Commutative rings in which every proper ideal is maximal

by

Fred I. Perticani (Minneapolis, Minn.)

Introduction. Following the line of Misan [16] and Szasz [18] we shall study here a class of rings having some strongly pathological properties.

All throughout we shall denote by A a *commutative ring with identity* 1 *in which every proper ideal* (i.e., different from (0) and A) *is maximal.*

We shall reduce the problem of characterizing all such rings to the more concrete one of computing the second cohomology group of a field with values in itself. More explicit definitions and statements are made below.

We will show that A has either one or two proper ideals, but not more. The later case is the easiest one and a full description of A is provided (see Theorem 1.4); the former is shown to be naturally related to an extension problem, (see Theorem 2.7) whence to the computation of cohomology groups. The choice of the particular cohomology theory to be used depends upon the generality of the approach—the complete solution being provided by the cohomology of Mac Lane (see [13], [14]), which is in fact a particular case of that of Shukla (see [17]). For the "splitting" case, the Hochshild cohomology [7], [8] is also available and the standard cohomology of abelian groups with trivial action (see Eilenberg–Mac Lane [3], [4], [5] and Hochshild–Serre [10]) may also be used to obtain some information.

1. First classification.

1.1. THEOREM. *A commutative ring with identity in which every proper ideal is maximal cannot have more than two different proper ideals.*

Proof. Assume I, J and K are three different ideals of A. Since $I+J$ properly contains I and I is maximal, $I+J$ must coincide with the whole ring A. Hence

$$1 = i+j, \text{ where } i \in I,\ j \in J\,. \tag{*}$$

Take now $k \in K$, $k \neq 0$. Clearly (*) implies $k = ki+kj$. But $KI \subset K \cap I$ and $KJ \subset K \cap J$. These intersections, being ideals properly contained in K, cannot be maximal, so that they must coincide with the zero ideal. This proves that $ki = kj = 0$ and *a posteriori* $k = ki + kj = 0$, a contradiction.

1.2. PROPOSITION. *If A contains two different proper ideals, then A is isomorphic to the direct product of them.*

Proof. Let I and J be the ideals. Then $I+J = A$, $I \cap J = 0$, and $IJ = 0$, so that $A \cong I \times J$.

1.3. PROPOSITION. *I and J are fields.*

Proof. Since $A \cong I \times J$, every ideal in the ring I (resp. J) is also an ideal of A. Hence I (resp. J) is a (commutative) ring with no proper ideals. Since $1 = i_0 + j_0$, $i_0 \in I$, $j_0 \in J$ and $IJ = 0$, we have for $i \in I$ (resp. $j \in J$)

$$i = ii_0 \quad (\text{resp. } j = jj_0) .$$

This proves that I (resp. J) is a ring with identity. But it is known (see Jacobson [12]) that a commutative ring with identity and no proper ideals must be a field. Hence I and J are fields.

1.4. THEOREM. *Every commutative ring with identity in which every proper ideal is maximal and having more than one proper ideal is isomorphic to the direct product of two fields.*

The proof follows from 1.1, 1.2 and 1.3.

2. Rings with only one proper ideal. In the last section we have considered the case of a ring with two proper ideals. As we have seen in 1.1 the only possible case left is that of a ring A containing only one proper ideal, which will be denoted by I.

2.1. PROPOSITION. *Let A be commutative ring with identity having only one proper ideal I. Then I is a "zero-ideal", i.e., $I^2 = 0$.*

Proof. Let S be the annihilator of I, i.e., $S = \{x \in A;\ xI = (0)\}$. Since S is an ideal, either $S = (0)$ or $S = I$ or $S = A$. The last possibility is ruled out by the fact that $1 \notin S$.

Assume then that $S = (0)$. Take any $z \in A$, $z \neq 0$; since $zI \neq 0$ and zI is an ideal contained in I, we must have $zI = I$. Take now $x \in I$ to be any nonzero element. Then, since $xI = I$, there exists $e \in I$ such that $xe = x$; e is different from 0. Consider the set $T = \{y \in A;\ ye = y\}$. T is a nonzero ideal (because $x \in T$), whence $I \subset T$, and in particular e is an identity for I. But now again, $xI = I$ implies the existence of an $x' \in I$ such that $xx' = e$ and this proves that the ring I is in fact a *field*.

Define now a mapping $\varphi\colon A \to I$ by $\varphi(a) = ae$. Since $\varphi(a+b) = (a+b)e = ae+be = \varphi(a)+\varphi(b)$ and $\varphi(ab) = (ab)e = abe^2 = aebe = \varphi(a)\varphi(b)$ we see that φ is a ring homomorphism. It is obviously onto since $x \in I$ implies $\varphi(x) = x$. Let B denote the kernel of φ. B being an ideal of A, it must coincide with either (0), or I or A; but $\varphi(e) = e \neq 0$ makes the two last cases impossible, so that $B = (0)$ and φ is also one-to-one, i.e., an isomorphism. This is a contradiction, because I is a field and A (having a proper ideal) is not. The contradiction followed from the assumption $S = (0)$. Thus the only possibility left is $S = I$, which, in other words means that I is a "zero-ideal": if $x, y \in I$, then $xy = 0$.

2.2. THEOREM. *Let A be a commutative ring with identity having only one proper ideal. Then A is an extension of a zero ring by a field, i.e., there exists an exact sequence of rings*

$$(**) \qquad 0 \to I \xrightarrow{j} A \xrightarrow{\pi} F \to 0$$

where I is a zero ring and F is a field.

Proof. Let I denote the proper ideal of A and take $F = A/I$. Since $I^2 = 0$ (proposition 2.1), I is a zero ring and since I is maximal, F is a field. The canonical mappings $I \to A$ and $A \to F$ provide the other data of the required exact sequence.

It is known (see Mac Lane [13]) that whenever a sequence like $(**)$ above is given, the product of A determines a structure of linear space for I over the field F, in the following way. First identity I with its image under j; so I becomes an ideal of A. Now for every $f \in F$ and $b \in I$, define fb as $fb = ab$, where a is any element of A mapped by π on f. Let us see that this action is well defined: if a, a' are mapped on f, then $a - a'$ belongs to the kernel of π, namely to I, and since $I^2 = 0$, we have $(a-a')\, b = 0$ for every $b \in I$. This means that $ab = a'b$ and so the element ab depends only on b and the class of a modulo I. It is now very easy to show that the action of F on I determines a structure of F-linear space for I. The reader will provide the details.

2.3. DEFINITION. The action of F on I and the structure of F-linear space of I described above will be referred to as the *action* and *the structure determined by the sequence* $(**)$.

2.4. PROPOSITION. *Under the hypothesis of Theorem 2.2, I (with the structure induced by $(**)$) is a one-dimensional linear space over the field F.*

Proof. Let $H \subset I$ be an F-subspace and $a \in A$. Then $aH = \pi(a)H \subset H$, and so H is an ideal of A. Therefore H coincides with (0) or with I. This shows that the only F-subspaces of I are the trivial ones, and hence I must be one dimensional.

2.5. COROLLARY: *Under the hypothesis of 2.2 the additive groups I and $F = A/I$ are isomorphic.*

A reciprocal of the sequence of statements made above is also true.

2.6. PROPOSITION. *Let*

$$({}^{*}_{**}) \qquad 0 \to I \xrightarrow{j} A \xrightarrow{\pi} F \to 0$$

be an exact sequence of commutative rings such that

1) *I is a zero ring, i.e., $I^2 = 0$;*

2) *F is a field;*

3) *I, with the structure determined by $(*{*}*)$ is a one-dimensional linear space over F.*

Then A is a commutative ring with identity which has only one proper ideal, namely $j(I)$.

Proof. The fact that A must have an identity (even under more general hypotheses) is known (see Mac Lane [13]). We shall sketch here a proof. First, identify I (using j) to $j(I)$. Observe now that I is a maximal ideal of A (F is a field); therefore, any ideal of A containing properly I must coincide with A itself. Choose $e \in A$ such that $\pi(e) = 1$ is the identity of F and consider the ideal Ae. Since $ex = x$ for every $x \in I$ (because $x = 1x = \pi(e)x = ex$), it is clear that $Ae \supset I$. As $1 = \pi(e) = \pi(e)\pi(e) = \pi(e^2) \in \pi(Ae)$, we see that $\pi(Ae) \neq 0$.

Then $Ae \neq I$, and so $Ae = A$. Choose now $k \in A$ such that $ke = e$ and define $J = \{y \in A;\ ky = y\}$. If $x \in I$, then $ex = x$, so that $kx = k(ex) = (ke)x = ex = x$. This means that $I \subset J$. Since $e \in J$ and $e \notin I$, the ideal J contains I *properly*, and hence $J = A$, or, in other words, k is an identity for A.

We shall see now that I is the only proper ideal of A. Assume that $J \subset A$ is an ideal. Since I is a one dimensional F-linear space, if $J \subset I$ and $J \neq I$, then necessarily $J = (0)$. We denote by H the intersection $J \cap I$. As we have seen, either $H = I$ or $H = (0)$, i.e., either $I \subset J$ or $I \cap J = (0)$. In the first case, I being maximal, the only two possibilities are $I = J$ or $A = J$. Assume now the second case: $I \cap J = (0)$. Using again the maximality of I we conclude that $I + J = A$, but for the trivial case $J = (0)$. Denoting as above by k the identity of A, if $A = I + J$ there is a decomposition $k = i_0 + j_0$, $i_0 \in I$, $j_0 \in J$. This implies that $k = k^2 = j_0^2$, because $i_0 j_0 \in IJ \subset I \cap J = (0)$ and $i_0^2 \in I^2 = 0$. But then $k = j_0^2 \in J$ and so $J = A$. We thus conclude that the only ideals of A are (0), I and A itself.

As a final remark, let us observe that Propositions 2.4 and 2.6 and Theorem 2.2 together yield the following characterization of A:

2.7. THEOREM. *All commutative rings with unit having only one proper ideal may be obtained as extensions of suitable one-dimensional vector spaces over some field (considered as zero rings) by the same field in such a way that the linear structure coincides with the structure determined by the exact sequence defining the extension. Moreover, all such extensions are rings of that sort.*

3. Cohomological considerations. The cohomology theories available today for algebraic structures are all intended to describe (through the first or the second cohomology group) the sets of extensions of a given object by another, also given. For the case of groups see [3], [4], [5], [10]. When extending the method to categories, other than groups, one can choose between assuming that the additive structure is trivial (i.e., when it "splits", or, as one says—see [11]—the case of "inessential extensions") and then a new cohomology theory is looked for in order to investigate the other operations (that is the case of associative algebras considered in [7], [8] and of Lie algebras, [9]) or that, on the contrary, the additive structure is described by a non trivial cohomology class. The reader will find references and descriptions of some examples in [15]. In particular one can consider the case of rings. The problem of finding all the extensions of a ring by another has been considered for long under various special conditions and for many different purposes. For example the one that appears under the name of "crossed products" ([11], [1]) whose applications to the study of simple algebras is done in [1]. A generalization of such "crossed products" has been given by Bovdi, [2]. The extension problem with any kernel I has been first discussed by Everett [6] and a generalization is the object of an article by Szendrei [19] (see also [20] for similar treatments). The final form of some of these investigations has been given by the theory of cohomology of Mac Lane–Shukla [13], [17].

Let us denote by $H^n(F, I)$ the n-th cohomology group, in the sense of Mac Lane–Shukla, of the ring F with values in the bimodule I.

3.1. THEOREM (MAC LANE [13]). *There exists one-to-one correspondence between the set of classes of equivalent extensions of the zero ring I by F which determine the given structure of F-bimodule for I and the second cohomology group $H^2(F, I)$.* (One such extension is called "special", in [17]).

A natural definition of *symmetric cochain* may be given in this theory, and thence the notion of *symmetric cohomology groups* $H_s^n(F, I)$ is obtained.

We shall say that F is a *homomorphic image* of A if there exist a ring homomorphism $A \to F$ which is *onto*.

3.2. COROLLARY. *Let F be any commutative field. The set of classes of isomorphic commutative rings with identity having only one proper ideal and having F as a homomorphic image is in one-to-one and onto correspondence with $H_s^2(F, I)$, where I is any one-dimensional vector space over F.*

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On the lattice of left annihilators of certain rings

by

M. F. Janowitz (Amherst, Mass.)

§ 1. Introduction. In this note we shall explore the connection between algebraic equivalence in a Rickart ring and certain lattice theoretic properties of its lattice of left annihilators—our goal being to place the whole theory in a lattice theoretic rather than a ring theoretic setting. In the case of the projection lattice of a von Neumann algebra, it is shown that the usual dimension relation of $*$-equivalence may be realized as perspectivity in a certain associated lattice. The parallel between von Neumann's dimension theory for a continuous geometry and the one for von Neumann algebras thus becomes apparent in that both are seen to be intrinsic—based on perspectivity.

§ 2. Rickart rings. Following terminology introduced by S. Maeda [8], we agree to call a ring $\mathfrak{A}$ a *Rickart ring* in case it satisfies the following two conditions:

(R_r) *The right annihilator of every element is the principal right ideal generated by an idempotent.*

(R_l) *The left annihilator of every element is the principal left ideal generated by an idempotent.*

For examples we refer the reader to Kaplansky [5] as well as S. Maeda [8]. Given the Rickart ring $\mathfrak{A}$, let $L(x)$ denote the left annihilator of x, $R(x)$ its right annihilator, $\mathfrak{L}(\mathfrak{A}) = \{L(x)\colon x \in \mathfrak{A}\}$ and $\mathfrak{R}(\mathfrak{A}) = \{R(x)\colon x \in \mathfrak{A}\}$. If $\mathfrak{L}(\mathfrak{A})$ and $\mathfrak{R}(\mathfrak{A})$ are each partially ordered by set inclusion, by [8], Theorem 1.1, p. 512, they form dual isomorphic relatively complemented lattices with 0 and 1. Our goal in this section is to extend [8], Lemma 4.3, p. 517.

First we need some additional terminology. Two elements e, f of a lattice L are said to form a *modular pair*, denoted $M(e,f)$, in case $a \leqslant f \Rightarrow a \vee (e \wedge f) = (a \vee e) \wedge f$; they form a *dual modular pair*, in symbols $DM(e,f)$, if $a \geqslant f \Rightarrow a \wedge (e \vee f) = (a \wedge e) \vee f$. In a lattice L with 0, two elements e and f are called *perspective* and written $e \sim f$ in case there is an element x such that $e \vee x = f \vee x$ with $e \wedge x = f \wedge x = 0$; they are called *strongly perspective* and denoted $e \sim^s f$ when they are perspective in