Proof. It follows from ([2], Corollary 12) that M is contradictible. According to [7], $C(M)$ has f.p.p. iff $M\times I$ has f.p.p. Since $M\times I$ is a product of smooth dendroids, we apply Theorem 3 and conclude that $C(M)$ has f.p.p.

References

[1] K. Borsuk, *A Theorem on fixed points*, Bull. Acad. Polon. Sci. 2 (1954), pp. 17–20.

[2] J. J. Charatonik and C. Eberhart, *On smooth dendroids*, Fund. Math. 67 (1970), pp. 297–322.

[3] H. Cook, *Tree-likeness of dendroids and λ-dendroids*, Fund. Math. 68 (1970), pp. 19–22.

[4] C. A. Eberhart and J. B. Fugate, *Approximating Continua from within*, to appear in Fund. Math.

[5] J. B. Fugate, *Retracting fans onto finite fans*, Fund. Math. 71 (1971), pp. 113–125.

[6] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton 1948.

[7] R. J. Knill, *Cones, products and fixed points*, Fund. Math. 60 (1967), pp. 35–46.

UNIVERSITY OF KENTUCKY
Lexington, Kentucky

Reçu par la Rédaction le 14. 10. 1969

Some relations between k-analytic sets and generalized Borel sets

by

R. C. Willmott * (Kingston, Ont.)

§ 1. Introduction. A. H. Stone introduced the family of k-analytic sets, where k is an arbitrary cardinal, and established some of their properties ([7], hereafter referred to simply as Stone). He conjectured that the relationships between classical analytic and Borel sets, such as Souslin's theorem, that if a set and its complement are both analytic, they are both Borel, would generalize to relationships between k-analytic sets and k-hyperborel sets, defined as the smallest family of sets containing all closed sets and closed under intersections of $\aleph_0$ and unions of k of them. While this seems to be the correct Borel family there are difficulties in working with different cardinal numbers.

In this paper we establish some relations between Souslin$(\alpha)\mathfrak{F}$ sets (k-analytic sets if $k=\aleph_\alpha$) and generalized Borel families of sets which contain the k-hyperborel sets, but admit intersections of k elements. We remark that Maximoff [2] was led to a similar Borel family while studying a relation between Borel sets and sets analogous to k-analytic sets. The method used is a generalization of Lusin's theory of sieves. Most of the proofs in this paper are direct generalizations of proofs of Lusin [1].

Specifically, in § 3 is developed the basic sieve theory; in corollary 4, Souslin$(\alpha)\mathfrak{F}$ sets are characterized as the sifted sets of a certain class of sieves. In § 4 we establish the decomposition of sifted sets and their complements into disjoint Borel$(\alpha)\mathfrak{F}$ sets (see § 2 for definitions) and apply this result to express a Souslin$(\alpha)\mathfrak{F}$ set as a union and intersection of $\aleph_{\alpha+1}$ Borel$(\alpha)\mathfrak{F}$ sets (theorem 6). Finally in § 5 we show (theorem 8) that disjoint Souslin$(\alpha)\mathfrak{F}$ sets can be separated by disjoint Borel$(\alpha)\mathfrak{F}$ sets and use this result to prove (corollary 9) that if a set and its complement are Souslin$(\alpha)\mathfrak{F}$, then they are Borel$(\alpha)\mathfrak{F}$, and that (theorem 11) a continuous, one-to-one image of $I(\alpha)$ (a generalization of the irrationals, see § 2) is a Borel$(\alpha)\mathfrak{F}$ set.

* This paper was written while the author was a fellow of the 1969 Summer Research Institute of the Canadian Mathematical Congress.

As is true of most of Stone's results most theorems in this paper are significant only if $\aleph_\alpha^{\aleph_0} > \aleph_\alpha$. Stone points out ([7], p. 4) that this is true for arbitrarily large cardinals. The results reduce to the classical ones if $\aleph_\alpha = \aleph_0$.

§ 2. Preliminaries. In this section are collected definitions, notations, and some discussion of concepts and known results that will be needed later. $A \sim B$ will denote set difference; $\mathrm{Cl}A$, the closure of A; $\mathfrak{F}$, the family of closed sets of the topological space in question; if $A \subset X \times Y$, $\pi_X A$ is the projection of A on X. A topological space is of weight $\aleph_\alpha$ if $\aleph_\alpha$ is the least cardinal of a base for the topology. An ordinal number is considered to be the set of all ordinals less than itself. Thus ω is the set of all non-negative integers. ω_α denotes, as usual, the least ordinal of cardinal $\aleph_\alpha$.

$I(\alpha) = \omega_\alpha^\omega$, the set of all sequences of ordinals in ω_α. On $I(\alpha)$ we define the metric ϱ given by $\varrho(i, j) = 1/(n+1)$ if $i_0 = j_0, \ldots, i_{n-1} = j_{n-1}$; $i_n \neq j_n$. If $k = \aleph_\alpha$, $I(\alpha)$ is isometric to the space $B(k)$ of Stone. It is complete and of weight $\aleph_\alpha$. $I(0)$ is homeomorphic to the irrationals. For $i \in I(\alpha)$, $n \in \omega$, we define

$i|n = (i_0, i_1, \ldots, i_n)$, an ordered $(n+1)$-tuple of ordinals in ω_α, and $I(i|n) = \{j \in I(\alpha);\ j|n = i|n\}$,

called a *Baire interval of order* $n+1$. The Baire intervals form a base for the metric topology on $I(\alpha)$. They are open and closed and homeomorphic to $I(\alpha)$. Two Baire intervals are either disjoint or one is contained in the other.

Suppose $\mathfrak{H}$ is a family of sets. $\mathfrak{H}_\sigma$, as usual, will denote the set of countable unions of elements of $\mathfrak{H}$; $\mathfrak{H}_\Sigma$, the set of unions of $\aleph_\alpha$ elements of $\mathfrak{H}$, where the cardinal $\aleph_\alpha$ will be clear from the context.

Borelian$(\alpha)\mathfrak{H}$ is the smallest family which contains $\mathfrak{H}$ and is closed under unions and intersections of $\aleph_\alpha$ elements.

Borel$(\alpha)\mathfrak{H}$ is the smallest family which contains $\mathfrak{H}$ and is closed under unions of $\aleph_\alpha$ elements and under complementation with respect to $\bigcup_{A \in \mathfrak{H}} A$.

Evidently $\mathfrak{H}_\sigma \subset \mathfrak{H}_\Sigma \subset$ Borelian$(\alpha)\mathfrak{H} \subset$ Borel$(\alpha)\mathfrak{H}$. In a metric space (among others) Borelian$(\alpha)\mathfrak{F}$ = Borel$(\alpha)\mathfrak{F}$.

Souslin$(\alpha)\mathfrak{H}$ is the family of all sets A such that

$$A = \bigcup_{i \in I(\alpha)} \bigcap_{n \in \omega} E(i|n)$$

for some function E on $\{i|n\colon\ i \in I(\alpha), n \in \omega\}$ to $\mathfrak{H}$. If $k = \aleph_\alpha$, then Souslin$(\alpha)\mathfrak{F}$ is just the family of k-analytic sets of Stone. We have (see Stone, p. 34)

(1) $\mathfrak{H} \subset$ Souslin$(\alpha)\mathfrak{H}$;

(2) Souslin(α) Souslin$(\alpha)\mathfrak{H}$ = Souslin$(\alpha)\mathfrak{H}$;

(3) The intersection of every countable family of Souslin$(\alpha)\mathfrak{H}$ sets, and the union of every family of at most $\aleph_\alpha$ Souslin$(\alpha)\mathfrak{H}$ sets, are again in Souslin$(\alpha)\mathfrak{H}$.

We remark that if $\aleph_\alpha^{\aleph_0} = \aleph_\alpha$, then the union in the definition of Souslin$(\alpha)\mathfrak{H}$ is over $\aleph_\alpha$ sets and consequently Souslin$(\alpha)\mathfrak{H} \subset$ Borelian$(\alpha)\mathfrak{H}$. It is not true in general that Borelian$(\alpha)\mathfrak{H} \subset$ Souslin$(\alpha)\mathfrak{H}$ (It is, of course, true in the classical case, $\alpha = 0$.)

3. Sieves and a characterization of Souslin$(\alpha)\mathfrak{F}$ sets. We first establish a result on the projection of Souslin$(\alpha)\mathfrak{F}$ sets which is analogous to that in [3], from which the proof is taken. By theorem 19 of Stone, every complete metric space of weight $\leqslant \aleph_\alpha$ is a continuous image of $I(\alpha)$.

THEOREM 1. *Suppose Y is a continuous image of $I(\alpha)$, X is a topological space, and A is a Souslin$(\alpha)\mathfrak{F}$ set in $X \times Y$. Then $\pi_X A$ is a Souslin$(\alpha)\mathfrak{F}$ set in X.*

Proof. Let $Y = f[I(\alpha)]$, f a continuous function, and

$$A = \bigcup_{i \in I(\alpha)} \bigcap_{n \in \omega} A(i|n)\,.$$

where, for each $i \in I(\alpha)$, $A(i|0) \supset A(i|1) \supset A(i|2) \supset \ldots$ We have

$$A = A \cap \bigcup_{j \in I(\alpha)} (X \times \{f(j)\}) = \bigcup_{j \in I(\alpha)} \bigcup_{i \in I(\alpha)} \bigcap_{n \in \omega} \big[(X \times \{f(j)\}) \cap A(i|n)\big]\,,$$

and

$$\pi_X A \subset \bigcup_{j \in I(\alpha)} \bigcup_{i \in I(\alpha)} \bigcap_{n \in \omega} \mathrm{Cl}\pi_X\big[(X \times f[I(j|n)]) \cap A(i|n)\big] = B\,.$$

Clearly B is a Souslin$(\alpha)\mathfrak{F}$ set in X. We show $B \subset \pi_X A$. Let $z \in B$. Then for some $i', j' \in I(\alpha)$,

$$z \in \bigcap_{n \in \omega} \mathrm{Cl}\pi_X\big[(X \times f[I(j'|n)]) \cap A(i'|n)\big]\,.$$

Suppose

$$z \notin \pi_X[(X \times f\{j'\}) \cap \bigcap_{n \in \omega} A(i'|n)] \subset \pi_X A\,.$$

Then

$$(z, j') \in X \times Y \sim \bigcap_{n \in \omega} A(i'|n) = \bigcup_{n \in \omega} [X \times Y \sim A(i'|n)]\,,$$

and, for some $m \in \omega$,

$$(z, j') \in X \times Y \sim A(i'|m)\,,$$

an open set in $X\times Y$. Since the sequence $A(i'|n)$, $n\in\omega$ is decreasing, and f is continuous, we may choose $p\geqslant m$ and U open in X so that $z\in U$ and

$$(U\times f[I(j'|p)])\cap A(i'|p)=\emptyset .$$

But then

$$z\notin \mathrm{Cl}\pi_X\big[(X\times f[I(j'|p)])\cap A(i'|p)\big] ,$$

a contradiction. We conclude that $z\in\pi_X A$ and so that $B\subset\pi_X A$.

We now introduce the concept of a sieve in $X\times I(\alpha)$ and several allied ideas (see [4], § 7, for a discussion and some results concerning sieves). Let $\gg$ be the lexicographical order on the elements of $I(\alpha)$, i.e. for $i,j\in I(\alpha)$, $i\gg j$ if and only if $i_n>j_n$ where n is the first component where i and j differ, and $>$ is the usual order on the ordinals. A sieve is any set C in $X\times I(\alpha)$; the set sifted by C is the set E of points x in X for which $C^{(x)}=\{y\colon (x,y)\in C\}$ contains a strictly decreasing sequence $y_0\gg y_1\gg y_2\gg\ldots$ The complement $X\sim E$ of the sifted set is thus the set of points for which $C^{(x)}$ is well-ordered by $\gg$.

$Q(\alpha)$ is the subset of $I(\alpha)$ consisting of sequences which take only the value 0 from some point on. The cardinality of $Q(\alpha)$ is $\aleph_\alpha$, and so it is an $\mathfrak{F}_\Sigma$ in $I(\alpha)$. If A is Souslin$(\alpha)\mathfrak{F}$ in $X\times Q(\alpha)$, then A is Souslin$(\alpha)\mathfrak{F}$ in $X\times I(\alpha)$.

If $\mathfrak{H}$ is a family of subsets of X, a standard $\mathfrak{H}$-sieve in $X\times Q(\alpha)$ is a set C of the form

$$C=\bigcup_{i\in Q(\alpha)} H_i\times\{i\}$$

where for each $i\in Q(\alpha)$, $H_i\in\mathfrak{H}$ or $H_i=\emptyset$.

THEOREM 2. *Suppose $\mathfrak{H}$ is closed under finite intersections. If A is a Souslin$(\alpha)\mathfrak{H}$ set in X, then there is a standard $\mathfrak{H}$-sieve C in $X\times Q(\alpha)$ whose sifted set is A.*

Proof. We have

$$A=\bigcup_{i\in I(\alpha)}\bigcap_{n\in\omega} A(i|n) ,$$

where each set $A(i|n)\in\mathfrak{H}$. As $\mathfrak{H}$ is closed under finite intersections we may suppose that for each $i\in I(\alpha)$, $A(i|0)\supset A(i|1)\supset A(i|2)\supset\ldots$ For $i\in I(\alpha)$, $n\in\omega$, define

$$r(i|n)=(i_0,i_1,i_2,\ldots,i_{n-1},i_n+1,0,0,0,\ldots)\in Q(\alpha) ,$$

and set

$$C=\bigcup_{i\in I(\alpha)}\bigcap_{n\in\omega} A(i|n)\times\{r(i|n)\} ,$$

a standard $\mathfrak{H}$-sieve in $X\times Q(\alpha)$. It is easy to check that A is the set sifted by C.

THEOREM 3. *If C is a Souslin$(\alpha)\mathfrak{F}$ set in $X\times I(\alpha)$, then the set sifted by C is Souslin$(\alpha)\mathfrak{F}$ in X.*

Proof. (See proof of theorem 10 in [4].) Enumerate the points of $Q(\alpha)$: $r_0, r_1, r_2, \ldots, r_\omega, \ldots, r_\gamma, \ldots|\omega_\alpha$

Let δ be a sequence such that

$$r_{\delta_0}=(1,0,0,0,\ldots) ,$$
$$r_{\delta_1}=(0,1,0,0,0,\ldots) ,$$
$$r_{\delta_2}=(0,0,1,0,0,0,\ldots) ,$$
$$\vdots$$

We have $r_{\delta_0}\gg r_{\delta_1}\gg r_{\delta_2}\gg\ldots$ For $\eta,\gamma\in\omega_\alpha$, define

$$r_\gamma r_\eta=\begin{cases} r_\eta & \text{if}\quad r_\eta\ll r_\gamma ,\\ \max\{r_{\delta_n}\colon r_{\delta_n}\ll r_\gamma\} & \text{otherwise} .\end{cases}$$

For a finite number $r_{\beta_1}, r_{\beta_2},\ldots, r_{\beta_n}$,

$$r_{\beta_1}r_{\beta_2}\ldots r_{\beta_n}=\big(\ldots\big((r_{\beta_1}r_{\beta_2})r_{\beta_3}\big)\ldots\big)r_{\beta_n} .$$

Now for $i\in I(\alpha)$, $n\in\omega$, define

$$R(i|n)=\begin{cases}\{y\in I(\alpha)\colon r_{i_0}\leqq y\} & \text{if}\quad n=0 ,\\ \{y\in I(\alpha)\colon r_{i_0}r_{i_1}\ldots r_{i_n}\leqq y\ll r_{i_0}r_{i_1}\ldots r_{i_{n-1}}\} & \text{if}\quad n>0\end{cases}$$

an $\mathfrak{F}_\sigma$-set and hence a Souslin$(\alpha)\mathfrak{F}$ set in $I(\alpha)$.

Now for each $i\in I(\alpha)$, $n\in\omega$, set

$$A(i|n)=\pi_X[C\cap\{X\times R(i|n)\}] .$$

By theorem 1, $A(i|n)$ is Souslin$(\alpha)\mathfrak{F}$ in X, so

$$A=\bigcup_{i\in I(\alpha)}\bigcap_{n\in\omega} A(i|n)$$

is Souslin(α)Souslin$(\alpha)\mathfrak{F}$ and so Souslin$(\alpha)\mathfrak{F}$ in X. It is easy to check that $A=E$, the set sifted by C.

Combining theorems 2 and 3 we have the following characterization. A standard $\mathfrak{F}$-sieve in $X\times Q(\alpha)$ is an $\mathfrak{F}_\Sigma$-set in $X\times Q(\alpha)$ and hence Souslin$(\alpha)\mathfrak{F}$ in $X\times Q(\alpha)$.

COROLLARY 4. *A is a Souslin$(\alpha)\mathfrak{F}$ set in X if and only if it is the set sifted by a standard $\mathfrak{F}$-sieve in $X\times Q(\alpha)$.*

4. Constituents of sifted sets and their complements. We turn now to the notion of the constituents of a sifted set and its complement which is needed for the proof of our main theorem. Suppose C is a sieve in $X\times Q(\alpha)$, E is the sifted set, and $B=X\sim E$ the complement of the sifted set.

For $x \in B$, $C^{(x)}$ is well-ordered. Let $\gamma(x)$ be the corresponding ordinal number. We have $\gamma(x) < \omega_{\alpha+1}$.

For $x \in E$, $C^{(x)}$ is not well-ordered, but we may define a well-ordered "lower part". Let $S(i) = \{k \in I(\alpha)\colon k \leqq i\}$ and set, for $x \in E$

$$j(x) = \sup\{i \in I(\alpha)\colon C^{(x)} \cap S(i) \text{ is well-ordered}\}.$$

Then $j(x)$ is unique, and $C^{(x)} \cap S(j(x))$ is well-ordered. Let $\gamma(x)$ be the corresponding ordinal number. Again $\gamma(x) < \omega_{\alpha+1}$. For $\beta \in \omega_{\alpha+1}$ set

$$B_\beta = \{x \in B\colon \gamma(x) = \beta\}, \quad \text{the } \beta\text{-constituent of } B,$$
$$E_\beta = \{x \in E\colon \gamma(x) = \beta\}, \quad \text{the } \beta\text{-constituent of } E.$$

Then

$$B = \bigcup_{\beta \in \omega_{\alpha+1}} B_\beta, \qquad E = \bigcup_{\beta \in \omega_{\alpha+1}} E_\beta,$$

where these unions are disjoint.

THEOREM 5. *Suppose C is a standard Borelian$(\alpha)\mathfrak{H}$-sieve in $X \times Q(\alpha)$. Then the constituents E_β and B_β, $\beta \in \omega_{\alpha+1}$ of the sifted set and its complement are Borel$(\alpha)\mathfrak{H}$ sets.*

Proof. (See Lusin [1], p. 188.) Enumerating the segments composing C, we have

$$C = \bigcup_{\gamma \in \omega_\alpha} F(\gamma) \times \{r(\gamma)\},$$

where $F(\gamma)$ is Borelian$(\alpha)\mathfrak{H}$, $r(\gamma) \in Q(\alpha)$. Clearly the 0-constituent of B,

$$B_0 = X \sim \pi_X C = X \sim \bigcup_{\gamma \in \omega_\alpha} F(\gamma)$$

is a Borel$(\alpha)\mathfrak{H}$ set. For $\gamma \in \omega_\alpha$, let

$$C(\gamma) = \bigcup_{\substack{\eta \in \omega_\alpha \\ r(\eta) \lessdot r(\gamma)}} F(\eta) \times \{r(\eta)\},$$

the part of C strictly "below" $X \times \{r(\gamma)\}$. Then

$$\theta(\gamma) = F(\gamma) \sim \pi_X C(\gamma) = F(\gamma) \sim \bigcup_{r(\eta) \lessdot r(\gamma)} F(\eta)$$

is a Borel$(\alpha)\mathfrak{H}$ set, as is

$$S = \bigcup_{\gamma \in \omega_\alpha} \theta(\gamma).$$

The points of S are just those points x with the property that $C^{(x)}$ has a least point. Now the 0-constituent of E, $E_0 = X \sim (B_0 \cup S)$, is a Borel$(\alpha)\mathfrak{H}$ set.

We now define the derived sieve C' of C.

$$C' = \bigcup_{\gamma \in \omega_\alpha} [F(\gamma) \sim \theta(\gamma)] \times \{r(\gamma)\} = \bigcup_{\gamma \in \omega_\alpha} [F(\gamma) \cap \pi_X C(\gamma)] \times \{r(\gamma)\},$$

again a standard Borelian$(\alpha)\mathfrak{H}$-sieve in $X \times Q(\alpha)$. C' may be considered as being constructed from C by removing from each set $\{x\} \times C^{(x)}$ the point with the least ordinate, if such a point exists. Clearly the sifted set of C' is still E. The 0-constituents of E and B with respect to C' are equal respectively to $E_0 \cup E_1$ and $B_0 \cup B_1$ and are again Borel$(\alpha)\mathfrak{H}$ sets. As E_0 and B_0 are Borel$(\alpha)\mathfrak{H}$ sets, the same is true of E_1 and B_1.

We now form a transfinite sequence of derived sieves

$$C = C_0, C_1, C_2, \ldots, C_\omega, \ldots, C_\gamma, \ldots | \omega_{\alpha+1}$$

by setting, for β a non-limit ordinal,

$$C_\beta = (C_{\beta-1})'$$

and for β a limit ordinal

$$C_\beta = \bigcap_{\eta<\beta} C_\eta.$$

Then each sieve C_β is a standard Borelian$(\alpha)\mathfrak{H}$-sieve having E as the sifted set, and the 0-constituents of E and B with respect to C_β, which are Borel$(\alpha)\mathfrak{H}$ sets, are equal respectively to $\bigcup_{\eta<\beta} E_\eta$ and $\bigcup_{\eta<\beta} B_\eta$. It follows that all the constituents E_γ, B_γ, $\gamma \in \omega_{\alpha+1}$ are Borel$(\alpha)\mathfrak{H}$.

The following result is similar to that of Maximoff ([2], theorem 4, p. 547).

THEOREM 6. *Suppose $\mathfrak{H}$ is closed under finite intersections. If A is a Souslin(α)Borelian$(\alpha)\mathfrak{H}$ set in X, then there exist Borel$(\alpha)\mathfrak{H}$ sets B_γ, $\gamma \in \omega_{\alpha+1}$, and Borelian$(\alpha)\mathfrak{H}$ sets D_γ, $\gamma \in \omega_{\alpha+1}$, such that*

$$A = \bigcup_{\gamma \in \omega_{\alpha+1}} B_\gamma = \bigcap_{\gamma \in \omega_{\alpha+1}} D_\gamma.$$

Proof. By theorem 2, there is a standard Borelian$(\alpha)\mathfrak{H}$-sieve C in $X \times Q(\alpha)$ whose sifted set is A. It suffices to take the sets B_γ, $\gamma \in \omega_{\alpha+1}$, of theorem 5, and the sets

$$D_\gamma = \pi_X C_\gamma, \qquad \gamma \in \omega_{\alpha+1},$$

the sets C_γ also from theorem 5.

5. The separation theorem. The sieve C in $X \times I(\alpha)$ is said to be bounded on a set D disjoint from the sifted set of C if the set of ordinals $\gamma(x)$, $x \in D$ (used in defining the constituents) does not have $\omega_{\alpha+1}$ as its supremum.

THEOREM 7. *Suppose X is a complete metric space of weight $\leqslant \aleph_\alpha$, C is a standard Souslin$(\alpha)\mathfrak{F}$-sieve in $X \times I(\alpha)$, and A is a Souslin$(\alpha)\mathfrak{F}$ set in X disjoint from the set E sifted by C. Then C is bounded on A.*

Proof. (See Lusin [1], p. 183.) As in the proof of theorem 5 we may express

$$C = \bigcup_{\gamma \in \omega_\alpha} F(\gamma) \times \{r(\gamma)\},$$

$F(\gamma)$ Souslin$(\alpha)\mathfrak{F}$ in X. For each $\gamma \in \omega_\alpha$, let $H(\gamma) = A \cap F(\gamma)$. As a Souslin$(\alpha)\mathfrak{F}$ set in X is a continuous image of $I(\alpha)$ (Stone, theorem 19), let f and f_γ, $\gamma \in \omega_\alpha$ be continuous functions on $I(\alpha)$ such that $A = f[I(\alpha)]$ and $H(\gamma) = f_\gamma[I(\alpha)]$.

Suppose that C is not bounded on A. For $\gamma \in \omega_\alpha$ let $C(\gamma)$ be the part of C "below" $X \times \{r(\gamma)\}$ as in the proof of theorem 5. Choose $k_0 \in \omega_\alpha$ such that $C(k_0)$ is not bounded on $H(k_0)$. (If a bound existed for each $C(\gamma)$ on $H(\gamma)$, then we could choose an ordinal greater than all these bounds, but still less than $\omega_{\alpha+1}$, which would be a bound for C on A.) Now choose $i_0, i_0^{k_0} \in \omega_\alpha$ such that $C(k_0)$ is unbounded on

$$f[I(i_0)] \cap f_{k_0}[I(i_0^{k_0})] = D_0 \subset A,$$

where $I(i_0)$ and $I(i_0^{k_0})$ are Baire intervals of order 1. Such a choice is possible since $A \cap H(k_0)$ is a union of $\aleph_\alpha$ such intersections. Now choose $k_1 \in \omega_\alpha$ so that $r(k_1) \leqslant r(k_0)$ and $C(k_1)$ is not bounded on $D_0 \cap H(k_1)$. Again choose $i_1, i_1^{k_0}$, and $i_0^{k_1}, i_1^{k_1} \in \omega_\alpha$ so that $C(k_1)$ is unbounded on

$$D_1 = f[I(i_0, i_1)] \cap f_{k_0}[I(i_0^{k_0}, i_1^{k_0})] \cap f_{k_1}[I(i_0^{k_1}, i_1^{k_1})] \subset A \cap H(k_0) \cap H(k_1).$$

We thus construct by recursion the sequences $i, i^{k_0}, i^{k_1}, \ldots \in I(\alpha)$, and D of non-empty sets such that for each $n \in \omega$, $r(k_{n+1}) \leqslant r(k_n)$, and $C(k_n)$ is unbounded on

$$D_n = f[I(i|n)] \cap \ldots \cap f_{k_n}[I(i^{k_n}|n)] \subset A \cap H(k_0) \cap \ldots \cap H(k_n).$$

Let $x = f(i)$, $y_n = f_{k_n}(i^{k_n})$. Then by continuity of f and f_{k_n}, we have $x = y_n$, and thus that $x \in A$ and $x \in H(k_n) \subset F(k_n)$ for every n. But the $r(k_n)$ form a decreasing sequence, contradicting the fact that A is disjoint from the sieved set. We conclude that C is bounded on A.

Using theorems 2, 7, and 5 we have immediately

THEOREM 8. *Suppose A and B are disjoint Souslin$(\alpha)\mathfrak{F}$ sets in a complete metric space of weight $\leqslant \aleph_\alpha$. Then there exist disjoint Borel$(\alpha)\mathfrak{F}$ sets C, D such that $A \subset C$ and $B \subset D$.*

COROLLARY 9. *Suppose X is a complete metric space of weight $\leqslant \aleph_\alpha$. If both A and $X \sim A$ are Souslin$(\alpha)\mathfrak{F}$ sets in X, then A and $X \sim A$ are Borel$(\alpha)\mathfrak{F}$ sets.*

We remark that these results are trivial if $\aleph_\alpha^{\aleph_0} = \aleph_\alpha$ in which case a Souslin$(\alpha)\mathfrak{F}$ set is a Borel$(\alpha)\mathfrak{F}$ set. Unfortunately, the converse of corollary 9 is not necessarily true as a Borel$(\alpha)\mathfrak{F}$ set need not be Souslin$(\alpha)\mathfrak{F}$.

A standard proof (e.g. [5], theorem 123, p. 230) gives

COROLLARY 10. *If $\{A_\gamma: \gamma \in \omega_\alpha\}$ is a family of disjoint Souslin$(\alpha)\mathfrak{F}$ sets in a complete metric space of weight $\leqslant \aleph_\alpha$, then there exists a family $\{B_\gamma: \gamma \in \omega_\alpha\}$ of disjoint Borel$(\alpha)\mathfrak{F}$ sets such that for each $\gamma \in \omega_\alpha$, $A_\gamma \subset B_\gamma$.*

Using corollary 10, and the facts that a Souslin$(\alpha)\mathfrak{F}$ set is a continuous image of $I(\alpha)$ and a closed subset of $I(\alpha)$ is a retract of $I(\alpha)$ (Stone, theorems 19, p. 35 and 3, p. 8), the proof of theorem 124, p. 232 of Sierpiński [5] gives

THEOREM 11. *If X is a complete metric space of weight $\leqslant \aleph_\alpha$, and $f: J \to X$ is a continuous, one-to-one function on a closed subset J of $I(\alpha)$ into X, then $f[J]$ is a Borel$(\alpha)\mathfrak{F}$ set in X.*

References

[1] N. Lusin, *Leçons sur les ensembles analytiques et leurs applications*, Paris 1930.
[2] I. Maximoff, *Sur le système de Souslin d'ensembles dans l'espace transfini*, Bull. Amer. Math. Soc. 46 (1940), pp. 543–550.
[3] C. A. Rogers and R. C. Willmott, *On the projection of Souslin sets*, Mathematika 13 (1966), pp. 147–150.
[4] — *On the uniformization of sets in topological spaces*, Acta. Math. 120 (1968), pp. 1–52.
[5] W. Sierpiński, *General Topology*, Toronto 1952.
[6] M. Sion and R. C. Willmott, *On a definition of ordinal numbers*, Amer. Math. Mon. 69 (1962), pp. 381–386.
[7] A. H. Stone, *Non-separable Borel sets*, Rozprawy Matematyczne 28, Warszawa 1962.

QUEEN'S UNIVERSITY
Kingston, Ontario

Reçu par la Rèdaction le 14. 10. 1969