Some examples in the theory of Borel sets

by

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All spaces are normal, unless the contrary is explicitly stated, and "normal" implies "Hausdorff". Our notation and terminology about Borel sets follow [2]. In particular, α, β and γ are always ordinals less than the first uncountable ordinal. The real line with its usual topology is denoted $\boldsymbol{R}$ and the subspace of $\boldsymbol{R}$ consisting of the irrationals is denoted $\boldsymbol{P}$.

In perfectly normal spaces, Borel sets are particularly easy to handle because they can be resolved into an increasing transfinite sequence of classes: $G_0, G_1, \ldots, G_\alpha, \ldots$ The important property of perfectly normal spaces here is that every closed set in such a space is a G_δ. Less is needed: it is enough that every closed set be a G_α for some α. In Section 1, for each fixed α, we construct a space such that α is the smallest ordinal for which every closed set in the space is a G_α, answering a correspondence question of A. H. Stone. As an easy corollary, the disjoint union of the spaces thus constructed has the property that every open set is Souslin (generated from the closed sets by operation A), but in which there is an open set which is not an F_α for any α.

The problem of producing a space in which each closed set is a G_α for some α (depending on the closed set), but in which no α works for all closed sets, is badgered in Section 2. The resulting counterexample is Hausdorff, but fails to be normal and depends on the continuum hypothesis for its existence. It would be interesting to have a normal example.

In perfectly normal spaces (see [2], pp. 347–348), the family of Borel sets coincides with the smallest family containing the open sets and closed under countable intersection and countable *disjoint* union, and the same is true if the closed sets are used in place of the open sets. In Section 3 an example (one of those used in Section 1) is given to show that perfect normality is not needed for either of these assertions. Previous examples of a similar nature have been given by Frolik ([1], p. 166).

1. For each α, $1 < \alpha < \omega_1$, we will provide a (normal) space X_α in which α is the smallest ordinal such that every closed set is a G_α. As a set,

X_α will be the real line. Let A_α be a subset of the line which, in the usual topology, is a non-trivial G_α set (i. e., a G_α set but not a G_β set for any $\beta < \alpha$—such sets exist in the line, see [2], p. 371). The open sets in X_α are now defined to be the sets of the form $U \cup V$, where U is open in the usual line topology and V is any subset of $X_\alpha - A_\alpha$. [For $\alpha = 2$, A_α could be the rationals. The resulting space X_2 was first used by Michael [3]] (1). As Michael has pointed out, any space constructed in this way (by "discretizing" a set, in our case $X_\alpha - A_\alpha$) from a metric space is hereditarily paracompact and thus normal.

Now note that A_α is a G_α in X_α and has for its relative topology in X_α its usual metric topology. Then if F is closed in X_α, $F = [F \cap (X - A_\alpha)] \cup \cup (F \cap A_\alpha)$, the former being open in X_α and the latter being a G_δ set in A_α (since it is a closed subset of the metric space A_α) and thus a G_α in X_α. It follows that every closed set F is a G_α in X_α.

Next, if Q is an open set in X_α, it is easily seen that an open set P in the usual line topology exists for which $P \cap A_\alpha = Q \cap A_\alpha$ and $P \subset Q$. It follows that if A_α is a G_β in X_α, it is a G_β in the line, so this cannot happen for $\beta < \alpha$.

Now if X is the disjoint union of the spaces X_α just constructed, the open set $\bigcup_\alpha (X_\alpha - A_\alpha)$ in X is not an F_α for any α. But every open set in X is the union of $\aleph_1$ F_α sets (for variable α) and thus is a Souslin set.

This fulfills the promises made for Section 1 in the introduction.

2. The program here is to provide an example showing every closed set in a space can be represented as a G_α without it being possible to fix α. Some preliminary lemmas are needed; unfortunately, even with these, the resulting example has the lack of common decency to be non-normal.

LEMMA. *For each α, $1 \leqslant \alpha < \omega_1$, a non-trivial G_α set P_α can be found in $\boldsymbol{R}$ such that*

(1) $P_\alpha \cap P_\beta = \emptyset$, *if* $\alpha \neq \beta$,

(2) $P_1 \cup \ldots \cup P_\alpha$ *is not a* G_β *for* $\beta < \alpha$,

(3) $\bigcup_{1 \leqslant \alpha < \omega_1} P_\alpha = \boldsymbol{R}$ [*continuum hypothesis*].

Proof. For each $r \in P$, let $P(r) = \{(q, r) |\ q \in P\} \subset P \times P$. Assign each ordinal α, $1 \leqslant \alpha < \omega_1$, to some $P(r)$, hereafter called $P(r_\alpha)$. From [2], a non-trivial G_α subset P'_α of $P(r_\alpha)$ exists, and P'_α will still be a non-trivial G_α in $P \times P$. But $P \times P$ is homeomorphic to P (proof: the irrationals are a countable product of copies of the integers), say h is a homeomorphism of $P \times P$ onto P. Let $P^*_\alpha = h(P'_\alpha)$. Now P^*_α is a non-trivial G_α

(1) And I acknowledge a debt to J. R. Isbell and A. H. Stone, each of whom pointed out the similarity of my (then different) example for $\alpha = 2$ to Michael's example.

set in P and hence in $\boldsymbol{R}$, and the P^*_α are mutually disjoint. Moreover, if $P^*_1 \cup \ldots \cup P^*_\alpha$ is a G_β, then $P'_1 \cup \ldots \cup P'_\alpha$ is a G_β, and hence $P'_\alpha = (P'_1 \cup \cup \ldots \cup P'_\alpha) \cap P(r_\alpha)$ is a $G_\beta \cap G_\delta$, hence a G_δ, if $\beta \geqslant 1$. Thus this cannot happen for $\beta < \alpha$.

We have the properties of the theorem except for 3. To get this, assume the continuum hypothesis and write the real numbers as a transfinite sequence $x_1, x_2, \ldots, x_\alpha, \ldots, \alpha < \omega_1$. Let $P^+_\alpha = P^*_\alpha \cup \{x_\alpha\}$. Then P^+_α is still a non-trivial G_α and $P^+_3 \cup \ldots \cup P^+_\alpha$ differs from $P^*_1 \cup \ldots \cup P^*_\alpha$ by a countable set ($\therefore$ an F_σ) and hence cannot be a G_β. The P^+_α are not mutually disjoint, but if we now set $P_\alpha = P^+_\alpha - \{x_\beta |\ \beta < \alpha\}$, we obtain mutually disjoint sets P_α having all the properties required (by arguments similar to those already given).

We proceed to the construction of the counterexample of this section. For each α, $1 \leqslant \alpha < \omega_1$, let P_α be a non-trivial G_α subset of $\boldsymbol{R}$ having properties 1 through 3 of the Lemma above. We will retopologize the line, now called X, as follows: the neighborhoods of a point p in P_α will have the form $U - (\bigcup_{\beta<\alpha} P_\beta)$ where U is a usual linear neighborhood of p. It is clear, then, that every set open in the usual topology is open in this topology.

Setting $B_\alpha = P_1 \cup \ldots \cup P_\alpha$, it is clear that B_α is closed and a G_α in X. But it is not a G_β for $\beta < \alpha$, for if G is any open subset of X meeting B_α, each point p of $G \cap B_\alpha$ has a neighborhood $U_p - (\bigcup_{\beta<\gamma} P_\beta)$, where U is a linear neighborhood of p and $\gamma \leqslant \alpha$, contained in G. Let $H = \bigcup_{p \in G \cap B_\alpha} U_p$. Then, easily, if B_α is generated as a G_β in X by $G_1, G_2, \ldots$, it will be generated as a G_β in $\boldsymbol{R}$ by $H_1, H_2, \ldots$, and this is not possible, by Part 2 of the lemma, for $\beta < \alpha$.

Thus closed non-trivial G_α sets exist in X for arbitrarily high class $\alpha < \omega_1$. It remains to show that each closed set in X is a G_α for some α. We attack the open sets, showing each to be an F_α for large enough α.

Let W be open in X. For $p \in W \cap P_\alpha$, let $V_p = U_p - \bigcup_{\beta<\alpha} P_\beta$ be a neighborhood of p contained in W. For each α, $1 \leqslant \alpha < \omega_1$, let $U_\alpha = \bigcup_{p \in W \cap B_\alpha} U_p$. Then $U_1, U_2, \ldots$ is an increasing, transfinite sequence of open subsets of $\boldsymbol{R}$—since $\bigcup_{1 \leqslant \alpha < \omega_1} U_\alpha$ is Lindelöf, the sequence must, in reality, be countable. If we let $V_\alpha = \bigcup_{p \in W \cap B_\alpha} V_p$, it follows that, for some $\alpha_0 < \omega_1$, $W = V_1 \cup \ldots \cup V_{\alpha_0}$. But $V_\alpha = U_\alpha - (\bigcup_{\beta<\alpha} P_\beta)$ is easily an F_α in $\boldsymbol{R}$ and thus in X, so that $W = V_1 \cup \ldots \cup V_{\alpha_0}$ is an F_{α_0} in X.

Thus X has the Borel properties we asked for: every closed set is a G_α for some α, but closed sets exist for arbitrarily high α which are non-trivial G_α sets.

It is clear that X is Hausdorff, since X consists of the real line with a topology finer than usual. However, X cannot be regular, as the following argument shows.

Since P_1 is not closed in the usual line topology we can find a limit point p of P_1 such that $p \in P$ for some $\alpha > 1$. Let U be any open set in the X-topology containing P_1. Then for some open set U_0 in the usual line topology, $P_1 \subset U_0 \subset U$. Let V_0 be an open set in the usual line topology with $p \in V_0$. Then $U_0 \cap V_0 \neq \emptyset$, and we are done if we show $U_0 \cap (V_0 - \bigcup_{\beta<\alpha} P_\beta) \neq \emptyset$. But otherwise $U_0 \cap V_0$ is a non-empty (usual) open set contained in $\bigcup_{\beta<\alpha} P_\beta$, while by construction each P_β is nowhere dense in the line. Then $\bigcup_{\beta<\alpha} P_\beta$ would be a set of first category with interior in the line. This contradiction establishes that X cannot be regular.

3. In any space X, let $\mathcal{B}(X)$ and $\mathcal{K}(X)$ (or, just $\mathcal{B}$ and $\mathcal{K}$, if X is fixed) denote, respectively, the family of Borel sets and the smallest family of sets:

(1) containing the closed sets,

(2) closed under countable intersection,

(3) closed under countable disjoint union.

Our purpose in this section is to show the existence of non-perfectly normal spaces in which $\mathcal{B}$ and $\mathcal{K}$ coincide. It is convenient to denote countable disjoint union by $\bar{\sigma}$, so that the sets of $\mathcal{K}$ occur in a transfinite sequence: $F, F_{\bar{\sigma}}, F_{\bar{\sigma}\delta}, F_{\bar{\sigma}\delta\bar{\sigma}}, \ldots$

THEOREM. *$\mathcal{B} = \mathcal{K}$ for the space X_2 of Section 1 (with A_2 taken to be the set of rationals Q).*

Proof. For convenience, denote X_2 by X, A_2 by Q and $X - A_2$ (the irrationals) by P.

First, it is a simple matter to show that, for $q \in Q$, $X - \{q\}$ is an $F_{\bar{\sigma}}$. Next, if A and B are closed subsets of X, with $A \subset B$, then $B - A$ is an $F_{\bar{\sigma}\delta\bar{\sigma}}$. For

$$B - A = [B \cap (-A \cap P)] \cup [B \cap (-A \cap Q)],$$

where the union is disjoint. The second term is countable and thus an $F_{\bar{\sigma}}$. The first term can be handled by showing that, whenever $D \subset P$, D is an $F_{\bar{\sigma}\delta}$. Now $\bar{D} - D$ is countable (consisting only of rationals) and

$$D = \bigcap_{q \in \bar{D} - D} \bar{D} - \{q\},$$

so it suffices to note that, since $\bar{D} - q = [X - \{q\}] \cap \bar{D}$, $\bar{D} - \{q\}$ is an $F_{\bar{\sigma}}$.

Now if $\bigcup_{i=1}^{\infty} A_i$ is an F_σ in X, we may assume $A_1 \subset A_2 \subset \ldots$ Then, from the above,

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} [A_i - A_{i-1}]$$

is an $F_{\bar{\sigma}\delta\bar{\sigma}}$.

Since every subset of X is an $F_{\sigma\delta}$ (each $A \subset X_2$ can be represented as $(A \cap Q) \cup (A \cap P)$ where $A \cap Q$ is an F_σ and $A \cap P$ is open. Since open subsets of X_2 are $F_{\sigma\delta}$ by Section 1, A is an $F_{\sigma\delta}$ every subset of X is an $F_{\bar{\sigma}\delta\bar{\sigma}\delta}$. This proves the theorem.

It is much easier to prove that, in the same space $X = X_2$, the family of Borel sets coincides with the smallest family

(1) containing the open sets,

(2) closed under countable intersection,

(3) closed under countable disjoint union.

References

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