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On the topology of curves II

by

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In the present paper we investigate two classes of curves which we call Suslinian and finitely Suslinian, respectively (see § 1). To motivate the terms which have been chosen, let us point out that the properties attached to them resemble a property of ordered sets introduced by M. Ja. Suslin and related to the famed Suslin problem. Our properties are intended to complete the well-known classification of curves (see [4], p. 96, and [5], p. 99) in which the notion of rational curves plays an essential role. Rational curves possess a decomposition property (see [2], p. 211), and an analogue for Suslinian curves is suggested here; it is, however, proved only in the case of hereditarily unicoherent curves (see § 2). We also prove the existence of Suslinian curves which are not rational (see § 3). A part of the material covered by this paper was mimeographed in [3].

§ 1. The concept of Suslinian curves. A curve X will be called *Suslinian* provided each collection of pairwise disjoint subcurves of X is countable (1). A curve X is called *hereditarily decomposable* provided each subcurve Y of X is decomposable, i.e. representable as the union of two proper subcurves of Y.

1.1. *Each Suslinian curve is hereditarily decomposable.*

Proof. This is because an indecomposable continuum has uncountably many pairwise disjoint composants and each of them is dense. Composants of a curve are countable unions of some subcurves (see [2], p. 147).

A space is called *ponctiform* provided each continuum contained in it is degenerate.

1.2. *If a curve X admits a decomposition $X = P \cup Q$ where P is ponctiform and Q is countable, then X is Suslinian.*

(1) We recall that, in our terminology, a continuum means a compact connected metric space, and a curve means a 1-dimensional continuum. Therefore the curves are non-degenerate sets. A subcurve means a curve which is contained in a curve under consideration.

Proof. Given any collection of pairwise disjoint subcurves of X, only countably many of them might intersect Q. But no curve is contained in P, thus the collection is countable.

1.3. *Each rational curve is Suslinian.*

Proof. If X is a rational curve, then X admits a decomposition $X = P \cup Q$ where P is 0-dimensional and Q is countable. Since 0-dimensional spaces are ponctiform, 1.3 follows from 1.2 (see also [2], p. 211).

Now, observe that if a curve X is not Suslinian, then there exist a number $\varepsilon_0 > 0$ and an uncountable collection of pairwise disjoint subcurves of X which all have diameters greater than ε_0. A curve X will be called *finitely Suslinian* provided, for every $\varepsilon > 0$, each collection of pairwise disjoint subcurves of X having diameters greater than ε is finite. Thus finitely Suslinian curves are Suslinian. The latter statement can be refined on by means of 1.3 and 1.4 below, since hereditarily locally connected curves are rational (see [5], p. 94).

1.4. *Each finitely Suslinian curve is hereditarily locally connected.*

Proof. If a curve X is not locally connected, then there exists an infinite sequence $C_0, C_1, \dots$ of subcurves of X such that $C_0 = \operatorname{Lim} C_i$ and $C_0 \cap C_i = \emptyset$ for $i = 1, 2, \dots$ (see [2], p. 196). Such a sequence always has an infinite subsequence composed of elements C_{i_j} which are pairwise disjoint. Since C_0 is non-degenerate, diameters of C_{i_j} cannot converge to zero, and 1.4 is proved.

1.5. *In order that a plane curve X be hereditarily locally connected it is necessary and sufficient that X be finitely Suslinian.*

Proof. The sufficiency is shown in 1.4, and the necessity follows from the Gehman theorem (see [2], p. 366).

1.6. THEOREM. *In order that a curve X be regular it is necessary and sufficient that, for every $\varepsilon > 0$, there exist an integer n such that each collection of pairwise disjoint subcurves of X having diameters greater than ε consists of at most n elements.*

Proof. Since a curve X satisfying the condition from 1.6 is finitely Suslinian, X is locally connected by 1.4. A theorem on locally connected curves which fail to be regular (see [4], p. 216) now applies to show that the condition is sufficient. We prove that it is also necessary.

Suppose a curve X does not satisfy the condition from 1.6. Then there exist a number $\varepsilon_0 > 0$ and collections $\boldsymbol{C}_n$ of subcurves of X such that $\boldsymbol{C}_n$ consists of n^2 elements which all have diameters greater than ε_0 and are pairwise disjoint ($n = 1, 2, \dots$). Let $\boldsymbol{A}_n$ be a finite cover of X such that $\boldsymbol{A}_n$ consists of n elements and

$$\lim_{n\to\infty} \operatorname{Max}\{\operatorname{diam} A \colon A \in \boldsymbol{A}_n\} = 0$$

($n = 1, 2, \dots$). Consequently, there exists a set $A_n \in \boldsymbol{A}_n$ such that A_n meets at least n elements of $\boldsymbol{C}_n$ ($n = 1, 2, \dots$), and we can assume that the sets A_n converge to a point $p \in X$. Thus every neighbourhood of p in X meets n pairwise disjoint subcurves which have diameters greater than ε_0, for every $n = 1, 2, \dots$ We infer that no finite set in X separates p from the complement of the open ball with center p and radius $\frac{1}{2}\varepsilon_0$. It follows that the curve X is not regular which completes the proof of 1.6.

1.7. COROLLARY. *Each regular curve is finitely Suslinian.*

Remarks. The cone over the Cantor set is an example of a hereditarily decomposable curve which is not Suslinian; thus the implication 1.1 cannot be reversed. A partial reversal of 1.2 is done in 2.2 below, and in the last section of this paper we construct an example to show that 1.3 cannot be reversed. There exists an example of a hereditarily locally connected curve which is not finitely Suslinian (see [2], p. 196); thus 1.4 cannot be reversed. Finally, there also exists an example of a hereditarily locally connected plane curve which is not regular (ibidem, p. 210). By 1.5, the latter curve is finitely Suslinian; thus 1.7 cannot be reversed.

§ 2. Hereditarily unicoherent Suslinian curves.

A curve X is called *hereditarily unicoherent* provided each subcurve Y of X is unicoherent, i.e. non-representable as the union of two subcurves of Y whose common part is not connected. A curve X is called *tree-like* provided X admits finite open covers whose elements have arbitrarily small diameters and whose nerves are acyclic 1-dimensional polyhedra.

2.1. *Each hereditarily unicoherent Suslinian curve is tree-like.*

Proof. Since each hereditarily unicoherent and hereditarily decomposable curve is tree-like (see [1], p. 20), 2.1 follows from 1.1.

2.2. THEOREM. *In order that a hereditarily unicoherent curve X be Suslinian it is necessary and sufficient that X admit a decomposition $X = P \cup Q$ where P is ponctiform and Q is countable.*

Proof. The sufficiency is shown in 1.2. To prove the necessity, let us consider a hereditarily unicoherent Suslinian curve X and let $\{G_1, G_2, \dots\}$ be a countable open basis in X. Since X is Suslinian, the collection of all non-degenerate components of the closure of G_i in X is countable; let us denote these components by $C_{i1}, C_{i2}, \dots$ By 1.1, each irreducible continuum J contained in X is of type λ, i.e. there exists a monotone continuous mapping $g\colon J \to I$ of J onto the unit segment I of the real line such that $g^{-1}(t)$ has void interior in J for $t \in I$ (see [2], pp. 137–139 and 153–154). Then J is irreducible between any two points belonging to the end tranches of J, i.e. the sets $g^{-1}(0)$ and $g^{-1}(1)$, respectively. Let $\boldsymbol{J}_{ij}$ denote the collection of all irreducible continua J in X each of them having an end tranche $T(J)$ such that

$$C_{ij} \cap T(J) \neq \emptyset = C_{ij} \cap [J \setminus T(J)] \tag{i}$$

$(i, j = 1, 2, \ldots)$. We now define a relation $\boldsymbol{R}_{ij}$ in $\boldsymbol{J}_{ij}$ by setting $J\boldsymbol{R}_{ij}J'$ if and only if

$$[J \setminus T(J)] \cap [J' \setminus T(J')] \neq \emptyset \tag{ii}$$

for $J, J' \in \boldsymbol{J}_{ij}$. First we prove that $\boldsymbol{R}_{ij}$ is an equivalence relation $(i, j = 1, 2, \ldots)$.

Clearly, the relation $\boldsymbol{R}_{ij}$ is reflexive and symmetric. To verify that it is transitive, let us take irreducible continua $J, J', J'' \in \boldsymbol{J}_{ij}$ such that $J\boldsymbol{R}_{ij}J'$ and $J'\boldsymbol{R}_{ij}J''$. Thus $C_{ij} \cap J \neq \emptyset$, whence $C_{ij} \cup J$ is a continuum which meets J'. Since X is hereditarily unicoherent, the set

$$(C_{ij} \cup J) \cap J' = (C_{ij} \cap J') \cup (J \cap J')$$

is a continuum which meets $C_{ij} \cap J' \subset T(J')$, by (i). The latter continuum also intersects $J' \setminus T(J')$, by (ii). But if a continuum intersects both an end tranche and its complement in an irreducible continuum, it contains all tranches from a neighbourhood of the end tranche. We conclude that $J \cap J'$ contains a neighbourhood of the end tranche $T(J')$ in J'. The same argument applied to J'' instead of J shows that $J'' \cap J'$ contains a neighbourhood of $T(J')$ in J'. Consequently, $T(J')$ lies in the interior of $K = J \cap J' \cap J''$ in J', and therefore $T(J')$ has void interior in K. According to (i), the end tranche $T(J')$ intersects C_{ij}, and C_{ij} is disjoint with $J \setminus T(J)$. Since $T(J') \subset K \subset J$, we infer that the continuum $T(J')$ intersects $T(J)$ and has void interior in J. Hence $T(J') \subset T(J)$ (see [2], p. 153). Also, let us observe that we could as well take the triple (J', J, J) in lieu of the triple (J, J', J'') and then getting $T(J) \subset T(J')$. Thus $T(J') = T(J)$. The same argument shows that $T(J') = T(J'')$. We see that $T(J) = T(J'')$ has void interior in $K \subset J \cap J''$. This yields $J\boldsymbol{R}_{ij}J''$, and we have proved that $\boldsymbol{R}_{ij}$ is indeed an equivalence relation.

Moreover, we have proved that $J\boldsymbol{R}_{ij}J'$ implies $T(J) = T(J')$. Let us write $T(\boldsymbol{J}) = T(J)$ for $J \in \boldsymbol{J} \in \boldsymbol{J}_{ij}/\boldsymbol{R}_{ij}$, and let $q(\boldsymbol{J})$ be a point belonging to $T(\boldsymbol{J})$, for $\boldsymbol{J} \in \boldsymbol{J}_{ij}/\boldsymbol{R}_{ij}$. If $\boldsymbol{J}, \boldsymbol{J}' \in \boldsymbol{J}_{ij}/\boldsymbol{R}_{ij}$ and $\boldsymbol{J} \neq \boldsymbol{J}'$, we have $J \operatorname{non} \boldsymbol{R}_{ij} J'$ for $J \in \boldsymbol{J}$ and $J' \in \boldsymbol{J}'$. Thus the sets $J \setminus T(J)$ and $J' \setminus T(J')$ are disjoint, by (ii), and they contain non-degenerate continua. Since X is Suslinian, it follows that the collection $\boldsymbol{J}_{ij}/\boldsymbol{R}_{ij}$ is countable $(i, j = 1, 2, \ldots)$. We are going to show that the sets

$$Q = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \{q(\boldsymbol{J}) \colon \boldsymbol{J} \in \boldsymbol{J}_{ij}/\boldsymbol{R}_{ij}\} \tag{iii}$$

and $P = X \setminus Q$ fulfil requirements of 2.2. It is already shown that Q is countable.

Suppose Y is a subcurve of X. Then there exist an integer $i > 0$ and a point $p \in Y$ such that the closure of G_i does not contain p and the common part of G_i with Y has at least one non-degenerate component. Hence there exists an integer $j > 0$ such that $C_{ij} \cap Y \neq \emptyset$. Since C_{ij} is a component of the closure of G_i, we have $p \notin C_{ij}$. The sets $C_{ij} \cap Y$ and $\{p\}$ being disjoint and closed in the continuum Y, there exists a closed set $J \subset Y$ such that J is irreducibly connected between $C_{ij} \cap J$ and $\{p\}$ (see [2], p. 158). Thus $C_{ij} \cap J \neq \emptyset$ and if $q \in C_{ij} \cap J$, then J is irreducible between p and q (ibidem, p. 159). It follows that J is an irreducible continuum of type λ whose end tranches contain p and $C_{ij} \cap J$, respectively. Therefore J satisfies (i) whence $J \in \boldsymbol{J}_{ij}$. Taking $\boldsymbol{J} \in \boldsymbol{J}_{ij}/\boldsymbol{R}_{ij}$ such that $J \in \boldsymbol{J}$, we obtain

$$q(\boldsymbol{J}) \in T(\boldsymbol{J}) = T(J) \subset J \subset Y,$$

and thus $Q \cap Y \neq \emptyset$, by (iii). This shows P is ponctiform and the proof of 2.2 is complete.

A curve X is said to be a *dendroid* provided X is hereditarily unicoherent and arcwise connected. Each subcurve of a dendroid is also a dendroid.

2.3. COROLLARY. *In order that a dendroid X be Suslinian it is necessary and sufficient that X admit a decomposition $X = P \cup Q$ where P is ponctiform and Q is countable.*

Remark. In the next section we describe a dendroid to show that 2.3 cannot be strengthened by requiring that P is totally disconnected rather than ponctiform.

§ 3. An example of a Suslinian dendroid. A space is called *totally disconnected* provided each of its quasi-components is degenerate. Each totally disconnected space is ponctiform.

3.1. EXAMPLE. *There exists a Suslinian dendroid X such that $X \neq P \cup Q$ where P is totally disconnected and Q is countable.*

Proof. Let $\tau = (T, S, r)$ be a triple composed of a triangle T, a side S of T, and an end point r of S; so r is a vertex of T. If $p, q \in T$, we denote by $\overline{pq}$ the straight segment with end points p and q. We have $S = \overline{rr'}$ where r' is a vertex of T, and let us denote by r'' the third vertex of T, different from r and r'. Take points $p_1, p_2 \in S$ and $q_1, q_2 \in \overline{rr''}$ such that

$$\operatorname{dist}(p_1, r) = \tfrac{1}{3}\operatorname{dist}(r, r'), \quad \operatorname{dist}(p_2, r) = \tfrac{2}{3}\operatorname{dist}(r, r'),$$

$$\operatorname{dist}(q_1, r) = \tfrac{1}{3}\operatorname{dist}(r, r''), \quad \operatorname{dist}(q_2, r) = \tfrac{2}{3}\operatorname{dist}(r, r''),$$

and points $x_i \in \overline{r'r''}$ such that $\operatorname{dist}(x_i, r') = i^{-1}\operatorname{dist}(r', r'')$, for $i = 1, 2, \ldots$ Let y_{ij} be the intersection point of the segment $\overline{rx_{2i}}$ with the segment $\overline{p_j q_j}$, for $i = 1, 2, \ldots$ and $j = 1, 2$. We can find points $r_i \in \overline{y_{i1}y_{i2}}$ such that each point of the segment $\overline{p_1p_2}$ is the limit point of a subsequence of the sequence $r_1, r_2, \ldots$ Let $z_i \in \overline{rx_{2i-1}}$ be the point such that the segment $\overline{r_iz_i}$ is parallel to the side $\overline{r'r''}$. Denote by T'_i the triangle with vertexes r, r_i, z_i, by T''_i the triangle with vertexes r_i, x_{2i}, x_{2i-1}, and put

$$\mathfrak{T}(\tau) = \{(T'_i, \overline{rr_i}, r)\colon\ i = 1, 2, \ldots\} \cup \{(T''_i, \overline{r_ix_{2i}}, r_i)\colon\ i = 1, 2, \ldots\}.$$

Given a triple τ as above, we define collections $\mathfrak{T}_n$ of triples inductively by setting

$$\mathfrak{T}_0 = \{\tau\}, \qquad \mathfrak{T}_{n+1} = \bigcup_{\tau \in \mathfrak{T}_n} \mathfrak{T}(\tau),$$

for $n = 0, 1, \ldots$ Finally, we denote by X_n the union of all triangles in triples from $\mathfrak{T}_n$, and define

$$X = \bigcap_{n=0}^{\infty} \operatorname{cl} X_n.$$

Observe that the area of any triangle from $\mathfrak{T}(\tau)$ is not greater than $\frac{2}{3}$ area T. It follows that the diameters of triangles from $\mathfrak{T}_n$ converge to zero when n tends to the infinity.

One readily sees X is a dendroid. If X was not Suslinian, there would exist a number $\varepsilon_0 > 0$ and an uncountable collection $\boldsymbol{C}$ of pairwise disjoint subcurves of X such that $\operatorname{diam} C > \varepsilon_0$ for $C \in \boldsymbol{C}$. Let n be large enough so that the diameters of triangles from $\mathfrak{T}_n$ are less than ε_0. The set V_n of vertexes of triangles from $\mathfrak{T}_n$ is countable. Consequently, there exists an uncountable collection $\boldsymbol{C}' \subset \boldsymbol{C}$ such that $|\boldsymbol{C}'| \subset X \setminus V_n$ (²). But each continuum joining two triangles from $\mathfrak{T}_n$ in the closure of X_n meets V_n. Thus $|\boldsymbol{C}'| \subset X \setminus X_n$. On the other hand, the set $\operatorname{cl}(X \setminus X_n)$ is the union of sides of triangles from $\mathfrak{T}_0 \cup \ldots \cup \mathfrak{T}_{n-1}$, and so $\operatorname{cl}(X \setminus X_n)$ is a rational curve. This contradicts 1.3, since $|\boldsymbol{C}'| \subset \operatorname{cl}(X \setminus X_n)$ means that $\operatorname{cl}(X \setminus X_n)$ is not Suslinian. Therefore X must be Suslinian.

Before verifying the main decomposition property (see 3.1) of the dendroid X just constructed, we prove a lemma which deals with some special cuttings of X.

3.2. Lemma. *If a set $Z \subset X$ is compact 0-dimensional and Z cuts X between some points of the segment S, then Z is uncountable.*

Proof. We prove this lemma by showing that Z contains a Cantor set. In fact, the closure of the set $\{r_1, r_2, \ldots\}$ contains the segment $\overline{p_1p_2}$, thus there is an infinite subsequence $r_{i_1}, r_{i_2}, \ldots$ such that $r_{i_j} \notin Z$ for

(²) By $|\boldsymbol{C}|$ we denote the union of all sets belonging to $\boldsymbol{C}$.

$j = 1, 2, \ldots$ If Z cut the dendroid X between points of only finitely many segments in triples from $\mathfrak{T}(\tau)$, there would exist an integer j_0 such that the set

$$A_j = \overline{rx_{2i_j}} \setminus Z$$

would lie in one quasi-component of $X \setminus Z$ for $j \geqslant j_0$. But by the hypothesis, the set $S \setminus Z = \overline{rr'} \setminus Z$ does not lie in one quasi-component of $X \setminus Z$. This is a contradiction, since Z being 0-dimensional, A_j is dense in $\overline{rx_{2i_j}}$, and thus every point of $\overline{rr'}$ is the limit point of a sequence of points belonging to A_j $(j \geqslant j_0)$, respectively. Consequently, the set Z cuts X between some points of infinitely many segments in triples from $\mathfrak{T}(\tau)$. What we actually need is only that Z cuts X between some points of at least two segments S_0, S_1 in triples from $\mathfrak{T}(\tau)$. Let T_0, T_1 be corresponding triangles. We can find compact subsets Z_0, Z_1 of Z such that $Z_k \subset T_k$, the set Z_k contains no vertex of T_k, and Z_k cuts X between some points of the segment S_k $(k = 0, 1)$. Observe that then Z_0 and Z_1 are disjoint.

Let us denote

$$z_0(Z) = Z_0, \qquad z_1(Z) = Z_1,$$

and define compact sets $Z_{k_1\ldots k_n}$ $(k_j = 0, 1)$ inductively by

$$Z_{k_1\ldots k_n 0} = z_0(Z_{k_1\ldots k_n}), \qquad Z_{k_1\ldots k_n 1} = z_1(Z_{k_1\ldots k_n}),$$

for $n = 1, 2, \ldots$ Since $Z_{k_1\ldots k_n}$ is a subset of a triangle from $\mathfrak{T}_n$, we have

$$\lim_{n\to\infty} \operatorname{diam} Z_{k_1\ldots k_n} = 0$$

for any sequence of $k_j = 0, 1$. Thus

$$\bigcap_{n=1}^{\infty} \bigcup_{k_j = 0,1} Z_{k_1\ldots k_n} \subset Z$$

is a Cantor set, and 3.2 is proved.

Now, given a decomposition $X = P \cup Q$ where Q is a countable set, let us take arbitrary two points $x, y \in S \setminus Q$ and suppose there exists a closed-open subset F of P such that $x \in F$ and $y \notin F$. Then the sets

$$U = \{x \in X\colon\ \operatorname{dist}(x, F) < \operatorname{dist}(x, P \setminus F)\},$$
$$V = \{x \in X\colon\ \operatorname{dist}(x, F) > \operatorname{dist}(x, P \setminus F)\}$$

are both open in X, disjoint, and $x \in U$, $y \in V$. Thus $X \setminus (U \cup V)$ cuts the dendroid X between the points x and y. But we have $P \subset U \cup V$ because F is closed-open in P. Consequently, we get $X \setminus (U \cup V) \subset Q$ which contradicts 3.2. It follows that the uncountable set $S \setminus Q$ lies in one quasi-component of P, so that P cannot be totally disconnected. This completes the proof of 3.1.

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Completely regular mappings with compact ANR fiber

by

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1. Introduction. Completely regular mappings were first considered by Dyer and Hamstrom in 1958 [2]. They are discrete analogues of locally trivial projections. If the homeomorphism group of the fiber is locally path-connected, and if the fiber is locally compact and separable, then a completely regular mapping is a Serre fibration [10].

In this paper, we will prove that if $p: E \to B$ is completely regular with fiber F, where F is a compact ANR and B is finite-dimensional and locally compact, then p is a Hurewicz fibration. We have thus improved a result of Michael ([9], p. 381) which would imply that p is a Serre fibration.

2. Definitions and notation.

(2.1) A continuous surjection $p: E \to B$ is *completely regular* if E and B are metric, and if for each $b \in B$ and $\varepsilon > 0$, there exists $\delta(b, \varepsilon) > 0$ such that if $d(b, b') < \delta(b, \varepsilon)$, there exists a homeomorphism h: $p^{-1}(b) \to p^{-1}(b')$, such that $d(x, h(x)) < \varepsilon$ for all $x \in p^{-1}(b)$.

The space B will always be assumed to be connected. Thus all fibers are homeomorphic, and we will denote this common fiber by F.

A topological space X is *locally n-connected* ($\mathbf{LC}^n$) if, given $x \in X$ and an open neighborhood U of x, there exists an open set V with $x \in V \subseteq U$, such that if f: $S^m \to V$ is a map ($m \leqslant n$), then f extends to F: $B^{m+1} \to U$.

Let $\{S_\alpha\}(\alpha \in A)$ be a collection of subsets of X. $\{S_\alpha\}$ is *equi*-$\mathbf{LC}^n$ [8] if, given $x \in X$ and an open neighborhood U of x, there exists an open set V with $x \in V \subseteq U$, such that if f: $S^m \to V \cap S_\alpha$ is a map ($m \leqslant n$ and $\alpha \in A$), then f extends to F: $B^{m+1} \to U \cap S_\alpha$.

Let Y be a topological space, and let $\mathcal{S}(Y)$ denote the collection of subspaces of Y. A function φ: $X \to \mathcal{S}(Y)$ is called a *lower semi-continuous carrier* (*l.s.c. carrier*) [7] if, given $x \in X$ and an open subset U of Y with $\varphi(x) \cap U \neq \emptyset$, then there exists an open neighborhood V of x, such that if $x' \in V$, then $\varphi(x') \cap U \neq \emptyset$.

(2.2) Note that if p: $E \to B$ is continuous and open, then the function taking b to $p^{-1}(b)$ is a l.s.c. carrier from B to $\mathcal{S}(E)$ ([7], p. 382).