# On game-theoretic methods in the theory of Souslin sets

by

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**1. Introduction.** In this note, we shall use the methods of Blackwell [1] to prove the Coreduction Principle (stated below) for Souslin sets in certain topological spaces and also establish a result on the constituents (defined below) of a Souslin set.

Let $Y$ be a topological space. A subset $A$ of $Y$ is said to be a *Souslin set* if there exists a system $\{A_{n_1 n_2 \ldots n_k}\}$, indexed by all finite sequences of natural numbers, of closed subsets of $Y$ such that

$$A = \bigcup_{\{n_k\}} \bigcap_{k=1}^{\infty} A_{n_1 n_2 \ldots n_k}$$

where the union extends over all sequences of natural numbers.

A subset $A$ of $Y$ is said to be a *bi-Souslin set* if both $A$ and $Y-A$ are Souslin sets.

An alternative way of describing Souslin sets is through sieves. Denote by $Q$ the set of all rationals in the open interval $(0,1)$, and label the elements of $Q$ as $r_1, r_2, \ldots$ (we shall hold fixed throughout the paper this particular labelling of the elements of $Q$). Any system $\{W_r, r \in Q\}$, indexed by the elements of $Q$, of subsets of $Y$ will be called a *sieve*. By *the set sifted by the sieve* $\{W_r, r \in Q\}$ is meant the set of all $y \in Y$ such that there is a sequence $\{r_{n_k}\}$ (possibly depending on $y$) of elements of $Q$ such that $r_{n_1} > r_{n_2} > \ldots$ and $y \in W_{r_{n_k}}$ for all $k \geqslant 1$. The alternative way of describing Souslin sets is this: $A$ is a Souslin subset of $Y$ if and only if there is a sieve $\{W_r, r \in Q\}$ of closed subsets of $Y$ such that $A$ is the set *sifted* by $\{W_r, r \in Q\}$ (cf. Theorems 9 and 10 in [5], p. 25).

Let $A$ be a Souslin subset of $Y$ and let $\{W_r, r \in Q\}$ be a sieve such that $A$ is the set sifted by $\{W_r, r \in Q\}$. For each ordinal $\alpha < \omega_1$ (= the first uncountable ordinal), let $A_\alpha$ be the set of all $y \in Y$ such that the set $\{r \in Q\colon y \in W_r\}$, when equipped with the usual order on the rationals, is of ordinal type $\alpha$. The sets $\{A_\alpha\colon \alpha < \omega_1\}$ are called the *constituents* of the Souslin set $A$ relative to the sieve $\{W_r, r \in Q\}$.

The aim of this paper is to prove by game-theoretic methods the following theorems.

THEOREM 1 (COREDUCTION PRINCIPLE). *Let $Y$ be a topological space in which every open set is a Souslin set. If $A$, $B$ are Souslin sets in $Y$, then there exist Souslin sets $E$, $F$ in $Y$ such that $A \subset E$, $B \subset F$, $A \cap B = E \cap F$ and $E \cup F = Y$.*

The classical analogue of Theorem 1 (that is, with $Y$ a Polish space and $A$, $B$ analytic subsets of $Y$) was established by Kuratowski [3]. Blackwell [1] used game-theoretic methods to prove the classical result. We shall imitate Blackwell's methods to prove Theorem 1.

THEOREM 2. *Let $Y$ be a topological space in which every open set is a Souslin set. Let $A$ be a Souslin set in $Y$ and let $\{W_r, r \in Q\}$ be any sieve of closed subsets of $Y$ such that $A$ is the set sifted by $\{W_r, r \in Q\}$. Then the constituents of $A$ (relative to $\{W_r, r \in Q\}$) are bi-Souslin subsets of $Y$.*

Theorem 2 was proved by methods quite different from ours by Rogers and Willmott (see corollary to Theorem 12 in [5], p. 30).

In the next section, we build up the machinery needed to prove Theorems 1 and 2.

**2. Sieves and games.** Let $Y$ be a topological space and let $\{W_r, r \in Q\}$ and $\{Z_r, r \in Q\}$ be two sieves of subsets of $Y$. Following Blackwell [1], we associate with each $y \in Y$ a two-person game $G(y)$ as follows: Players I and II choose rationals from $Q$ alternately, player I being the first to make a choice, each choice being made with complete information about previous choices of both players. A play $\pi = (r_{m_1}, r_{n_1}, r_{m_2}, r_{n_2}, \ldots)$ is a *win for player* I in $G(y)$ if there is a natural number $k$ such that $r_{m_1} > r_{m_2} > \ldots > r_{m_k}$, $y \in W_{r_{m_i}}$, $i = 1, 2, \ldots, k$, $r_{n_1} > r_{n_2} > \ldots > r_{n_{k-1}}$, $y \in Z_{r_{n_i}}$, $i = 1, 2, \ldots, k-1$, and either $r_{n_k} \geqslant r_{n_{k-1}}$ or $y \notin Z_{r_{n_k}}$. The play $\pi$ is a *win for player* II in $G(y)$ if there is a $k \geqslant 1$ such that $r_{m_1} > r_{m_2} > \ldots > r_{m_{k-1}}$, $y \in W_{r_{m_i}}$, $i = 1, 2, \ldots, k-1$, $r_{n_1} > r_{n_2} > \ldots > r_{n_{k-1}}$, $y \in Z_{r_{n_i}}$, $i = 1, 2, \ldots, k-1$, and either $r_{m_k} \geqslant r_{m_{k-1}}$ or $y \notin W_{r_{m_k}}$. Finally the play $\pi$ ends in a *draw* in $G(y)$ if for every $k \geqslant 1$, $r_{m_k} > r_{m_{k+1}}$, $y \in W_{r_{m_k}}$, $r_{n_k} > r_{n_{k+1}}$, and $y \in Z_{r_{n_k}}$.

Thus, each player at each stage tries to produce a rational $r \in Q$ which is strictly smaller than his previous choices and such that $y \in W_r$ or $y \in Z_r$ according as whether player I plays or player II plays. The first player to fail in this loses in the game $G(y)$. If neither player fails, it is a draw.

Let $P_1$ be the collection of all finite sequences of elements of $Q$ (including the empty sequence, which we denote by $e$) of even length, let $P_2$ be the collection of all finite sequences of elements of $Q$ of odd length, and let $P = P_1 \cup P_2$. By a *strategy* (in any of the games $G(y)$) for player I (II) is meant a function from $P_1$ ($P_2$) to $Q$. Denote the set of all strategies for players I and II by $\Phi$ and $\Psi$, respectively; that is, $\Phi = Q^{P_1}$ and $\Psi = Q^{P_2}$. Equip $\Phi$ and $\Psi$ with the product of discrete topologies on $Q$. Since $P_1$ and $P_2$ are countably infinite, we note that $\Phi$ and $\Psi$ are homeomorphic to $N^N$, where $N$ is the set of all natural numbers and $N^N$ is equipped with the product of discrete topologies on $N$.



A strategy $\varphi$ for player I and a strategy $\psi$ for player II uniquely determine a play $(r_{m_1}, r_{n_1}, r_{m_2}, r_{n_2}, \ldots)$ as follows:

$$r_{m_1} = \varphi(e),$$

$$r_{n_k} = \psi(r_{m_1}, r_{n_1}, \ldots, r_{m_{k-1}}, r_{n_{k-1}}, r_{m_k}), \quad k \geqslant 1,$$

and

$$r_{m_{k+1}} = \varphi(r_{m_1}, r_{n_1}, \ldots, r_{m_k}, r_{n_k}), \quad k \geqslant 1.$$

We shall denote the play determined by player I using the strategy $\varphi$ and player II using the strategy $\psi$ by $\langle \varphi, \psi \rangle$. We say that $\varphi^* \in \Phi$ is a *winning* strategy in $G(y)$ for player I if for every $\psi \in \Psi$, the play $\langle \varphi^*, \psi \rangle$ is a win for player I in $G(y)$. Call a strategy $\varphi^* \in \Phi$ a *drawing* strategy for player I in $G(y)$ if for every $\psi \in \Psi$, the play $\langle \varphi^*, \psi \rangle$ is a win for player I in $G(y)$ or the play $\langle \varphi^*, \psi \rangle$ ends in a draw in $G(y)$. Analogous definitions apply to winning and drawing strategies for player II.

We now prove a lemma which will be used in the sequel.

LEMMA. *Let $Y$ be a topological space in which every open set is a Souslin set. Let $\{W_r, r \in Q\}$ and $\{Z_r, r \in Q\}$ be two sieves of closed subsets of $Y$. Define:*

$$E = \{y \in Y\colon \textit{player I has a drawing strategy in } G(y)\}$$

*and*

$$F = \{y \in Y\colon \textit{player II has a drawing strategy in } G(y)\}.$$

*Then $E$ and $F$ are Souslin subsets of $Y$.*

(Here, of course, $G(y)$, $y \in Y$, are the games associated, as above, with the sieves $\{W_r, r \in Q\}$ and $\{Z_r, r \in Q\}$ of the lemma).

Proof. We shall prove that $E$ is a Souslin set. An analogous proof works for $F$.

Let $H = \{(y, \varphi) \in Y \times \Phi\colon \varphi$ is a drawing strategy for player I in $G(y)\}$. Observe that $E$ is the projection of $H$ to $Y$. Thus, if we can prove that $H$ is a Souslin subset of $Y \times \Phi$, it will follow by a result of Rogers and Willmott [4] that $E$ is a Souslin set in $Y$. In fact, we shall now show that $H$ is bi-Souslin in $Y \times \Phi$.

With each sequence $(r_{m_1}, r_{n_1}, \ldots, r_{m_{k-1}}, r_{n_{k-1}}) \in P_1$ (when $k = 1$, the sequence $(r_{m_1}, r_{n_1}, \ldots, r_{m_{k-1}}, r_{n_{k-1}})$ is to be interpreted as the empty sequence), we associate sets $K(r_{m_1}, r_{n_1}, \dots, r_{m_{k-1}}, r_{n_{k-1}})$, $L(r_{m_1}, r_{n_1}, \dots, r_{m_{k-1}}, r_{n_{k-1}})$ and $M(r_{m_1}, r_{n_1}, \dots, r_{m_{k-1}}, r_{n_{k-1}})$ as follows:

$$K(r_{m_1}, r_{n_1}, \dots, r_{m_{k-1}}, r_{n_{k-1}})$$
$$= [\bigcap_{i=1}^{k-1} W_{r_{m_i}} \cap \bigcap_{i=1}^{k-1} Z_{r_{n_i}}] \times [\bigcap_{i=1}^{k-1} \{\varphi \in \Phi\colon \varphi(r_{m_1}, r_{n_1}, \dots, r_{m_{i-1}}, r_{n_{i-1}}) = r_{m_i}\}] \text{ if } k > 1$$
$$= Y \times \Phi \quad \text{if } k = 1 .$$

$$L(r_{m_1}, r_{n_1}, \dots, r_{m_{k-1}}, r_{n_{k-1}})$$
$$= \bigcup_{r \in Q} [W_r^c \times \{\varphi \in \Phi\colon \varphi(r_{m_1}, r_{n_1}, \dots, r_{m_{k-1}}, r_{n_{k-1}}) = r\}], \qquad k \geqslant 1 .$$

$$M(r_{m_1}, r_{n_1}, \dots, r_{m_{k-1}}, r_{n_{k-1}})$$
$$= \bigcup_{r \in Q(r_{m_1}, r_{n_1}, \dots, r_{m_{k-1}}, r_{n_{k-1}})} [Y \times \{\varphi \in \Phi\colon \varphi(r_{m_1}, r_{n_1}, \dots, r_{m_{k-1}}, r_{n_{k-1}}) = r\}]$$

where

$$Q(r_{m_1}, r_{n_1}, \dots, r_{m_{k-1}}, r_{n_{k-1}}) = \{r \in Q\colon r \geqslant r_{m_{k-1}}\} \quad \text{if } k > 1$$
$$= \emptyset \quad \text{if } k = 1 .$$

(union over the empty set is to be interpreted as the empty set). It is easy to see that the sets

$$K(r_{m_1}, r_{n_1}, \dots, r_{m_{k-1}}, r_{n_{k-1}}),$$
$$L(r_{m_1}, r_{n_1}, \dots, r_{m_{k-1}}, r_{n_{k-1}}),$$
$$M(r_{m_1}, r_{n_1}, \dots, r_{m_{k-1}}, r_{n_{k-1}})$$

are all bi-Souslin in $Y \times \Phi$. Finally, note that

$$H^c = \bigcup_{s \in \bar{P}_1} [K(s) \cap [L(s) \cup M(s)]]$$

where

$$\bar{P}_1 = \bigcup_{k=1}^{\infty} [\{(r_{m_1}, r_{n_1}, \dots, r_{m_k}, r_{n_k}) \in P_1\colon r_{m_i} > r_{m_{i+1}}, r_{n_i} > r_{n_{i+1}},$$
$$i = 1, 2, \dots, k-1\}] \cup \{e\} .$$

Since $\bar{P}_1$ is countable, it follows that $H^c$ is bi-Souslin in $Y \times \Phi$, and so $H$ is bi-Souslin in $Y \times \Phi$. This completes the proof of the lemma.

## 3. Proof of theorems.

Proof of Theorem 1. Let $\{W_r, r \in Q\}$ and $\{Z_r, r \in Q\}$ be sieves of closed subsets of $Y$ such that $A, B$ are, respectively, the sets sifted by $\{W_r, r \in Q\}$ and $\{Z_r, r \in Q\}$. Let $\{A_\alpha, \alpha < \omega_1\}$ and $\{B_\beta, \beta < \omega_1\}$ be the constituents of $A, B$ with respect to the sieves $\{W_r, r \in Q\}$ and $\{Z_r, r \in Q\}$, respectively. For each $y \in Y$, let $G(y)$ be the game associated with the



sieves $\{W_r, r \in Q\}$ and $\{Z_r, r \in Q\}$ as in Section 2. Let $E$ and $F$ be the sets defined in the lemma of Section 2. We shall prove that the sets $E, F$ have the required properties.

First, by the lemma of Section 2, $E$ and $F$ are Souslin subsets of $Y$. Next, we note that

(1) $$E = A \cup [\bigcup_{\alpha < \omega_1} (A_\alpha \cap \bigcup_{\beta < \alpha} B_\beta)]$$

and

(2) $$F = B \cup [\bigcup_{\beta < \omega_1} (B_\beta \cap \bigcup_{\alpha \leqslant \beta} A_\alpha)] .$$

To see this, let $y \in A$. Then there exists a sequence $\{r_{m_k^*}\}$ of elements of $Q$ such that for every $k \geqslant 1$, $r_{m_k^*} > r_{m_{k+1}^*}$ and $y \in W_{r_{m_k^*}}$. Now consider a strategy $\varphi^*$ for player I defined by:

$$\varphi^*(r_{i_1}, r_{i_1}', \dots, r_{i_{k-1}}, r_{i_{k-1}}') = r_{m_k^*} .$$

It is easy to see that $\varphi^*$ is a drawing strategy for player I in the game $G(y)$, so $y \in E$. Next suppose that $y \in A_\alpha \cap B_\beta$, where $\beta < \alpha < \omega_1$. Set $H_1 = \{r \in Q\colon y \in W_r\}$ and $H_2 = \{r \in Q\colon y \in Z_r\}$. Then $H_1$ and $H_2$ are of ordinal types $\alpha$ and $\beta$, respectively. Since $\beta < \alpha$, there is a similarity mapping (that is, a one-to-one and order-preserving mapping) $g$ which takes $H_2$ onto a proper segment of $H_1$. Choose an element $r^* \in H_1 - g(H_2)$ and define a strategy $\varphi^{*\prime}$ for player I (in the game $G(y)$) as follows:

$$\varphi^{*\prime}(e) = r^*$$

and

$$\varphi^{*\prime}(r_{i_1}, r_{i_1}', \dots, r_{i_k}, r_{i_k}') = g(r_{i_k}') \quad \text{if } r_{i_k}' \in H_2$$
$$= r' \quad \text{if } r_{i_k}' \in Q - H_2 ,$$

where $r'$ is a fixed but arbitrary element of $Q$. It is not difficult to see that $\varphi^{*\prime}$ is a winning strategy for player I in the game $G(y)$, so $y \in E$. Thus $E \supset A \cup [\bigcup_{\alpha < \omega_1} (A_\alpha \cap \bigcup_{\beta < \alpha} B_\beta)]$. Conversely, suppose

$$y \notin A \cup [\bigcup_{\alpha < \omega_1} (A_\alpha \cap \bigcup_{\beta < \alpha} B_\beta)] .$$

We distinguish two cases.

Case 1. $y \in B$. As $y \notin A$, it follows that $y \in A_\alpha$ for some $\alpha < \omega_1$. As $H_1$ is well-ordered while $H_2$ is not, it is clear that player II has a winning strategy in $G(y)$. Indeed, the set $H_2$ contains a strictly decreasing sequence $\{r_{n_k^*}\}$ so that the strategy $\psi^*$ for player II defined by

$$\psi^*(r_{i_1}, r_{i_1}', \dots, r_{i_{k-1}}, r_{i_{k-1}}', r_{i_k}) = r_{n_k^*}$$

wins for player II in the game $G(y)$. Hence $y \notin E$.

Case 2. $y \notin B$. It now follows that $y \in A_\alpha \cap B_\beta$ where $\alpha \leqslant \beta < \omega_1$. Hence there is a similarity mapping $g'$ from $H_1$ onto a segment of $H_2$.

Define a strategy $\psi^{*\prime}$ for player II as follows:

$$\psi^{*\prime}(r_{i_1}, r'_{i_1}, \ldots, r_{i_{k-1}}, r'_{i_{k-1}}, r_{i_k}) = g'(r_{i_k}) \quad \text{if } r_{i_k} \in H_1,$$
$$= r' \quad \text{if } r_{i_k} \in Q - H_1,$$

where $r'$ is a fixed but arbitrary element of $Q$. It is clear that $\psi^{*\prime}$ is a winning strategy for player II in the game $G(y)$, so that $y \notin E$. We have thus proved that

$$E \subset A \cup [\bigcup_{\alpha<\omega_1} (A_\alpha \cap \bigcup_{\beta<\alpha} B_\beta)],$$

from which equation (1) follows. Equation (2) follows analogously.

It is now straightforward to derive from equations (1)–(2) that $A \subset E$, $B \subset F$, $A \cap B = E \cap F$ and $E \cup F = Y$. This completes the proof of Theorem 1.

Proof of Theorem 2. Fix an ordinal $\alpha_0 < \omega_1$ and choose a subset $T$ of $Q$ so that $T$ is of ordinal type $\alpha_0$. Define $Z_r = Y$ if $r \in T$ and $Z_r = \emptyset$ if $r \notin T$. If $B$ is the set sifted by $\{Z_r, r \in Q\}$, then plainly $B = \emptyset$. Moreover, $B_\beta = \emptyset$ if $\beta \neq \alpha_0$ and $\beta < \omega_1$ and $B_\beta = Y$ if $\beta = \alpha_0$, where $\{B_\beta, \beta < \omega_1\}$ are the constituents of $B$ relative to the sieve $\{Z_r, r \in Q\}$. Let $\{A_\alpha, \alpha < \omega_1\}$ be the constituents of $A$ relative to the sieve $\{W_r, r \in Q\}$. For each $y \in Y$, let $G(y)$ be the game associated with the sieves $\{W_r, r \in Q\}$ and $\{Z_r, r \in Q\}$ as in Section 2. Let $E, F$ be the sets defined in the lemma of Section 2.

By the lemma of Section 2, $E$ and $F$ are Souslin subsets of $Y$. Moreover, the proof of Theorem 1 shows that

$$E = A \cup \bigcup_{\alpha>\alpha_0} A_\alpha$$

and

$$F = \bigcup_{\alpha\leqslant\alpha_0} A_\alpha .$$

It follows that $\bigcup_{\alpha\leqslant\alpha_0} A_\alpha$ is a bi-Souslin subset of $Y$, since $E \cup F = Y$ and $E \cap F = \emptyset$. As $\alpha_0$ was arbitrary, we have proved that for every ordinal $\delta < \omega_1$, $\bigcup_{\beta\leqslant\delta} A_\beta$ is bi-Souslin. Consequently,

$$A_\alpha = \bigcup_{\beta\leqslant\alpha} A_\beta - \bigcup_{\delta<\alpha} (\bigcup_{\beta\leqslant\delta} A_\beta)$$

is a bi-Souslin subset of $Y$. This completes the proof of Theorem 2.

Remark 1. Theorem 2 can be proved by means of classical methods as follows. Let $D$ be the Cantor set, which we shall think of as a countable product of copies of the two—element set $\{0, 1\}$. Define a sieve $\{P_r, r \in Q\}$ of closed subsets of $D$ as follows: $P_{r_n} = \{t \in D\colon t_n = 1\}$ where $t_n$ denotes the $n$th coordinate of $t$. Let $G$ be the set sifted by $\{P_r, r \in Q\}$ and let $\{G_\alpha, \alpha < \omega_1\}$ be the constituents of $G$ relative to $\{P_r, r \in Q\}$. Then it is known that the sets $G_\alpha$ are Borel subsets of $D$ (see [2], p. 272). Now consider the characteristic function (in the sense of Marczewski) of the sieve $\{W_r, r \in Q\}$, that is, $f(y) = \sum_{n=1}^{\infty} \frac{2}{3^n} I_{W_{r_n}}(y)$, $y \in Y$, where $I_{W_{r_n}}$ is the indicator of the set $W_{r_n}$. It is easy to verify that the function $f$ is measurable between the spaces $(Y, \boldsymbol{S})$ and $(D, \boldsymbol{B})$, where $\boldsymbol{S}$ is the $\sigma$-algebra of bi-Souslin subsets of $Y$ and $\boldsymbol{B}$ the $\sigma$-algebra of Borel subsets of $D$. Moreover, for each $\alpha < \omega_1$, $A_\alpha = f^{-1}(G_\alpha)$ (cf. [2], p. 408). Consequently each $A_\alpha$ is bi-Souslin in $Y$.



Remark 2. It is true that Theorem 1 can also be obtained by imitating Kuratowski's method in [3]. But this involves suitably modifying the sieves $\{W_r, r \in Q\}$ and $\{Z_r, r \in Q\}$ with which we started and then the sets $E$ and $F$ are no longer as naturally related to the orginial sieves as in our proof.

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