, \chi)| \left|1 - \exp\left\{-it \log\left(1 + \frac{1}{n}\right)\right\}\right|.$$

Let t be such that with the above c_2 and γ

$$1 + t^2 \leqslant \exp\left\{\left(\frac{2}{c_2} \log t\right)^{1+\gamma}\right\} \tag{5.5}$$

and N_1, N_2 such that

$$\exp\left\{\max\left(\left(\frac{2}{c_2} \log t\right)^{1+\gamma}, \left(\frac{2}{c_2} \log t \log k\right), \log \omega(k)\right)\right\} \leqslant \frac{N}{2} \leqslant N_1 < N_2 \leqslant N. \tag{5.6}$$

Hence

$$J_2 < c_6 N t \exp\left\{-c_2 \frac{\log N}{\max(\log k, \log^{\gamma/(1+\gamma)} N)}\right\}.$$

From (5.6) we get

$$\exp\left\{-c_2\frac{\log N}{\max(\log k,\log^{\gamma/(1+\gamma)}N)}\right\}\leqslant\frac{1}{t^2}.$$

Hence

$$J_2<c_7\frac{N}{t}. \tag{5.7}$$

The terms J_1, J_3 can also be estimated from above by $c\dfrac{N}{t}$ (see [4], p. 157).

From (5.4) it follows the estimate

$$|f_{N_1N_2}(\chi,t)|<c_8\frac{N}{t} \tag{5.8}$$

if (5.5) and (5.6) are satisfied.

Consider

$$1<\sigma\leqslant\tfrac{3}{2},\qquad t\geqslant 2. \tag{5.9}$$

By partial summation we get from (5.8)

$$\Big|\sum_{N_1\leqslant n\leqslant N_2}\chi(n)\frac{\Lambda(n)}{n^s}\Big|<c_9\frac{N^{1-\sigma}}{t}. \tag{5.10}$$

6. Choose

$$\eta\geqslant\exp\left\{\max\left(\left(\frac{2}{c_2}\log t\right)^{1+\gamma},\left(\frac{2}{c_2}\log t\log k\right),\log\omega(k)\right)\right\}. \tag{6.1}$$

Applying (5.10) with

$$N_1^j=\eta\cdot 2^j,\qquad N_2^j=\eta\cdot 2^{j+1},\qquad j=0,1,2,\ldots \tag{6.2}$$

we have

$$\Big|\sum_{n\geqslant\eta}\chi(n)\frac{\Lambda(n)}{n^s}\Big|<c_{10}\frac{\eta^{1-\sigma}}{t(\sigma-1)}. \tag{6.3}$$

Suppose

$$\xi\geqslant\exp\left\{\max\left(\left(\frac{2}{c_2}\log t\right)^{1+\gamma},\left(\frac{2}{c_2}\log t\log k\right),\log\omega(k)\right)\right\}, \tag{6.4}$$

where r denotes a positive integer.

Multiplying (6.3) by $\dfrac{1}{\eta}\log^r\left(\dfrac{\eta}{\xi}\right)$ and integrating over $\langle\xi,+\infty)$ we get

$$\Big|\sum_{n\geqslant\xi}\frac{\chi(n)\Lambda(n)}{n^s}\log^{r+1}\left(\frac{n}{\xi}\right)\Big|<\frac{c_{11}}{t(\sigma-1)}\,\frac{(r+1)!\,\xi^{1-\sigma}}{(\sigma-1)^{r+1}}. \tag{6.5}$$

From Lemma 2 it follows

$$\Big|\sum_{\varrho}\frac{\xi^{\varrho-s}}{(\varrho-s)^{r+2}}-E_0\frac{\xi^{1-s}}{(1-s)^{r+2}}\Big|<c_{12}\left(\frac{\xi^{1-\sigma}}{t(\sigma-1)^{r+2}}+\frac{\log k\,(t+2)}{\xi}\right). \tag{6.6}$$

Hence (see [4], p. 190)

$$\Big|\sum_{\varrho}\frac{\xi^{\varrho-s}}{(\varrho-s)^{r+2}}\Big|<c_{13}\frac{\xi^{1-\sigma}}{t}\left(\frac{1}{(\sigma-1)^{r+2}}+\log k(t+2)\right). \tag{6.7}$$

7. Supposing our Theorem 1 is false there exist zeros

$$\varrho^*=\sigma^*+it^*,\qquad t^*\to\infty$$

of $L(s,\chi)$, $\chi\bmod k$, for which

$$\sigma^*>1-\frac{1}{30}\,\frac{(c_2/2)^{1+\gamma}}{\max\{\max(1,(c_2/2)^{1+\gamma})\log k,\log^\gamma t^*\}}. \tag{7.1}$$

Let t^* be such that

$$t^*>\max\left\{e^e,\exp\left(40^{1/\gamma}\left(\frac{c_2}{2}\right)^{(1+\gamma)/\gamma}\right),\exp\left(\frac{2}{c_2}\right)^{1+\gamma},k^{1/9},\frac{1}{k}\exp\frac{\log\omega(k)}{\log k}\right\}=T_0. \tag{7.2}$$

Put

$$s=s_1=1+\frac{(c_2/2)^{1+\gamma}}{\log^\gamma t^*}+it^*=\sigma_1+it^*, \tag{7.3}$$

$$\xi=\exp\big((r+2)\lambda\big), \tag{7.4}$$

where

$$10\log t^*\leqslant r+2\leqslant 12.5\log t^* \tag{7.5}$$

and

$$\lambda=\left(\frac{2}{c_2}\right)^{1+\gamma}\max\left\{\max\left(1,\left(\frac{c_2}{2}\right)^{1+\gamma}\right)\log k,\log^\gamma t^*\right\}. \tag{7.6}$$

It is easily seen that the condition (5.5) and (6.4) are satisfied and the condition (5.9) owing to (7.2) becomes

$$1<\sigma_1=1+\left(\frac{c_2}{2}\right)^{1+\gamma}\frac{1}{\log^\gamma t^*}<1+\frac{1}{40}. \tag{7.7}$$

Hence inequality (6.7) can be used. Multiplying (6.7) by

$$|\xi^{s_1-\varrho^*}(s_1-\varrho^*)^{r+2}|=\xi^{\sigma_1-\sigma^*}(\sigma_1-\sigma^*)^{r+2},$$

we get

$$\Big|\sum_{\varrho}\xi^{\varrho-\varrho^*}\left(\frac{s_1-\varrho^*}{s_1-\varrho}\right)^{r+2}\Big|<c_{13}\frac{\xi^{1-\sigma^*}}{t}\left(\left(\frac{\sigma_1-\sigma^*}{\sigma_1-1}\right)^{r+2}+(\sigma_1-\sigma^*)\log k\,(t+2)\right). \tag{7.8}$$

Under (7.2), (7.5) and (7.7) we have

$$(\sigma_1-1)^{r+2}\log k(t+2) < \left(\frac{1}{40}\right)^{r+2}\log t^{9}(t+2) = O(1),$$

and from (7.1), (7.3) and (7.5)

$$\left(\frac{\sigma_1-\sigma^*}{\sigma_1-1}\right)^{r+2} < \left(1+\frac{1-\sigma^*}{\sigma_1-1}\right)^{r+2} < \left(1+\frac{1}{30}\right)^{r+2} < t^{*5/12}.$$

Therefore from (7.8) we get the estimate

$$\left|\sum_{\varrho}\xi^{\varrho-\varrho^*}\left(\frac{s_1-\varrho^*}{s_1-\varrho}\right)^{r+2}\right| < c_{14}\frac{\xi^{1-\sigma^*}}{t^{*7/12}}. \tag{7.9}$$

8. Let us estimate the part of the sum (7.9) for which $t_\varrho \geqslant t^*+8$. Owing to (7.5) and [1], Theorem (3.3), p. 220, we get

$$S_1 = \left|\sum_{\substack{\varrho \\ t_\varrho \geqslant t^*+8}}\xi^{\varrho-\varrho^*}\left(\frac{s_1-\varrho^*}{s_1-\varrho}\right)^{r+2}\right| < c_{15}\,\xi^{1-\sigma^*}\frac{1}{t^*}.$$

Similarly we can prove that the sums for

$$t^*+6(\sigma_1-\sigma^*) \leqslant t_\varrho \leqslant t^*+8, \qquad 0 < t_\varrho \leqslant t^*-6(\sigma_1-\sigma^*),$$

and

$$|t_\varrho - t^*| \leqslant 6(\sigma_1-\sigma^*), \qquad \sigma_\varrho < 1-3(\sigma_1-\sigma^*)$$

are absolutely less than

$$c_{16}\,\xi^{1-\sigma^*}\frac{1}{t^*}.$$

Therefore, owing to (7.9) it follows

$$V = \left|\sum_{\substack{|t_\varrho - t^*| \leqslant 6(\sigma_1-\sigma^*) \\ \sigma_\varrho \geqslant 1-3(\sigma_1-\sigma^*)}}\left(e^{\lambda(\varrho-\varrho^*)}\frac{s_1-\varrho^*}{s_1-\varrho}\right)^{r+2}\right| < c_{17}\frac{\xi^{1-\sigma^*}}{t^{*7/12}}. \tag{8.1}$$

9. To estimate V from below we use Lemma 1. We choose

$$z_j = e^{\lambda(\varrho-\varrho^*)}\,\frac{s_1-\varrho^*}{s_1-\varrho}$$

and

$$m = 10\log t^*. \tag{9.1}$$

In order to define N we have to estimate the number of roots of $L(s,\chi)$, χ mod k, in the region

$$1-3(\sigma_1-\sigma^*) \leqslant \sigma < 1, \qquad |t-t^*| \leqslant 6(\sigma_1-\sigma^*). \tag{9.2}$$

We will apply Lemma 4. Let $s_0 = s_1 = \sigma_1 + it^*$, $\mu = \left(\frac{c_2}{2}\right)^{1+\gamma}\frac{1}{\log^{\gamma} t^*}$ and let d to be the maximal distance of points in (9.2) from s_1. Hence

$$d = |(\sigma_1-1)+3(\sigma_1-\sigma^*)+i6(\sigma_1-\sigma^*)|.$$

From (7.1) we have

$$1-\sigma^* < \frac{1}{30}\mu, \qquad \sigma_1-\sigma^* = (\sigma_1-1)+(1-\sigma^*) < \frac{31}{30}(\sigma_1-1).$$

Hence $d < 7.5\mu < 8\mu$. So the region (9.2) is contained in the circle $|s-s_1| \leqslant 8\mu$.

Owing to Lemma 4 we can take

$$N = 250\mu\log t^* + c_{18}\log\log t^*,$$

if the interval $(m, m+N)$ is contained in (7.5).

This condition is satisfied by such N because

$$N < 2.5\log t^* \quad \text{for} \quad t^* > \max\big(T_0, c_{19}(\gamma)\big) = T_1.$$

From Lemma 1 there exist $\nu = r+2$ that

$$V \geqslant \left(\frac{250\mu\log t^* + c_{18}\log\log t^*}{48e^2\cdot 15\log t^*}\right)^{250\mu\log t^* + c_{18}\log\log t^*}$$

$$> \left(\frac{1}{48e^2}\,\frac{250\mu\log t^*}{15\log t^*}\right)^{250\mu\log t^*}\cdot\left(\frac{c_{18}\log\log t^*}{15\log t^*}\right)^{c_{18}\log\log t^*}$$

$$> \exp\left\{-250\gamma\left(\frac{c_2}{2}\right)^{1+\gamma}\log^{1-\gamma}t^*\cdot\log\log t^* - c_{20}(\gamma)\log^{1-\gamma}t^*(\log\log t^*)^2\right\}.$$

From this estimate and from (8.1) it follows

$$\xi^{1-\sigma^*} > \frac{t^{*7/12}}{c_{17}}\times$$

$$\times\exp\left\{-250\gamma\left(\frac{c_2}{2}\right)^{1+\gamma}\log^{1-\gamma}t^*\cdot\log\log t^* - c_{20}(\gamma)\log^{1-\gamma}t^*(\log\log t^*)^2\right\} > t^{*7/16}$$

for $t^* > \max\big(T_1, c_{21}(\gamma)\big)$.

Taking into account (7.4)–(7.6), we get

$$\frac{7}{16}\log t^* < (1-\sigma)(r+2)\lambda$$

$$< (1-\sigma^*)\cdot 12.5\log t^*\cdot\left(\frac{2}{c_2}\right)^{1+\gamma}\max\left\{\max\left(1,\left(\frac{c_2}{2}\right)^{1+\gamma}\right)\log k,\ \log^\gamma t^*\right\}.$$

Hence

$$1-\sigma^* > \frac{7}{200}\cdot\left(\frac{c_2}{2}\right)^{1+\gamma}\frac{1}{\max\{\max(1,(c_2/2)^{1+\gamma})\log k,\ \log^\gamma t^*\}}$$

and this leads to a contradiction to (7.1). This proves Theorem 1.

10. Proof of Theorem 2. We know already that if (3.6) is satisfied with a γ', $0<\gamma'\leqslant 1$, then (3.5) follows with $\gamma = \dfrac{1}{1+\gamma'}$. This is the first part of the proof.

In the second part we have to show that if $0<\gamma\leqslant 1$ and

$$|\Delta(x,k,l)| < c_1\frac{x}{\varphi(k)}\exp\left\{-c_2\frac{\log x}{\max(\log k,\ \log^{\gamma/1+\gamma}x)}\right\} \tag{10.1}$$

for $x\geqslant\omega(k)$ and $0<l\leqslant k$, $(l,k)=1$, where

$$\exp\log^2 k\leqslant\omega(k)\leqslant\exp(A\log^2 k), \tag{10.2}$$

then $\prod_{\chi \bmod k} L(s,\chi)\neq 0$, $s\neq\beta_1$, in the region

$$\sigma > 1-\frac{c_3}{\max\{\log k,\ \log^\gamma(|t|+3)\}}, \tag{10.3}$$

where the constant c_3 depends only on A, c_1, c_2 and γ.

From (10.1), (10.2) and Theorem 1 we have $\prod_{\chi \bmod k} L(s,\chi)\neq 0$ in the region

$$\sigma > 1-\frac{c_4}{\max\{\log k,\ \log^\gamma|t|\}},\qquad t\geqslant\max(c_5,k^A). \tag{10.4}$$

Let k be chosen such that $k^A\geqslant c_5$. In this case the region (10.4) can obviously be defined as

$$\sigma > 1-\frac{c_4}{\max\{\log k,\ \log|t|\}},\qquad t>k^A. \tag{10.5}$$

It is easy to verify that from Lemma 3 it follows that $\prod_{\chi \bmod k} L(s,\chi)\neq 0$ in the region

$$\sigma > 1-\frac{c_6}{\max\{\log k,\ \log|t|\}},\qquad t\geqslant 3. \tag{10.6}$$

From (10.5) and (10.6) we have $\prod_{\chi \bmod k} L(s,\chi)\neq 0$ for $k^A\geqslant c_5$ in the region

$$\sigma > 1-\frac{c_7}{\max\{\log k,\log^\gamma|t|\}},\qquad t\geqslant 3. \tag{10.7}$$

It remains to discuss the case $1\leqslant k\leqslant c_5^{1/A}$. In this case from (10.4) it follows that $\prod_{\chi \bmod k} L(s,\chi)\neq 0$ in the region

$$\sigma > 1-\frac{c_4}{\max\{\log k,\ \log^\gamma|t|\}},\qquad t\geqslant c_5. \tag{10.8}$$

From (10.7), (10.8) and [1], pp. 118–122, Theorem 2 follows.

References

[1] K. Prachar, *Primzahlverteilung*, Berlin 1957.

[2] W. Staś, *Über die Dichte der Nullstellen der Dirichletschen L-Funktionen*, Acta Arith. 6 (1961), pp. 313–323.

[3] T. Tatuzawa, *On the number of primes in an arithmetic progression*, Jap. J. Math. 21 (1951), pp. 93–111.

[4] P. Turán, *Eine Neue Methode in der Analysis und deren Anwendungen*, Budapest 1953 und die chinesische umgearbeitete und erweiterte Auflage dieses Buches, Peking 1956.

[5] — *On the remainder-term of the prime-number formula, II*, Acta Math. Hung. 1 (1950), pp. 155–166.

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