References

[1] A. Appert and Ky-Fan, *Espaces topologiques intermédiaires*, Actual. Sci. Ind. 1121, Paris 1951.
[2] M. Fréchet, *Les espaces abstraits*, Paris 1951.
[3] Z. P. Mamuzič, *Introduction to General Topology*, Groningen 1963.
[4] B. Schweizer and A. Sklar, *Statistical metric spaces*, Pacific J. Math. (10) 1 (1960), pp. 313–334.
[5] E. Thorp, *Generalized topologies for statistical metric spaces*, Fund. Math. 51 (1962), pp. 9–21.

Reçu par la Rédaction le 4. 12. 1969

Consistency statements in formal theories*

by

R. G. Jeroslow (Minneapolis, Minn.)

In this paper, we establish several results regarding the behavior of consistency statements in formal theories in the language of arithmetic; extensions to other "larger" languages are usually straightforward. The paper continues work begun by S. Feferman in [1], employing as its chief device self-referential statements or processes, as in [1], [4], [5], [6].

In particular, we will find that a very weak theory may be able to prove its own (RE) consistency (see Theorem 1.5); that any reflexive theory containing Peano arithmetic arises from adding to some theory the (RE) statement of its own consistency (see Theorem 1.4); that the addition of a consistency statement to a theory can substantially alter the Lindenbaum algebras of the theory, and in fact render impossible homomorphisms of these algebras which commute with a finite (and specified) number of quantifiers (see Theorem 4.1). We also explore the degrees of relative interpretability between that of a theory and the theory plus its consistency (Theorems 3.1, 3.2).

We assume that the reader is familiar with the paper of Feferman [1]; when we do not specify a convention that we use, it is to be found in that reference, which we shall call "Feferman's paper".

1. Let a theory $\mathcal{A}$ be given possessing A as an axiomatization (i.e., A is a set of Gödel numbers of formulas which axiomatize $\mathcal{A}$); let $\alpha(\mathrm{w})$ be a formula in the language of arithmetic which *designates* A in the following sense:

$$\alpha(\bar{n}) \quad \text{if and only if} \quad n \in A .$$

Then the construction of Feferman ([1], Def. 4.1) assigns to this designator $\alpha(\mathrm{w})$ the formula $\mathrm{Prf}_\alpha(\mathrm{y}, \mathrm{x})$ in the two free variables x, y, and this formula intuitively "says" that x is a (Gödel number of a) proof

* Theorems 1.4, 1.5, 2.1, 2.2 (with Lemma 2.1) and 4.1 are results from the author's doctoral dissertation at Cornell (September 1969), which was written under the guidance of Professor Anil Nerode and was supported by a National Science Foundation Fellowship. All other results were obtained at the University of Minnesota.

of (the formula with Gödel number) y in the system axiomatized by the designator $\alpha(\mathrm{w})$. By $\mathrm{Prf}_{[z]}(y, x)$ we shall abbreviate the result of inserting the formula $\mathrm{w} = \mathrm{z}$ for $\alpha(\mathrm{w})$ in the usual construction; by $\mathrm{Con}_{[z]}$ we abbreviate the consistency statement of Feferman deriving from the designator $\mathrm{w} = \mathrm{z}$ (see [1], Definition 4.9 (ii)).

Let q_0 be the number of the only axiom of Robinson's theory $\mathfrak{Q}$.

We now introduce a theory $\mathfrak{U}$ in an extended language for arithmetic containing, among other function symbols, the function symbols $*$, lh, $\dot{\leftrightarrow}$, $(\ )_i$, and sub, with the intuitive meanings as follows: $x{*}y$ is the result of concatenating the sequence with number y after the sequence with number x; $\mathrm{lh}(x)$ is the length of the sequence with number x; $x \dot{\leftrightarrow} y$ is the number of the formula obtained when the bi-conditional is placed between the formula with number x and the formula with number y; $\mathrm{sub}(x, y)z$ is the result of substituting the numeral for x at all free occurrences in formula number z of the y-th variable (this would be v_y); $(x)_i$ is the exponent of the ith prime in the number x. In the following, $\mathrm{Pr}_{[z]}(y)$ abbreviates $(\exists x)\,\mathrm{Prf}_{[z]}(y, x)$. Among the axioms of $\mathfrak{U}$ are the following sentences:

1. $(\forall y)(\forall z)\big(\mathrm{Pr}_{[z]}(y) \to \mathrm{Pr}_{[\bar{q}_0]}(\mathrm{sub}(z, 2)\,\mathrm{sub}(y, 1)\,\overline{\mathrm{Pr}_{[z]}(y)})\big)$.
2. $(\forall z)\big(\mathrm{Pr}_{[z]}(\bar{q}_0) \to (\forall y)\big(\mathrm{Pr}_{[\bar{q}_0]}(y) \to \mathrm{Pr}_{[z]}(y)\big)\big)$.
3. $(\forall z)(\forall x)(\forall y)\big(\mathrm{Pr}_{[z]}(x \dot{\leftrightarrow} y) \to \big(\mathrm{Pr}_{[z]}(x) \leftrightarrow \mathrm{Pr}_{[z]}(y)\big)\big)$.
4. $$(\forall x)(\forall y)(\forall z)(\forall u)(\forall v)(\forall w)\left(\begin{aligned}&\mathrm{lh}(x) = u \wedge \mathrm{lh}(y) = v \wedge \mathrm{lh}(z) = w \to \\ &\big(\mathrm{lh}((x{*}y){*}z) = u + v + w\big) \wedge \\ &(\forall i)\big(1 \leqslant i \leqslant u \to ((x{*}y){*}z)_i = x_i\big) \wedge \\ &(\forall i)\big(u + 1 \leqslant i \\ &\qquad \leqslant u + v \to ((x{*}y){*}z)_i = y_{i-u}\big) \wedge \\ &(\forall i)\big(u + v + 1 \leqslant i \leqslant u + v + w \\ &\qquad \to ((x{*}y){*}z)_i = z_{i-u+v}\big)\end{aligned}\right)$$
5. $$(\forall i)(\forall u)(\forall v)(\forall w)\left(\begin{aligned}&(1 \leqslant i \leqslant u + v + w) \leftrightarrow \\ &(1 \leqslant i \leqslant u) \vee \\ &(u + 1 \leqslant i \leqslant u + v) \vee \\ &(u + v + 1 \leqslant i \leqslant u + v + w)\end{aligned}\right)$$

In addition to these axioms, $\mathfrak{U}$ contains a statement for every function symbol which appears in it to the effect that this function is equivalent to the usual encoding in Robinson's system $\mathfrak{Q}$ of its primitive recursive graph; e.g., if $\mathrm{sub}(x, y, z, w)$ is the usual Gödel encoding of the (real world) function for which sub is a symbol, $\mathfrak{U}$ will contain the axiom

$$(\forall x)(\forall y)(\forall z)(\forall w)\big(\mathrm{sub}(x, y, z, w) \leftrightarrow \mathrm{sub}(x, y)z = w\big). \tag{1}$$

In $\mathfrak{U}$ we also place names for functions needed to define the functions mentioned above, along with the primitive recursive schema which define each such function from those previous. In addition, $\mathfrak{U}$ is to contain the equality axioms for all of its function symbols, as well as all the axioms of Robinson's $\mathfrak{Q}$.

As we shall see below in Theorem 1.1, the purpose of constructing $\mathfrak{U}$ is to provide a finite theory in which Gödel's Second Underivability Theorem can be arithmetized. Feferman in his paper gives a way of assigning to each formula in the extended language a formula in the ordinary language of arithmetic, so that, taking the axioms of $\mathfrak{U}$ one by one, we obtain axioms for some finite theory $\mathfrak{U}'$ in the ordinary language, and $\mathfrak{U}$ is a conservative extension of $\mathfrak{U}'$. We shall not differentiate between the theories $\mathfrak{U}$ and $\mathfrak{U}'$, and we shall call them both $\mathfrak{U}$.

Before proceeding, we wish to note the following result. If $\alpha(\mathrm{w})$ is any arithmetic formula and $\mathfrak{B}$ any theory containing $\mathfrak{U}$ (1), then we have

$$\vdash_{\mathfrak{B}} (\forall t)\big(\mathrm{Pr}_\alpha(t) \to (\forall y)\big(\mathrm{Pr}_{[t]}(y) \to \mathrm{Pr}_\alpha(y)\big)\big). \tag{2}$$

The reasoning behind (2) is based upon a study of the construction of the formula Pr_α. In the system $\mathfrak{B}$ one can argue that, if one had a proof of t in the system designated by α and a proof of y in the system with only axiom t, then, by concatenating the proof in the α-system of t with the number of the statement $t \to t$ and then concatenating this with the proof of y from axiom t, we would have a proof of y in the α-system. The reasoning of $\mathfrak{B}$ is the following: up to the proof of t in the α-system there is nothing to check; the statement $t \to t$ is an axiom of the Predicate Calculus; after the statement $t \to t$ we can use the same justification for a step in the proof as was used in the system with axiom t, except in the case that the justification is that the formula is an axiom; in this latter case, it must be t, and then the justification comes from modus ponens with the previous occurrences of t and $t \to t$. To reason thus, $\mathfrak{B}$ uses the axioms 4, 5 above.

Once we have (2), an investigation of the construction of Con_α will reveal that, if $\mathfrak{B}$ contains $\mathfrak{U}$,

$$\vdash_{\mathfrak{B}} (\forall t)\big(\mathrm{Pr}_\alpha(t) \to (\mathrm{Con}_\alpha \to \mathrm{Con}_{[t]})\big), \tag{3}$$

for $\alpha(\mathrm{w})$ any arithmetic formula. We need (3) in Theorem 1.1. In the following, $\mathfrak{T}$ is Peano arithmetic.

THEOREM 1.1. *$\mathfrak{U}$ is a finite subtheory of $\mathfrak{T}$ which contains $\mathfrak{Q}$. If $\mathfrak{J}$ is any consistent, r.e. theory extending $\mathfrak{U}$ and $\mathrm{t}(\mathrm{w})$ any RE designator for some axiomatization of $\mathfrak{J}$, then $\mathrm{Con}_{\mathrm{t}}$ cannot be proven in $\mathfrak{J}$.*

(1) I.e., all the theorems of $\mathfrak{U}$ are theorems of $\mathfrak{B}$. We also write this $\mathfrak{B} \supseteq \mathfrak{U}$ or $\mathfrak{U} \subseteq \mathfrak{B}$.

Proof. The fact that $\mathcal{U}$ is a finite theory extending $\mathcal{T}$ derives from its construction, and the fact that $\mathcal{U}$ is a subtheory of $\mathcal{T}$ derives from the fact that all the axioms (1) and other axioms used in $\mathcal{U}$ can be proven in $\mathcal{T}$ using some mathematical inductions.

We next note that, if b_0 is the number of a consistent statement with at least the strength of $\mathcal{U}$, then by referring to the proof of Theorem 5.6 of Feferman, we can adapt it to show that $\mathrm{Con}_{[b_0]}$ is not provable in the system $\mathcal{B}$ axiomatized by b_0, using axioms 1, 5, 3 above.

Now, let $\mathcal{J}$ be any r.e. consistent theory extending $\mathcal{U}$ and $t(w)$ an *RE* designator for $\mathcal{J}$. If Con_t were provable in $\mathcal{J}$, then it would be provable in a finite subtheory $\mathcal{B}$ of $\mathcal{J}$ which is axiomatized by a single axiom b_0. Further, we can assume that $\mathcal{B}$ contains $\mathcal{U}$. Since $\mathcal{B}$ contains $\mathcal{Q}$, we have $\vdash_{\mathcal{B}}\mathrm{Pr}_t(b_0)$ and hence, by (3), $\vdash_{\mathcal{B}}\mathrm{Con}_t\to\mathrm{Con}_{[b_0]}$. Since we have $\vdash_{\mathcal{B}}\mathrm{Con}_t$, we have also $\vdash_{\mathcal{B}}\mathrm{Con}_{[b_0]}$. We saw that this was impossible in the last paragraph, and hence Con_t cannot be provable in $\mathcal{J}$. Q.E.D.

Let u_0 be the number of the sole axiom of $\mathcal{U}$ (obtained by conjoining the axioms given above for $\mathcal{U}$).

COROLLARY 1.1. *For any RE formula* $t(w)$,

$$\vdash_{\mathcal{T}}\mathrm{Pr}_t(u_0)\to\big(\mathrm{Con}_t\to\neg\mathrm{Pr}_t(\mathrm{Con}_t)\big)\,.$$

Sketch of Proof. The entire proof of Theorem 1.1 can be arithmetized. One point that requires comment is the arithmetization of the statement that an *RE* designator $t(w)$ numerates in $\mathcal{J}$ the theory which it describes, but Feferman's Corollary 5.5 shows that

$$\vdash_{\mathcal{T}}t(w)\to\mathrm{Pr}_{[q_0]}\big(\mathrm{sub}(w,3)\overline{t(w)}\big)\,,$$

where (say) w is the third variable of logical systems. Q.E.D.

THEOREM 1.2. *Let* $t(w)$, $s(w)$ *be formulas in one free variable* w, n_0 *a number, and suppose the formula* $v(w)$ *is given by*

$$t(w)\vee(\exists u)\big((w=n_0\dot{+}u)\wedge s(u)\big)\,.$$

Then we have

$$\vdash_{\mathcal{T}}\mathrm{Con}_t\wedge\neg\mathrm{Pr}_t(n_0)\to\mathrm{Con}_v\,.$$

Proof. If the argument which follows in the next paragraphs is arithmetized in $\mathcal{T}$, which can easily be done, we obtain Theorem 1.2.

If the theory $\mathcal{V}$ described by $v(w)$ were inconsistent, by the Deduction Theorem we would have, for some finite set of axioms $\sigma_1,\ldots,\sigma_l$ in the theory $\mathcal{S}$ described by $s(w)$,

$$\vdash_{\mathcal{J}}(\lambda\to\sigma_1)\wedge\ldots\wedge(\lambda\to\sigma_l)\to 0=1\wedge\neg 0=1\,,$$

where λ is the formula with number n_0 and $\mathcal{J}$ is the theory designated by $t(w)$. Rearranging, we obtain,

$$\vdash_{\mathcal{J}}\neg(\lambda\to\sigma_1)\vee\ldots\vee\neg(\lambda\to\sigma_l)\,,$$

and then, using the fact that $\lambda\to\sigma_i$ is co-provable with $\neg\lambda\vee\sigma_i$, and Boolean distributivity, we obtain,

$$\vdash_{\mathcal{J}}\lambda\wedge\Big(\bigvee_{i=1}^{l}\neg\sigma_i\Big)\,.$$

In particular, we have

$$\vdash_{\mathcal{J}}\lambda\,,$$

which is contrary to our hypothesis. Q.E.D.

A theory, such as $\mathcal{U}$, with the property that, for any supertheory $\mathcal{J}$ and *RE* $t(x)$ numerating an axiomatization of $\mathcal{J}$ in $\mathcal{J}$, we cannot have Con_t provable in $\mathcal{J}$, we shall call an *essentially-G* theory. Our object in Theorem 1.1 was simply to explicitly construct one such theory; we do not known if there is an elegant form of an essentially-*G* theory, in the way that Robinson's $\mathcal{Q}$ is an elegant theory which numerates all recursive relations and defines all recursive functions.

The existence of finitely axiomatizable essentially-*G* theories has been known among researchers for some time, as G. Sacks remarked when the author mentioned Theorem 1.1 to him. We need the stronger fact that the existence of such theories can be verified in Peano, which is Corollary 1.1, in the proof of the next result.

THEOREM 1.3. *Let* $\mathcal{S}$ *be an r.e. supertheory of* $\mathcal{T}$. *Then the following are equivalent for any subtheory* $\mathcal{J}$ *of* $\mathcal{S}$ *which contains* $\mathcal{U}$:

(1) *There exists an RE formula* $t(w)$ *which designates* $\mathcal{J}$ *for which we have*

$$\vdash_{\mathcal{S}}\mathrm{Con}_t\,.$$

(2) *There exists a supertheory* $\mathcal{V}$ *of* $\mathcal{J}$, *an RE formula* $v(w)$ *which designates* $\mathcal{V}$, *and an RE formula* $t(w)$ *which designates* $\mathcal{J}$, *the two formulas related by the condition*

$$\vdash_{\mathcal{Q}}(\forall w)\big(t(w)\to v(w)\big) \tag{4}$$

and such that $\mathcal{V}\cup\{\mathrm{Con}_v\}$ *axiomatizes precisely* $\mathcal{S}$.

Proof. Using equation (4) the proof that (2) implies (1) is trivial, since (2) gives $\vdash_{\mathcal{S}}\mathrm{Con}_v$ and $\mathcal{S}\supseteq\mathcal{T}$.

To see that (1) implies (2), we use the fact that $\mathcal{J}$ contains $\mathcal{U}$, Corollary 1.1, and the fact that $\mathcal{S}$ contains $\mathcal{T}$, to note that

$$\vdash_{\mathcal{S}}\mathrm{Con}_t\to\neg\mathrm{Pr}_t(\mathrm{Con}_t)\,.$$

Then, since $\vdash_{\mathcal{S}} \mathrm{Con}_t$ by hypothesis, we evidently have $\vdash_{\mathcal{S}} \neg \mathrm{Pr}_t(\mathrm{Con}_t)$. If n_0 is the number of Con_t, $s(w)$ any *RE* designator for $\mathcal{S}$, we may create the formula $v(w)$ as indicated in Theorem 1.2. By putting existential quantifiers to the front of $v(w)$ and using properties of the *PR* conjunction and disjunction formulas which are provable in $\mathcal{Q}$, we find $v(w)$ equivalent in $\mathcal{Q}$ to an *RE* formula, which we designate also by $v(w)$. By Theorem 1.2, since we have $\vdash_{\mathcal{S}} \mathrm{Con}_t$ and $\vdash_{\mathcal{S}} \neg \mathrm{Pr}_t(n_0)$, we have $\vdash_{\mathcal{S}} \mathrm{Con}_v$.

The fact that $\vdash_{\mathcal{Q}} (\forall w)\big(t(w) \to v(w)\big)$ is immediate. Let $\mathcal{V}$ be the system designated by $v(w)$. By examining those axioms described by $v(w)$, we see that every axiom of $\mathcal{V}$ is a theorem of $\mathcal{S}$, and since $\vdash_{\mathcal{S}} \mathrm{Con}_v$, evidently $\mathcal{S}$ is a supertheory of $\mathcal{V} \cup \{\mathrm{Con}_v\}$. However, in $\mathcal{V} \cup \{\mathrm{Con}_v\}$ we can evidently prove Con_t, since $\mathcal{V}$ contains $\mathcal{Q}$ and $\vdash_{\mathcal{Q}} (\forall w)\big(t(w) \to v(w)\big)$, Once Con_t is proven in $\mathcal{V} \cup \{\mathrm{Con}_v\}$, by the very form of the axioms for $\mathcal{V}$. we can obtain any axiom of $\mathcal{S}$ by a single modus ponens. Q.E.D.

No doubt various versions of Theorem 1.3, with hypotheses altered to discuss numeration properties, as well as various strengthenings of Theorem 1.3, are possible. The reader may wish to compare our theorem with Feferman's Theorem 6.7 in [1]. Our version is designed primarily to obtain the following corollary.

COROLLARY 1.2. *If $\mathcal{S}$ is an r.e., reflexive theory extending $\mathcal{T}$, then, given any finite subtheory $\mathcal{F}$ of $\mathcal{S}$, there exists a supertheory $\mathcal{V}$ of $\mathcal{F}$ and an RE designator* $v(w)$ *for $\mathcal{V}$ such that $\mathcal{V} \cup \{\mathrm{Con}_v\}$ axiomatizes precisely* $\mathcal{S}$.

Proof. Let f_0 be the sole axiom of $\mathcal{F} \cup \mathcal{U}$. Since $\mathcal{S}$ is reflexive, $\vdash_{\mathcal{S}} \mathrm{Con}_{f_0}$, and then our result follows by Theorem 1.3. Q.E.D.

In particular, if $\mathcal{F}$ is the Predicate Calculus, we can find a $\mathcal{V}$ fulfilling Corollary 1.3 for $\mathcal{S}$. However, the reader will note that, by our construction, the theory $\mathcal{V}$ will contain the theory $\mathcal{U}$. Since $\mathcal{U}$ has a good deal of proof-theoretic strength, it is of interest to ask: how weak can a theory $\mathcal{V}$ be and yet have an *RE* designator $v(w)$ such that $\mathcal{V} \cup \{\mathrm{Con}_v\}$ axiomatizes $\mathcal{S}$? The answer to that question is surprising and derives from a previously unnoticed feature of the Gödel encodings of recursively enumerable properties.

THEOREM 1.4. *Let a true, r.e. theory $\mathcal{S}$ extending $\mathcal{T}$ be given. There exists an r.e. theory $\mathcal{V}$ possessing an RE designator* $v(w)$, *such that*:

(1) $\mathcal{V}$ *is a subtheory of* $\mathcal{S}$.

(2) $v(w)$ *numerates an axiomatization of* $\mathcal{V}$ *in* $\mathcal{V}$.

(3) $\mathcal{V}$ *represents all recursive functions and numerates all r.e. sets.*

(4) $\mathcal{V} \cup \{\mathrm{Con}_v\}$ *axiomatizes precisely* $\mathcal{S}$.

(5) $\mathcal{V}$ *does not have the power to prove* $\neg 0 = 1$.

Proof. In the one-element structure for the language of arithmetic (the *trivial structure*) the Gödel encodings of the initial primitive recursive functions hold universally. If we then understand the formula $x < y$ to abbreviate $(\exists z)(x + z' = y)$, then the encoding of the beta-function is also universally valid.

Inductively, we note that, if the functions entering into a composition schema have Gödel encodings which are universally valid in the trivial structure, the same is true of their composition; and similarly, if the functions entering into a primitive recursion schema have universally valid Gödel encodings, so does the function resulting from their primitive recursion. This establishes the universal validity in the trivial structure of all encodings of graphs of primitive recursive functions.

To encode the result of applying the least number operator, we do not use the usual encoding. Instead, we encode the fact that a function is not zero by writing that it takes a value, and this value is other than zero—the latter clause obtained from an encoding of the primitive recursive function which detects inequality. By this encoding, any value of any function is different from zero (in the trivial structure), and using this new formula in the encoding of the least number operator at the position where a negation of the old formula occurs in the usual treatment, we obtain the result that, if the function entering into a least-number schema has an encoding which is universally valid in the trivial structure, so does the result of applying the least-number operator to this function.

In this manner, all formulas encoding all recursive functions are shown to be universally valid in the one element structure. We may then provide $\mathcal{T}$ with a definition of truth in the trivial model by providing in $\mathcal{T}$ the definition of a primitive recursive function having the proper truth values on atomic formula, and which derives truth values on other formula by the obvious quantifier elimination procedure for the trivial structure. In this manner, $\mathcal{T}$ can "see" that it is impossible for a statement and its negation to both be true; that truth is preserved under logical deduction (for, in essence, by quantifier elimination we are dealing with the propositional logic); and that all our encodings of recursive functions are universally valid.

Let A_1 be the set containing all numeric substitutions into encodings of recursive functions and relations, such that the resulting statements are actually true in the standard integers; A_1 is an r.e. set. We may also take A_1 to include unicity of value statements, and it is still true in the trivial structure.

If one examines Feferman's construction of $\mathrm{Prf}_\alpha(y, x)$, it will be seen that this formula is universally valid in the trivial structure, regardless of α. As a consequence, Feferman's formula Con_α is universally false, regardless of α. Let $T(e, w, y)$ be the usual *PR* Turing predicate of Kleene, so that $(\exists y)\, T(e, w, y)$ holds iff w is in the e-th r.e. set W_e.

By Con_e we abbreviate Con_α where $(\exists y)T(e, w, y)$ is $\alpha(w)$ (the numeral e having been substituted for the variable e).

By $A_2(e)$ let us understand the set of all axioms of the form $\mathrm{Con}_e \to \sigma$, where σ is an axiom of $\mathcal{S}$. Note that $A_2(e)$ is also r.e., and in fact there is a recursive function $f(e)$ with a suitable definition in $\mathcal{S} \supseteq \mathcal{F}$ such that

$$W_{f(e)} = A_1 \cup A_2(e)$$

and

$$\vdash_{\mathcal{S}} (\forall e)\, \mathrm{Con}_{f(e)} ,$$

the latter since $\mathcal{S}$ can "see" that $W_{f(e)}$ is true in the trivial structure.

By the Recursion Theorem, we can find a number e_0 such that $W_{f(e_0)} = W_{e_0}$, and with f suitably chosen (which can be done) this fact can be verified in $\mathcal{F}$; i.e., if $d_0 = f(e_0)$, then

$$\vdash_{\mathcal{F}} (\exists y)T(e_0, w, y) \leftrightarrow (\exists y)T(d_0, w, y) .$$

Evidently, then, we have $\vdash_{\mathcal{S}} \mathrm{Con}_{e_0}$. With this one fact established, we set $\mathcal{V}$ equal to the theory with axioms in W_{e_0}, and $v(w)$ equal to $(\exists y)T(e_0, w, y)$, and we complete the proof that $\mathcal{V} \cup \{\mathrm{Con}_v\}$ axiomatizes precisely $\mathcal{S}$ by the same devices as in the proof of Theorem 1.3. The numeration and representation properties of $\mathcal{V}$ are a little more delicate: the presence of A_1 in $\mathcal{V}$ insures that all true statements are present, and the fact that $\mathcal{V}$ is a subtheory of the true theory $\mathcal{S}$ insures that no false numerations occur. Obviously, $\mathcal{V}$ cannot prove $\neg 0 = 1$, since $\mathcal{V}$ is true in the trivial structure and $\neg 0 = 1$ is false there. Q.E.D.

In Feferman's paper, the fact that $\mathcal{F}$ possesses the axiom schemata given in Feferman's Corollary 5.5 is used in a crucial way to show that, if $\alpha(w)$ is *RE* and numerates some axiomatization of a theory $\mathcal{A}$ in $\mathcal{A}$, then $\mathcal{A}$ cannot prove Con_α—Gödel's famous Second Underivability Theorem. It is of interest to investigate what can occur in extraordinarily weak systems in which the axiom schemata in question may not be present, and it is plausible that such weak systems can prove their own consistency even with $\alpha(w)$ *RE*. R. Platek tells us that it is unknown whether or not $\mathrm{Con}_{[q_0]}$ can be proven in $\mathcal{Q}$.

We can contribute only a theory $\mathcal{V}$ which can prove its own consistency using *RE* $v(w)$, when, by the consistency statement, we mean the following statement Con'_α (which differs from Feferman's Con_α):

$$(\forall y)\big(\mathrm{Pr}_\alpha(y) \to \mathrm{Neq}(y, \overline{0 = 1 \wedge \neg 0 = 1})\big) ,$$

where $\mathrm{Neq}(u, v)$ is the standard encoding of the primitive recursive condition $u \neq v$.

THEOREM 1.5. *There exists an r.e. theory $\mathcal{V}$ possessing an RE designator $v(w)$ such that:*

(1) *$\mathcal{V}$ is a subtheory of $\mathcal{F}$.*

(2) *$v(w)$ numerates an axiomatization of $\mathcal{V}$ in $\mathcal{V}$.*

(3) *$\mathcal{V}$ represents all recursive functions and numerates all r.e. sets.*

(4) $\vdash_{\mathcal{V}} \mathrm{Con}'_v$.

Proof. We proceed as in the proof of Theorem 1.4 up through the creation of the set A_1. Then, instead of $A_2(e)$, we define the set $A_3(e)$ having only the axiom Con'_e. As before, we find a recursive function $g(e)$ such that

$$W_{g(e)} = A_1 \cup A_3(e) ,$$

and then find a fixed point e_0 such that $W_{g(e_0)} = W_{e_0}$. This entire discussion can be arithmetized in $\mathcal{F}$ because Con'_e is universally valid in the trivial structure, regardless of e. $\mathcal{V}$ is then taken to be the fixed point theory W_{e_0}. The details are checked as in Theorem 1.4. Q.E.D. (²)

2. The discovery that the consistency of reflexive theories extending $\mathcal{F}$ can indeed be proven within these theories, when this consistency is suitably expressed (see Theorem 5.9 of Feferman's paper), led Feferman to a partial characterization of the types of descriptions of the theories which, when used in building the consistency statement in the usual manner, would yield the expected Gödel Second Underivability Theorem (i.e., the consistency thus expressed cannot be proven within the system).

The issue is that a non-standard designator $s(w)$ may so mysteriously describe $\mathcal{S}$ that $\mathcal{S}$ can prove consistent whatever $s(w)$ may designate, not "knowing" that $s(w)$ designates $\mathcal{S}$ itself.

Certainly, if a designator is provably equivalent in $\mathcal{S}$ to some standard designator, and the consistency statement based on the latter is not provable, then, since $\mathcal{S} \supseteq \mathcal{F}$, neither is the consistency statement based on the former. Feferman, however, succeeded in showing more, namely, that so long as the designator is an *RE* formula, the consistency statement based on it cannot be proven in the system $\mathcal{S}$, even though the non-standard *RE* designator cannot be proven within $\mathcal{S}$ to axiomatize the same theory as the standard one (see Feferman's Theorem 5.6).

So far as the author knows, no broader characterization than "*RE*-formula", which insures the Gödel Second Underivability Theorem, has been forthcoming since Feferman's work. It is our aim in this section

(²) Actually, in Theorem 1.5, $\mathcal{V}$ can be taken to be finitely axiomatizable. In fact, a suitable "fixed point extension" of $Q \vee (\forall x)(\forall y)(x = y)$ will do, where Q is the sole axiom of Robinson's Q.

to show that there cannot be a "broadest" constructive characterization which is necessary as well as sufficient.

To begin our work, let us define

$$U = \{\alpha|\ \alpha \text{ numerates some axiomatization of } \mathcal{S} \text{ in } \mathcal{S}\}\,.$$

THEOREM 2.1. *If $\mathcal{S}$ is a true theory extending $\mathcal{T}$, then U is π_2^0-complete.*

Proof. Let $s(x)$ designate some RE axiomatization of $\mathcal{S}$. A Tarski–Kuratowski computation shows that U is π_2^0. If we can 1-reduce a complete π_2^0 set to U, we shall be done.

Now if $T_{\forall\exists}$ denotes all true π_2^0 sentences, one easily shows that $T_{\forall\exists}$ is π_2^0-complete.

Let β be any sentence (true of false) in π_2^0-form; and suppose

$$\beta = (\forall y)(\exists z)\alpha(y, z)$$

where we may suppose $\alpha(y, z)$ is PR. (This form can be found effectively from a form for β in which only a recursive matrix appears.) To β, let us assign the formula $\varphi_\beta(x)$ given by

$$s(x)\wedge(\forall y \leqslant x)(\exists z)\alpha(y, z)\,.$$

Then one can easily show, using the fact that $\mathcal{S}$ is not finitely axiomatizable, that

$$\beta \in T_{\forall\exists} \leftrightarrow \varphi_\beta \in U\,.$$

Q.E.D.

If we require minimally of a potential description of an axiomatization of $\mathcal{T}$ that it numerate $\mathcal{T}$ in $\mathcal{T}$, Theorem 2.1 shows the essential non-constructivity of detecting such descriptions. If we require further that a description actually designate an axiomatization of $\mathcal{T}$, the set V of all such designators can be easily proven recursively isomorphic to the set of all true arithmetic statements, again a highly non-constructive set.

Let $R = \{\alpha|\ \alpha \text{ is } RE\}$. Feferman's result (his Theorem 5.6) is that

$$\alpha \in U \cap R \rightarrow not\text{-} \vdash_{\mathcal{T}} \mathrm{Con}_\alpha\,, \tag{1}$$

indicating a type of constructivity which might be possible—i.e., that, given that $\alpha \in U$, it is a recursive property of α which determines the behavior of Con_α. However, as the next two results show, even this semi-constructivity of the type in (1) cannot be necessary and sufficient.

In what follows, we shall need the result that, if $\chi(u, v, w, \ldots)$ is an arithmetic formula in free variables $u, v, w, \ldots$ then there exists a formula $\varphi(v, w, \ldots)$ in the free variables $v, w, \ldots$, with number $\overline{\varphi}$, such that

$$\vdash_{\mathcal{Q}}\varphi(v, w, \ldots)\leftrightarrow\chi(\overline{\varphi}, v, w, \ldots);$$

this result we refer to as the Fixed Point Theorem. Feferman in his Lemma 5.1 has a proof of the Fixed Point Theorem for only one free variable u, but his technique easily obtains this more general result.

LEMMA 2.1. *Let $\mathcal{S}$ be a true, r.e., reflexive supertheory of $\mathcal{T}$. There exists a formula $\alpha(x, y)$ in arithmetic in the two free variables x, y only, which is a Δ_2^0 formula such that:*

(1) *For each n, $\alpha(n, x)$ designates $\mathcal{S}$.*

(2) *For each n, $\alpha(n, x)$ numerates an axiomatization of $\mathcal{S}$ in $\mathcal{S}$.*

(3) $n \in K \leftrightarrow \vdash_{\mathcal{S}} \mathrm{Con}_{\alpha(n,\cdot)}$.

(In the above K is, of course, the complete r.e. set.)

Proof. Let $s(w)$ be a PR designator for $\mathcal{S}$. Let $\chi(u, v)$ be the formula

$$(\forall y)\neg T(u, u, y)\wedge(\exists z)\mathrm{Prf}_s\big(\mathrm{sub}(u, 0)v, z\big)\,, \tag{2}$$

and, by the Fixed Point Theorem, let $\varphi(u)$ be a formula with number $\overline{\varphi}$ such that

$$\vdash_{\mathcal{Q}}\varphi(u)\leftrightarrow\chi(u, \overline{\varphi})\,. \tag{3}$$

Finally, let $\alpha(u, x)$ be the formula

$$s^*(x)\vee[x = \overline{0 = 1}\wedge\chi(u, \overline{\varphi})] \tag{4}$$

where $s^*(x)$ is the formula for $\mathcal{S}$ insured by Feferman's Theorem 5.9.

CLAIM 1. *For each n, $\alpha(\overline{n}, x)$ designates $\mathcal{S}$.*

Proof of Claim 1. If $n \in K$, then $(\exists y)T(n, n, y)$ is true, hence $\chi(\overline{n}, \overline{\varphi})$ is false by Eq. (2) and hence $\alpha(\overline{n}, x)$ does indeed designate $\mathcal{S}$, by Eq. (4). (s^* surely designates $\mathcal{S}$, since it bi-numerates $\mathcal{S}$, and $\mathcal{S}$ proves only true statements.)

If $n \notin K$, then $\chi(\overline{n}, \overline{\varphi})$ cannot be true, for if it were true, then

$$(\exists z)\mathrm{Prf}_s\big(\mathrm{sub}(\overline{n}, 0)\overline{\varphi}, z\big)$$

is true, i.e.,

$$\vdash_{\mathcal{S}}\varphi(\overline{n})$$

and hence, by Eq. (3),

$$\vdash_{\mathcal{S}}\neg\chi(\overline{n}, \overline{\varphi});$$

this is an impossibility, since in $\mathcal{S}$ only true statements are provable. Hence, if $n \notin K$, $\chi(\overline{n}, \overline{\varphi})$ is again false, and again $\alpha(\overline{n}, x)$ designates $\mathcal{S}$.

CLAIM 2. *For each n, $\alpha(\overline{n}, x)$ numerates $\mathcal{S}$ in $\mathcal{S}$.*

Recalling that s^* bi-numerates $\mathcal{S}$ in $\mathcal{S}$, we see that, if $m \neq \overline{0 = 1}$, then

$$\vdash_{\mathcal{S}}\alpha(\overline{n}, \overline{m})\leftrightarrow\vdash_{\mathcal{S}} s_*(m)\leftrightarrow m \text{ is an axiom of } \mathcal{S}\,.$$

It is then left only to show that

$$\text{not-} \vdash_{\mathcal{S}} \alpha(\mathrm{n}, \overline{0=1})$$

which is evident, since, by Claim 1, $\alpha(\bar{\mathrm{n}}, \mathrm{x})$ designates $\mathcal{S}$, and in $\mathcal{S}$ no false statement can be proven.

CLAIM 3. $\mathrm{n} \in \mathrm{K} \leftrightarrow \vdash_{\mathcal{S}} \mathrm{Con}_{\alpha(\bar{\mathrm{n}},\cdot)}$.

In one direction we have

$$\begin{aligned}
(5)\qquad & \vdash_{\mathcal{S}} \mathrm{Con}_{\alpha(\bar{\mathrm{n}},\cdot)} \to \vdash_{\mathcal{S}} \neg\, \alpha(\bar{\mathrm{n}}, \overline{0=1}) \\
& \to \vdash_{\mathcal{S}} \neg\, \chi(\bar{\mathrm{n}}, \bar{\varphi}) \\
& \to \vdash_{\mathcal{S}} \neg\big[(\forall \mathrm{y}) \neg\, \mathrm{T}(\bar{\mathrm{n}}, \bar{\mathrm{n}}, \mathrm{y}) \wedge (\exists \mathrm{z}) \mathrm{Prf}_{\mathrm{s}}\big(\mathrm{sub}(\bar{\mathrm{n}}, 0)\bar{\varphi}, \mathrm{z}\big)\big].
\end{aligned}$$

From the second line in Eq. (5) and by Eq. (3),

$$(6)\qquad \vdash_{\mathcal{S}} \varphi(\bar{\mathrm{n}}).$$

Now Eq. (6) and the PR nature of s gives (using Feferman's Theorem 5.4, Lemma 3.9, and Theorem 4.5),

$$(7)\qquad \vdash_{\mathcal{S}} (\exists \mathrm{z}) \mathrm{Prf}_{\mathrm{s}}\big(\mathrm{sub}(\bar{\mathrm{n}}, 0)\bar{\varphi}, \mathrm{z}\big).$$

Eqs. (5) and (7) together give

$$\begin{aligned}
\vdash_{\mathcal{S}} \mathrm{Con}_{\alpha(\bar{\mathrm{n}},\cdot)} & \to \vdash_{\mathcal{S}} \neg(\forall \mathrm{y}) \neg\, \mathrm{T}(\bar{\mathrm{n}}, \bar{\mathrm{n}}, \mathrm{y}) \\
& \to \vdash_{\mathcal{S}} (\exists \mathrm{y}) \mathrm{T}(\bar{\mathrm{n}}, \bar{\mathrm{n}}, \mathrm{y}).
\end{aligned}$$

Then, since $\mathcal{S}$ proves only true statements, we evidently must have $\mathrm{n} \in K$.

In the opposite direction,

$$\begin{aligned}
\mathrm{n} \in K & \to \vdash_{\mathcal{S}} (\exists \mathrm{y}) T(\bar{\mathrm{n}}, \bar{\mathrm{n}}, \mathrm{y}) \\
& \to \vdash_{\mathcal{S}} \neg\, \chi(\bar{\mathrm{n}}, \bar{\varphi}) && \text{(using Eq. (2))} \\
& \to \vdash_{\mathcal{S}} (\forall \mathrm{x})\big(\mathrm{s}^*(\mathrm{x}) \leftrightarrow \alpha(\bar{\mathrm{n}}, \mathrm{x})\big) && \text{(using Eq. (4))} \\
& \to \vdash_{\mathcal{S}} \mathrm{Con}_{\mathrm{s}^*} \leftrightarrow \mathrm{Con}_{\alpha(\bar{\mathrm{n}},\cdot)} \\
& \to \vdash_{\mathcal{S}} \mathrm{Con}_{\alpha(\bar{\mathrm{n}},\cdot)}
\end{aligned}$$

since, by Theorem 5.4 of Feferman, we indeed have

$$\vdash_{\mathcal{S}} \mathrm{Con}_{\mathrm{s}^*}.$$

By inspection, $\chi(\mathrm{u}, \mathrm{v})$ is Δ_2^0, and the construction in Feferman [1] $\mathrm{s}^*(\mathrm{x})$ gives a π_1^0 formula; hence $\alpha(\mathrm{x}, \mathrm{y})$ is Δ_2^0. Q.E.D.

THEOREM 2.2. *Let $\mathcal{S}$ be any true, r.e., reflexive supertheory of $\mathcal{T}$. Let R be any r.e. set such that*

$$\alpha \in R \cap U \to \text{not-} \vdash_{\mathcal{S}} \mathrm{Con}_\alpha.$$

Then, given an index e *of R, we can effectively find a formula $\beta(\mathrm{x})$ such that*

$$\beta \in U - R \quad \text{and} \quad \text{not-} \vdash_{\mathcal{S}} \mathrm{Con}_\beta.$$

Proof. We can effectively find an index f_0 for

$$\{\mathrm{n} \mid \overline{\alpha(\bar{\mathrm{n}}, \mathrm{x})} \in R\}.$$

By Lemma 2.1, $W_{\mathrm{f}_0} \subseteq \widetilde{K}$. From the productivity of $\widetilde{K}$, $\mathrm{f}_0 \in \widetilde{K} - W_{\mathrm{f}_0}$. Then we evidently have

$$\overline{\alpha(\bar{\mathrm{f}}_0, \mathrm{x})} \notin R,$$

$$\overline{\alpha(\bar{\mathrm{f}}_0, \mathrm{x})} \in U \quad \text{(by Lemma 2.1 (2))}$$

and not- $\vdash_{\mathcal{S}} \mathrm{Con}_{\alpha(\bar{\mathrm{f}}_0,\cdot)}$ (using Lemma 2.1 (3) and the fact that $\mathrm{f}_0 \in \widetilde{K}$). Q.E.D.

It is clear, by examining the proof of Theorem 2.2, that its conclusion can be strengthened to

$$\beta \in V - R \quad \text{and} \quad \text{not-} \vdash_{\mathcal{S}} \mathrm{Con}_\beta.$$

Next, we note that if we set

$$P = \{\bar{\alpha} \mid \text{not-} \vdash_{\mathcal{S}} \mathrm{Con}_\alpha\}$$

then P is π_1^0 and we have

$$\alpha \in P \cap U \leftrightarrow \text{not-} \vdash_{\mathcal{S}} \mathrm{Con}_\alpha$$

so that the most obvious strengthening of Theorem 2.2 is not possible.

Virtually the same results hold if we completely abandon proof-theoretic properties and require only the designator property, since the formulas we have created are both designators as well as numerators.

3. In this section, we turn our attention to another result of Feferman (see his Theorem 6.5) which states that, if $\mathcal{S} \supseteq \mathcal{T}$, and $\mathrm{s}(\mathrm{x})$ is an RE numerator for $\mathcal{S}$, then $\mathcal{S} \cup \{\mathrm{Con}_{\mathrm{s}}\}$ is not relatively interpretable in $\mathcal{S}$. Our aim is to further explore the degrees of relative interpretability which lie between $\mathcal{S}$ and $\mathcal{S} \cup \{\mathrm{Con}_{\mathrm{s}}\}$.

We assume that the reader is familiar with the definition of relative interpretability in the form we use it (see Feferman, p. 49) and the fact that, if $\mathcal{A} \leqslant \mathcal{B}$ is taken to denote the relative interpretability of $\mathcal{A}$ in $\mathcal{B}$, the relation $\leqslant$ is transitive and symmetric. The equivalence classes under the relation $\leqslant$ we call the degrees of relative interpretability; such a degree is called r.e. if one of its elements is an r.e. theory.

We warn the reader that, in our treatment, we cavalierly ignore certain fine points, the main one being that arithmetic is viewed (for purposes of degrees of relative interpretability) as entirely a relational language.

By a *relativizing map* we mean a choice of one-place predicate (the relativizing predicate) and a choice, for each relation symbol, of a predicate in the corresponding free variables. The result of applying a relativizing map to a formula is as specified inductively by the rules of Feferman, taking due care to avoid clash of variables. Since the language of arithmetic has only finitely many relation symbols, a relativizing map is a finite series of choices, and hence can be "coded" into a natural number by some standard choice of encoding; we shall also speak of this natural number as a relativizing map. We note that a relativizing map provides a relative interpretation of $\mathcal{A}$ in $\mathcal{B}$ exactly in the case that the result of applying the map to any axiom of $\mathcal{A}$ is a theorem of $\mathcal{B}$; thus, in the instance that $\mathcal{A}$ is finitely axiomatized and $\mathcal{B}$ is r.e., the fact that $\mathcal{A}$ is interpretable in $\mathcal{B}$ is r.e., and the fact that a specific number is the number of a proof in $\mathcal{B}$ of the result of applying specific relativizing map to the sole axiom of $\mathcal{A}$ is primitive recursive, provided $\mathcal{B}$ is axiomatized primitive recursively.

In Lemma 3.1, which follows next, we adapt an argument of Montague in [4]. As in Theorem 2.2, the technique of proof is by invoking the Fixed Point Theorem upon a suitable formula. Before we can begin the proof of Lemma 3.1, however, we shall need a little terminology.

Fix a formula $\psi(x)$ in one free variable x; by a *ψ-disjunct* δ we mean a disjunction of the form

$$\delta = \pm\psi(\bar{n}_1) \vee \pm\psi(\bar{n}_2) \vee \pm \ldots \vee \pm\psi(\bar{n}_l)$$

where the sign $\pm$ indicates that either the term following appears alone (if $+$) or appears prefaced by a negation sign (if $-$). In δ all the numbers $n_1, \ldots, n_l$ must be distinct.

We say that a number n *appears positively* in δ if $\psi(\bar{n})$ is one of the ψ-terms occurring in δ which is not prefaced by a negation; n *appears negatively* if $\psi(\bar{n})$ appears in δ prefaced by a negation.

In what follows, we shall have occasion to treat the formulas $\psi(\bar{0}), \psi(\bar{1}), \psi(\bar{2}), \ldots$ as if they were atomic letters of a propositional logic. Let a ψ-disjunct δ be given as well as a conjunction γ of disjuncts $\delta_1, \ldots, \delta_n$; i.e., $\gamma = \delta_1 \wedge \ldots \wedge \delta_n$. Then it is decidable whether or not $\gamma \to \delta$ holds in the propositional logic just mentioned, and in fact, if $\gamma \to \delta$ is false a partial truth valuation of propositional letters in which γ is true and δ is false can be found. If we order partial truth valuations by their sequence numbers, then the least valuation which makes γ true and δ false (if any) can be found primitive recursively in γ, δ.

LEMMA 3.1. *If $\mathcal{S} \supseteq \mathcal{T}$ is r.e. and ω-consistent and* $s(x)$ *is an RE formula numerating some axiomatization of* $\mathcal{S}$ *in* $\mathcal{S}$, *a formula* $\psi(x)$ *can be found primitive recursively from the number of* $s(x)$ *such that:*

(1) *If δ is a ψ-disjunct and γ a conjunct of ψ-disjuncts, then $\mathcal{S} \cup \{\delta\}$ is relatively interpretable in $\mathcal{S} \cup \{\gamma\}$ if and only if $\gamma \to \delta$ is provable in the propositional logic based on the letters $\psi(\bar{0}), \psi(\bar{1}), \ldots$*

(2) $\vdash_{\mathcal{S}} \mathrm{Con}_s \to (\underline{\forall} x)\psi(x)$.

Proof. Given $s(x)$, we can construct a *PR* formula $s'(x)$ by the method of Craig which axiomatizes the same theory as $s(x)$, and for which this latter fact is actually provable in $\mathcal{S}$ (see Feferman Theorem 4.13).

Let $\mathrm{Re1}(r, s, t, y, z)$ be a *PR* formula encoding the relation "δ is a ψ-disjunct and γ a conjunct of ψ-disjuncts, where δ has number r, γ has number s, and ψ has number t; y is a relativizing map and z an $s'(x)$-proof of the y-relativization of δ in $\mathcal{S} \cup \{\gamma\}$; and furthermore, $\gamma \to \delta$ is not provable in the propositional logic based on the letters $\psi(\bar{0}), \psi(\bar{1}), \ldots$."

Let $\mathrm{Re2}(x, r, s, t)$ be a *PR* formula encoding the relation, "δ is a ψ-disjunct and γ is a conjunct of ψ-disjuncts, where δ has number r, γ has number s, and ψ has number t; and either x appears (as a numeral) in neither δ or γ; or else x appears in δ and appears negatively; or else x appears in γ alone, and the ψ-term in which the numeral x is inserted obtains the value "true" in the least valuation (if any) making γ true and δ false."

We now define the formula $\Gamma(x, t)$ to be

$$(\forall w)\left[\mathrm{Re1}\big((w)_0, (w)_1, t, (w)_2, (w)_3\big) \wedge (\forall u < w) \neg \mathrm{Re1}\big((u)_0, (u)_1, t, (u)_2, (u)_3\big) \to \mathrm{Re2}\big(x, (w)_0, (w)_1, t\big)\right].$$

By the Fixed Point Theorem, we obtain a formula $\psi(x)$ in one free variable x, with number $\bar{\psi}$, such that

$$\vdash_{\mathcal{Q}} \Gamma(x, \bar{\psi}) \leftrightarrow \psi(x). \tag{1}$$

We now prove the first assertion of the Lemma. Certainly, if $\gamma \to \delta$ if provable in propositional logic, then $\mathcal{S} \cup \{\delta\}$ is relatively interpretable (by the identity map) in $\mathcal{S} \cup \{\gamma\}$. The converse is established by supposing it false and reaching a contradiction. Suppose that there exists δ, γ such that $\mathcal{S} \cup \{\delta\}$ is relatively interpretable in $\mathcal{S} \cup \{\delta\}$, and yet $\gamma \to \delta$ is not provable in propositional logic.

Then there exists a number h such that $\mathrm{Re1}\big((\bar{h})_0, (\bar{h})_1, \bar{\psi}, (\bar{h})_2, (\bar{h})_3\big)$, and such that h is the least number for which this is true; and both these facts are verifiable in Robinson's $\mathcal{Q}$.

Hence, by the definition of $\Gamma(x, t)$,

$$\vdash_{\mathcal{Q}} \Gamma(x, \bar{\psi}) \leftrightarrow \mathrm{Re2}\big(x, (\bar{h})_0, (\bar{h})_1, \bar{\psi}\big),$$

and so, by Eq. (1),

$$\vdash_{\mathcal{Q}} \psi(\mathrm{x}) \leftrightarrow \mathrm{Re}2(\mathrm{x}, (\bar{\mathrm{h}})_0, (\bar{\mathrm{h}})_1, \bar{\psi}) . \tag{2}$$

Fix a number n. If n appears in δ and in fact appears positively, then by Eq. (2) above,

$$\vdash_{\mathcal{Q}} \neg\psi(\bar{\mathrm{n}}) ,$$

while if n appears in δ and appears negatively,

$$\vdash_{\mathcal{Q}} \psi(\bar{\mathrm{n}}) .$$

In either case, $\psi(\bar{\mathrm{n}})$ is proven in $\mathcal{Q}$ with polarity (i.e., positivity or negativity) opposite its appearance in δ. Thus, in $\mathcal{Q}$ the disjunct δ is refutable, and so $\mathcal{S} \cup \{\delta\}$ is inconsistent. Since $\mathcal{S} \cup \{\delta\}$ is relatively interpretable in $\mathcal{S} \cup \{\gamma\}$, the latter must also be inconsistent.

Furthermore, the least valuation which makes γ true and δ false must value all n which occur positively in δ as false, and all n which appear negative in δ as true, since this is the only valuation which makes the disjunction δ false. In particular, the least valuation agrees with the valuations given by $\mathcal{Q}$ for these n occurring in δ.

If n occurs in γ alone, and the least valuation making γ true and δ false values n as true, then by the definition of Re2 we have $\vdash_{\mathcal{Q}} \psi(\mathrm{n})$; while, if n is valued as false, we have $\vdash_{\mathcal{Q}} \neg\psi(\mathrm{n})$. Thus, the valuation given by $\mathcal{Q}$ to the n occurring in δ or γ agrees with the least valuation which makes γ true and δ false; hence in particular $\vdash_{\mathcal{Q}} \gamma$.

We conclude that $\mathcal{S} \cup \{\gamma\}$ and $\mathcal{S}$ are the same theory, and since $\mathcal{S}$ is consistent, $\mathcal{S} \cup \{\gamma\}$ is also consistent. This is a contradiction.

The second part of the Lemma is proven by arithmetizing in $\mathcal{S}$ the proof of the first part, which we just gave. The only details whose arithmetization might present difficulties are where we discuss numeration properties of $\mathcal{Q}$; but in these instances, Feferman's schemata in his Corollary 5.5 is invoked.

Then, reasoning in $\mathcal{S}$, we conclude that the first part of the Lemma is correct provided $\mathcal{S}$ is consistent, and hence, provided $\mathcal{S}$ is consistent, no x ever appears in any δ or γ in the situation discussed, so that $\psi(\mathrm{x})$ is true for any x. Q.E.D.

It is perhaps of interest to note that the first part of Lemma 3.1 holds if only $\mathcal{S} \supseteq \mathcal{Q}$, and in this case $(\forall \mathrm{x})\psi(\mathrm{x})$ is true, whether or not provable in $\mathcal{S} \cup \{\mathrm{Con}_{\mathcal{S}}\}$.

Let us call a form in propositional letters *positive* if no negation sign appears in it. Then Lemma 3.1 gives the following result.

THEOREM 3.1. *If $\mathcal{S} \supseteq \mathcal{T}$ is r.e. and consistent and* $\mathrm{s}(\mathrm{x})$ *is an RE formula numerating some axiomatization of $\mathcal{S}$ in $\mathcal{S}$, then a* 1-1 *mapping M can be defined from the propositional logic on denumerably many letters $L_1, L_2, L_3, \ldots$ to r.e. degrees of relative interpretability of supertheories of $\mathcal{S}$, such that M reverses order.*

Furthermore, the range of M on the positive forms is contained the degrees of r.e. subtheories of $\mathcal{S}$ which are also subtheories of $\mathcal{S} \cup \{\mathrm{Con}_{\mathcal{S}}\}$.

Proof. If two propositional forms in the letters $L_1, L_2, \ldots$ are logically equivalent in propositional logic, then the same holds true of the corresponding forms in $\psi(\bar{0}), \psi(\bar{1}), \ldots$ (where $\psi(\mathrm{x})$ is the formula mentioned in Lemma 3.1), and hence, if the two corresponding forms in letters $\psi(\bar{0}), \psi(\bar{1}), \ldots$ be designated λ' and σ', $\mathcal{S} \cup \{\lambda'\}$ and $\mathcal{S} \cup \{\sigma'\}$ are interdeducible theories and so belong to the same r.e. degree of relative interpretability.

Let the equivalence class of a propositional form λ be denoted by $[\lambda]$, and let λ' be the corresponding form in the letters $\psi(\bar{0}), \psi(\bar{1}), \ldots$. By the previous paragraph, the mapping

$$M([\lambda]) = \text{the degree of the theory } \mathcal{S} \cup \{\lambda'\}$$

is well-defined.

Lemma 3.1 (1) shows that M is 1-1 by the following reasoning. If λ, σ are forms in the letters $L_1, L_2, \ldots$ such that $M([\lambda]) = M([\sigma])$, then we may as well take both λ, σ to be in conjunctive normal form. Let λ', σ' correspond to λ, σ respectively in the letters $\psi(\bar{0}), \psi(\bar{1}), \ldots$ Suppose that σ' is the conjunction of the ψ-disjuncts $\delta_1, \ldots, \delta_n$, so that $\sigma' = \delta_1 \wedge \delta_2 \wedge \ldots \wedge \delta_n$. Since $\mathcal{S} \cup \{\sigma'\} \leqslant \mathcal{S} \cup \{\lambda'\}$ by hypothesis, then $\mathcal{S} \cup \{\delta_i\} \leqslant \mathcal{S} \cup \{\lambda'\}$ holds for each $i = 1, \ldots, n$, and hence by Lemma 2.1 (1) we have $\lambda' \to \delta_i$ in the propositional logic for $i = 1, \ldots, n$. Therefore we have $\lambda' \to \sigma'$; by symmetry we have also $\sigma' \to \lambda'$, and hence $\sigma \leftrightarrow \lambda$ holds in the propositional logic, so that $[\sigma] = [\lambda]$.

Similar reasoning will show that M is order-reversing. Then an application of Lemma 3.1 (2) will show that the value of M on a positive form is an r.e. degree of a theory intermediate between $\mathcal{S}$ and $\mathcal{S} \cup \{\mathrm{Con}_{\mathcal{S}}\}$. Q.E.D.

The following general theorem has obvious particular application to the degrees between those of $\mathcal{S}$ and $\mathcal{S} \cup \{\mathrm{Con}_{\mathcal{S}}\}$; it is a type of density result for degrees. The very fact that it, as well as our Theorem 3.1, is justified by techniques entirely different from those used in the theory of Turing degrees, should indicate that it is not wise to make too direct an analogy between the two types of degrees, despite the fact that one is occasionally lucky. None of this, of course, is to belie the use of more sophisticated techniques adopted from the Turing degrees; the author is not familiar enough with them to comment further.

In what follows, $\mathcal{A} \not\leqslant \mathcal{B}$ signifies that $\mathcal{A}$ is not interpretable in $\mathcal{B}$.

THEOREM 3.2. *Let $\mathcal{S} \supseteq \mathcal{F}$ be r.e. Suppose that $\mathcal{T} \supseteq \mathcal{S}$ is r.e. and that we have $\mathcal{T} \nleqslant \mathcal{S}$. Then there exists a theory $\mathcal{U} \supseteq \mathcal{S}$, such that $\mathcal{T} \supseteq \mathcal{U}$, and furthermore*

$$\mathcal{T} \nleqslant \mathcal{U}, \text{ and } \mathcal{U} \nleqslant \mathcal{S}.$$

Proof. By a theorem of Orey (see Feferman, Theorem 6.9), $\mathcal{T} \nleqslant \mathcal{S}$ implies that there exists a finite subtheory $\mathcal{F} \subseteq \mathcal{T}$ such that $\mathcal{F} \nleqslant \mathcal{S}$. Let $\mathcal{F}$ have λ as its only axiom.

Let a *PR* formula $\mathrm{s}(\mathrm{x})$ numerating an axiomatization of $\mathcal{S}$ in $\mathcal{Q}$ be given. Let this formula be employed in the construction of a formula $\mathrm{Re1}(\mathrm{w},\mathrm{x},\mathrm{z})$ which bi-numerates in the relation: "w is a relativizing map and z is a proof in the system $\mathrm{s}(\mathrm{x})$ of the w-relativization of the axioms of $\mathcal{Q}$ conjoined with $\lambda \vee \psi$, where x is the number of ψ".

Let us similarly construct a formula $\mathrm{Re2}(\mathrm{w},\mathrm{x},\mathrm{y},\mathrm{z})$ which bi-numerates in $\mathcal{Q}$ the relation: "w is a relativizing map and z is a proof in $\mathcal{Q}$ conjoined with $\lambda \vee \psi$ conjoined with γ of the w-relativization of λ, where x is the number of ψ and y the number of γ".

Let $\Gamma(\mathrm{x})$ be the formula

$$(\forall \mathrm{t})\Big[\mathrm{Re1}\big((\mathrm{t})_0, \mathrm{x}, (\mathrm{t})_2\big) \to (\exists \mathrm{u} < \mathrm{t})\big(\mathrm{Re2}\big((\mathrm{u})_0, \mathrm{x}, (\mathrm{u})_2, (\mathrm{u})_3\big) \wedge \mathrm{Prf}_{\mathrm{s}}\big((\mathrm{u})_2, (\mathrm{u})_4\big)\big)\Big]$$

where, of course, $\mathrm{Prf}_{\mathrm{s}(\mathrm{x},\mathrm{y})}$ is Feferman's proof predicate.

By the Fixed Point Theorem, a sentence ψ with number $\overline{\psi}$ can be found so that

$$(3) \qquad \vdash_{\mathcal{Q}} \psi \leftrightarrow \Gamma(\overline{\psi}).$$

CLAIM 1. $\mathcal{Q} \cup \{\lambda \vee \psi\}$ *is not relatively interpretable in* $\mathcal{S}$.

Suppose the contrary. Then there exists a number h so that $\mathrm{Re1}\big((\overline{\mathrm{h}})_0, \psi, (\overline{\mathrm{h}})_2\big)$ is true, and such that h is the least such number. These latter facts are verifiable in $\mathcal{Q}$, and hence, by the definition of $\Gamma(\mathrm{x})$, we have

$$(4) \qquad \vdash_{\mathcal{Q}} \Gamma(\overline{\psi}) \leftrightarrow (\exists \mathrm{u} < \overline{\mathrm{h}})\big(\mathrm{Re2}\big((\mathrm{u})_0, \overline{\psi}, (\mathrm{u})_2, (\mathrm{u})_3\big) \wedge \mathrm{Prf}_{\mathrm{s}}\big((\mathrm{u})_2, (\mathrm{u})_4\big)\big).$$

Now, either $(\exists \mathrm{u} < \overline{\mathrm{h}})\big(\mathrm{Re2}\big((\mathrm{u})_0, \overline{\psi}, (\mathrm{u})_2, (\mathrm{u})_3\big) \wedge \mathrm{Prf}_{\mathrm{s}}\big((\mathrm{u})_2(\mathrm{u})_4\big)\big)$ is true or it is false.

If it is true, then it is provable in $\mathcal{Q}$, and so $\vdash_{\mathcal{Q}} \Gamma(\overline{\psi})$ by (4), giving $\vdash_{\mathcal{Q}} \psi$ by (3). Also, since it is true, there exists a theorem γ of $\mathcal{S}$ such that

$$(5) \qquad \{\lambda\} \leqslant \mathcal{Q} \cup \{\lambda \vee \psi\} \cup \{\gamma\}.$$

But then $\vdash_{\mathcal{Q}} \psi$ shows that $\vdash_{\mathcal{Q}} \lambda \vee \psi$ and hence $\mathcal{Q} \cup \{\lambda \vee \psi\} \cup \{\gamma\}$ has the same strength as $\mathcal{Q} \cup \{\gamma\}$, a subtheory of $\mathcal{S}$, so by (5), we have

$$\{\lambda\} \leqslant \mathcal{S}$$

which is contrary to the choice of λ.

Hence, it must be that $(\exists \mathrm{u} < \overline{\mathrm{h}})\big(\mathrm{Re2}\big((\mathrm{u})_0, \overline{\psi}, (\mathrm{u})_2, (\mathrm{u})_3\big) \wedge \mathrm{Prf}_{\mathrm{x}}\big((\mathrm{u})_2, (\mathrm{u})_4\big)\big)$ is false, and so its negation is provable in $\mathcal{Q}$, and hence (4) gives $\vdash_{\mathcal{Q}} \neg\Gamma(\overline{\psi})$, from which we have $\vdash_{\mathcal{Q}} \neg\psi$ by (3). Therefore we have $\vdash_{\mathcal{Q}} \lambda \vee \psi \leftrightarrow \lambda$, so that $\mathcal{Q} \cup \{\lambda \vee \psi\}$ and $\mathcal{Q} \cup \{\lambda\}$ have the same strength. By the hypothesis of our proof by contradiction, we have

$$\mathcal{Q} \cup \{\lambda \vee \psi\} \leqslant \mathcal{S}$$

and, by what we have just seen, this implies

$$\mathcal{Q} \cup \{\lambda\} \leqslant \mathcal{S}$$

which is contrary to our choice of λ.

In either case, we have a contradiction, and so the claim is proven.

CLAIM 2. $\{\lambda\}$ *is not relatively interpretable in any theory of the form* $\mathcal{Q} \cup \{\lambda \vee \psi\} \cup \{\gamma\}$, *with* γ *a theorem of* $\mathcal{S}$.

The claim is true, because, supposing the contrary, we see that for some number h, $\mathrm{Re2}\big((\overline{\mathrm{h}})_0, \overline{\psi}, (\overline{\mathrm{h}})_2, (\overline{\mathrm{h}})_3\big) \wedge \mathrm{Prf}_{\mathrm{s}}\big((\overline{\mathrm{h}})_2, (\overline{\mathrm{h}})_4\big)$ is true and h is the least number such that it is true. Then these latter facts would be verifiable in $\mathcal{Q}$, and hence, by the definition of $\Gamma(\mathrm{x})$, we would have

$$(6) \qquad \vdash_{\mathcal{Q}} \Gamma(\overline{\psi}) \leftrightarrow (\exists \mathrm{t} \leqslant \overline{\mathrm{h}})\,\mathrm{Re1}\big((\mathrm{t})_0, \overline{\psi}, (\mathrm{t})_2\big).$$

By Claim 1, $\mathcal{Q} \cup \{\lambda \vee \psi\} \nleqslant \mathcal{S}$ and so indeed $(\forall \mathrm{t} \leqslant \overline{\mathrm{h}})\,\neg\mathrm{Re1}\big((\mathrm{t})_0, \overline{\psi}, (\mathrm{t})_2\big)$ is true and also provable in $\mathcal{Q}$, and so we would have $\vdash_{\mathcal{Q}} \Gamma(\overline{\psi})$ by (6) and then $\vdash_{\mathcal{Q}} \psi$ by (3).

Then we would have $\vdash_{\mathcal{Q}} \lambda \vee \psi$, so that $\mathcal{Q} \cup \{\lambda \vee \psi\} \cup \{\gamma\}$ would have the same strength as $\mathcal{Q} \cup \{\gamma\}$, which is a subtheory of $\mathcal{S}$, and then since

$$\{\lambda\} \leqslant \mathcal{Q} \cup \{\lambda \vee \psi\} \cup \{\gamma\}$$

is supposed to be true, we would have

$$\{\lambda\} \leqslant \mathcal{S}$$

contrary to our choice of λ.

CLAIM 3. *If we set* $\mathcal{U} = \mathcal{S} \cup \{\lambda \vee \psi\}$ *then* $\mathcal{T} \supseteq \mathcal{U}$, *and also* $\mathcal{T} \nleqslant \mathcal{U}$, *and* $\mathcal{U} \nleqslant \mathcal{S}$.

For the fact that $\mathcal{T} \supseteq \mathcal{U}$ is immediate, while if $\mathcal{T} \leqslant \mathcal{U}$ then in particular

$$\{\lambda\} \leqslant \mathcal{S} \cup \{\lambda \vee \psi\}$$

and hence there is a theorem γ of $\mathcal{S}$ such that

$$\{\lambda\} \leqslant \{\gamma\} \cup \{\lambda \vee \psi\}$$

which contradicts the fact that

$$\{\lambda\} \nleqslant \mathcal{Q} \cup \{\lambda \vee \psi\} \cup \{\gamma\}$$

which we have obtained from Claim 2.

If $\mathcal{U} \leqslant \mathcal{S}$, then in particular,

$$\mathcal{Q} \cup \{\lambda \vee \psi\} \leqslant \mathcal{S}$$

which contradicts Claim 1.

Hence, indeed $\mathcal{J} \not\leqslant \mathcal{U}$ and $\mathcal{U} \not\leqslant \mathcal{S}$. Q.E.D.

By putting together Theorems 3.2 and 3.3 we see that the theories intermediate between $\mathcal{S}$ and $\mathcal{S} \cup \{\mathrm{Con}_{\mathcal{S}}\}$ give rise to r.e. degrees of relative interpretability which form a dense partial order and contains a sub-ordering isomorphic to the positive propositional logic. Furthermore, these degrees arising as values of the map M can be given a Boolean structure with meet, join, and complement, the latter operation not applicable to the degrees bounded by $\mathcal{S} \cup \{\mathrm{Con}_{\mathcal{S}}\}$.

By examining Lemma 3.1, one finds a mapping F from the power set of the integers $\mathcal{P}(N)$ into the degrees (not necessarily r.e.) between $\mathcal{S}$ and $\mathcal{S} \cup \{\mathrm{Con}_{\mathcal{S}}\}$ given by

$$F(T) = \mathcal{S} \cup \{\psi(\bar{\mathrm{n}}) |\ \mathrm{n} \in T\}$$

for $T \in \mathcal{P}(N)$. This mapping is 1-1, so that it embeds $\mathcal{P}(N)$. (This result was already obtained by Montague in [4], though he does not explicitly say so.)

Thus, the structure of the degrees of relative interpretability appears complex.

4. In this section, we study the effect that adding the consistency statement has on the finite polyadic Lindenbaum algebras of the theory $\mathcal{S}$. From a result of Pour-El and Kripke, since we assume $\mathcal{S} \supseteq \mathcal{P}$, the algebra of sentences will not be changed (see [2] for a statement and proof of their result). In what follows, $\mathcal{B}_n(\mathcal{S})$ will designate the polyadic algebra obtained by forming equivalence classes of formulas in the first $(n+1)$ free variables, under the equivalence of logical interdeducibility, defining the usual Boolean operations on these equivalence classes (so that $\mathcal{B}_n(\mathcal{S})$ becomes a Boolean algebra) and defining the action of the quantifiers $\mathbf{\exists}_{\mathrm{x}}, \mathbf{\exists}_{\mathrm{y}}, \ldots$ in the natural way, e.g.,

$$\mathbf{\exists}_{\mathrm{x}}([\lambda]) = [(\mathbf{\exists}\mathrm{x})\lambda]$$

(here $[\lambda]$ is the equivalence class of λ), thus giving a polyadic structure to $\mathcal{B}_n(\mathcal{S})$.

In his paper [3], Kleene showed that, for any r.e. theory $\mathcal{S}$ in arithmetic, there can be added to the relational language of arithmetic a finite number of new predicate symbols $Q_1, \ldots, Q_t$ (t depending on $\mathcal{S}$) for which there exists a single axiom involving $Q_1, \ldots, Q_t, S, \mathrm{Add}, \mathrm{Mult}, \mathrm{Z}$ (Z the condition uniquely satisfied by 0), that has precisely the relational theory corresponding to $\mathcal{S}$ as reduct; i.e., the formulas that are provable in this finitely-axiomatized theory and which contain S, Add, Mult, Z, as the only predicate letters are precisely the relational theory corresponding to $\mathcal{S}$.

Let $\Delta = \Delta(Q_1, \ldots, Q_t, S, \mathrm{Add}, \mathrm{Mult}, \mathrm{Z})$ be this single axiom; by the abbreviation $\Delta_\delta(\psi_1, \ldots, \psi_t, \sigma, \alpha, \mu)$ we denote the formula obtained from Δ when proper substitutions of the formulas $\psi_1, \ldots, \psi_t, \sigma, \alpha, \mu, \tau$ for the letters $Q_1, \ldots, Q_t, S, \mathrm{Add}, \mathrm{Mult}, \mathrm{Z}$ are performed and then all quantifiers are relativized to the formula $\delta(\mathrm{x})$; this is essentially a relative interpretation I of Δ where I assigns the cited formulas to the cited predicate letters and relativizes by the formula δ. (We have chosen to suppress the formula τ which corresponds to Z.)

Let the formula Δ have number a_0 and $T(\mathrm{x}, \mathrm{y})$ a standard PR formula which holds if y is the number of a formula in the relational language and x is the number of its arithmetic correspondent. Finally, let $Pr_\alpha(\mathrm{y})$ and Con_α be as in the fourth section of Feferman.

Theorem 4.1. *If $\mathcal{S}$ is a consistent, reflexive, r.e. theory and $\mathcal{S} \supseteq \mathcal{P}$, then there exists a number N and an RE formula $\mathrm{s}(\mathrm{x})$ which designates some axiomatization of $\mathcal{S}$ for which there is no Boolean homomorphism preserving (i.e., commuting with) all quantifiers $\mathbf{\exists}_0, \ldots, \mathbf{\exists}_N$ which maps $\mathcal{B}_N(\mathcal{S} \cup \{\mathrm{Con}_{\mathcal{S}}\})$ to $\mathcal{B}_N(\mathcal{S})$.*

Proof. We construct Δ as above; then the formula

$$(\mathbf{\exists}\dot{\mathrm{y}})\big(Pr_{\mathrm{z}=\bar{\mathrm{a}}_0}(\mathrm{y}) \wedge T(\mathrm{x}, \mathrm{y})\big)$$

has a logical equivalent $\mathrm{s}(\mathrm{x})$ in $\mathcal{P}$ which is an RE formula, and by Kleene's results it designates precisely all the theorems of $\mathcal{S}$.

With judicious construction of $T(\mathrm{x}, \mathrm{y})$ we have

$$\vdash_{\mathcal{S}} \mathrm{Con}_{\mathrm{s}} \rightarrow \mathrm{Con}_{\mathrm{z}=\bar{\mathrm{a}}_0}$$

and, from Feferman's Theorem 6.2 in [1],

$$\vdash_{\mathcal{S} \cup \{\mathrm{Con}_{\mathrm{z}=\bar{\mathrm{a}}_0}\}} I\big(\Delta(Q_1, \ldots, Q_t, S, \mathrm{Add}, \mathrm{Mult})\big)$$

for some relativizing map I. Combining these two results we see that, for some formulas $\delta, \psi_1, \ldots, \psi_2, \sigma, \alpha, \mu$, we have

$$\vdash_{\mathcal{S} \cup \{\mathrm{Con}_{\mathrm{s}}\}} \Delta_\delta(\psi_1, \ldots, \psi_t, \sigma, \alpha, \mu)\,.$$

Let $\Delta_\delta(\psi_1, \ldots, \psi_t, \sigma, \alpha, \mu)$ be put in a "special" Prenex Normal Form, in which only the quantifiers outside of $\psi_1, \ldots, \psi_t, \sigma, \alpha, \mu$ are "moved outward", these formulas are treated as irreducible predicate letters, and the resulting matrix is a Boolean combination of these formulas. Let the largest index of any variable "moved outward" to the prefix be N, so that x_N is in the prefix, but x_n is not if $n > N$. Suppose, for the sake of contradiction, that there exists a Boolean homomorphism

$$F\colon\ \mathcal{B}_N(\mathcal{S} \cup \{\mathrm{Con}_{\mathrm{s}}\}) \rightarrow \mathcal{B}_N(\mathcal{S})$$

which preserves all quantifiers $\mathbf{\exists}_0, \ldots, \mathbf{\exists}_N$.

CLAIM. $F([\Delta_\delta(\psi_1, \ldots, \psi_t, \sigma, \alpha, \mu)]) = [\Delta_{\delta'}(\psi_1', \ldots, \psi_t', \sigma', \alpha', \mu')]$.

Proof of Claim. Let $\Delta_\delta(\psi_1, \ldots, \psi_t, \sigma, \alpha, \mu)$ be put in the special Prenex Normal Form Ω; we see that $[\Omega] = [\Delta_\delta(\psi_1, \ldots, \psi_t, \sigma, \alpha, \mu)]$. It is easily seen that, for any formula φ in the first m free variables where $m \leqslant N$, that we have

$$F([(\exists x_1)\varphi]) = [(\exists x_1)\varphi']$$

and

$$F([(\forall x_1)\varphi]) = [(\forall x_1)\varphi'] \quad (1 \leqslant m)$$

for any formula φ' such that $F([\varphi]) = [\varphi']$; the first equation is immediate, since F commutes with $\exists_1$, and the second is almost as easy if one treats $(\forall x_1)\varphi$ as $\neg(\exists x_1)\neg\varphi$ and uses the fact that F is a Boolean homomorphism.

Thus, if, e.g., Ω is $(\forall x_{i_1})(\exists x_{i_2})(\forall x_{i_3}) \ldots \Gamma$ for some matrix Γ which is a Boolean combination of the formulas $\delta, \psi_1, \ldots, \psi_t, \sigma, \alpha, \mu$, we have

$$F([\Omega]) = [(\forall x_{i_1})(\exists x_{i_2})(\forall x_{i_3}) \ldots \Gamma']$$

where Γ' is any formula such that $F([\Gamma]) = [\Gamma']$. If we set $F([\delta]) = [\delta']$, $F([\psi_i]) = [\psi_i']$ for $i = 1, \ldots, t$, $F([\sigma]) = [\sigma']$, $F([\alpha]) = [\alpha']$, and $F([\mu]) = [\mu']$, then the Boolean nature of F shows that Γ' can be taken to be the same Boolean combination of $\delta', \psi_1', \ldots, \psi_t', \sigma', \alpha', \mu'$ as Γ is of $\delta, \psi_1, \ldots, \psi_t, \sigma, \alpha, \mu$. Thus, "undoing" the special Prenex Normal Form $(\forall x_{i_1})(\exists x_{i_2})(\forall x_{i_3}) \ldots \Gamma'$ we shall arrive at a formula $\Delta_{\delta'}(\psi_1', \ldots, \psi_t', \delta', \alpha', \mu')$ which is logically equivalent to it, and this completes the proof of the claim.

From the claim, and the fact that $F(\mathbf{1}) = \mathbf{1}$ and that

$$[\Delta_\delta(\psi_1, \ldots, \psi_t, \sigma, \alpha, \mu)] = \mathbf{1}$$

in the algebra $\mathcal{B}_N(\mathcal{S} \cup \{\mathrm{Con}_\mathcal{S}\})$, we must have

$$[\Delta_{\delta'}(\psi_1', \ldots, \psi_t', \sigma', \alpha', \mu')] = \mathbf{1}$$

in the algebra $\mathcal{B}_N(\mathcal{S})$, which means

$$\vdash_\mathcal{S} \Delta_{\delta'}(\psi_1', \ldots, \psi_t', \sigma', \alpha', \mu').$$

Thus, for some relative interpretation I',

$$\vdash_\mathcal{S} I'(\Delta(\psi_1, \ldots, \psi_t, S, \mathrm{Add}, \mathrm{Mult}));$$

this shows that, for some finite subtheory $\mathcal{F} \subseteq \mathcal{S}$ axiomatized by an axiom φ,

$$\vdash_\mathcal{S} I'(\Delta(\psi_1, \ldots, \psi_t, S, \mathrm{Add}, \mathrm{Mult})).$$

By Feferman's Theorem 6.4 in [1],

$$\vdash_\mathcal{S} \mathrm{Con}_{x=\bar{\varphi}} \to \mathrm{Con}_{x=\bar{a}_0};$$

since $\mathcal{S} \supseteq \mathcal{T}$, $\mathcal{S}$ is reflexive, so we have

$$\vdash_\mathcal{S} \mathrm{Con}_{x=\bar{\varphi}}$$

and hence

$$\vdash_\mathcal{S} \mathrm{Con}_{x=\bar{a}_0}.$$

Since it is clear that

$$\vdash_\mathcal{S} \mathrm{Con}_{x=\bar{a}_0} \to \mathrm{Con}_\mathcal{S}$$

we evidently have

$$\vdash_\mathcal{S} \mathrm{Con}_\mathcal{S}$$

which is an impossibility by Feferman's Theorem 5.6 in [1], since $s(x)$ is RE and numerates all theorems of $\mathcal{S}$ in $\mathcal{T}$. Q.E.D.

Since the whole problem of polyadic algebras of theories is wide open, it is almost humorous to list unsolved problems, but the ones listed below seem to be particularly worth investigating.

We ask:

(1) *If $\mathcal{S}, \mathcal{J} \supseteq \mathcal{T}$ and $\mathcal{J}$ has the uniform reflection principle over $\mathcal{S}$, is there a Boolean homomorphism from $\mathcal{B}_1(\mathcal{J})$ to $\mathcal{B}_1(\mathcal{S})$ preserving $\exists_0$ and $\exists_1$?* A. Nerode conjectures "no".

(2) *If $\mathcal{S}, \mathcal{J} \supseteq \mathcal{T}$ is there a Boolean isomorphism from $\mathcal{B}_0(\mathcal{S})$ to $\mathcal{B}_0(\mathcal{J})$ preserving $\exists_0$?* (This is essentially an open problem stated by Pour-El.) We conjecture "yes".

(3) *If $\mathcal{S} \supseteq \mathcal{T}$, is $\mathcal{B}_0(\mathcal{S})$ a universal monadic algebra?* We conjecture "yes".

(4) *Are there means of computing the algebras $\mathcal{B}_n(\mathcal{S})$, $n < \infty$, which rely only on a presentation of a given axiomatization of $\mathcal{S}$?*

References

[1] S. Feferman, *Arithmetization of metamathematics in a general setting*, Fund. Math. 49 (1960), pp. 35–92.

[2] R. Jeroslow, *Uses of self-reference in arithmetic*, doctoral dissertation at Cornell University, 1969, unpublished.

[3] S. C. Kleene, *Finite axiomatizability of theories in the predicate calculus using additional predicate symbols*, in *Two papers on the predicate calculus*, Memoirs of the AMS, no. 10, Providence, R. I., 1952, 68 pp.

[4] R. Montague, *Theories incomparable with respect to relative interpretability*, J. Symb. Logic 27 (1962), pp. 195–211.
[5] M. B. Pour-El, *Effectively extensible theories*, J. Symb. Logic 33 (1968), pp. 56–68,
[6] B. Rosser, *Extensions of some theorems of Gödel and Church*, J. Symb. Logic 1 (1936), pp. 87–91.

THE UNIVERSITY OF MINNESOTA

Reçu par la Rédaction le 20. 1. 1970

Shapes of compacta and ANR-systems

by

Sibe Mardešić and **Jack Segal** * (Zagreb and Seattle)

1. Introduction. It is well-known that local difficulties prevent a successful application of homotopy notions to arbitrary compacta. In an attempt to remedy this K. Borsuk introduced a theory of shapes of metric compacta [3, 4]. In this paper we give an alternate description of shapes and at the same time we generalize the theory to the non-metric case. Our approach is based on inverse systems of ANR's, one advantage of which is that it is more categorical. The actual proof that the two approaches are equivalent on metric compacta is given in a sequel to this paper [11]. As an application of our method we classify all P-adic solenoids and all (n-sphere)-like continua as to their shape. It is also shown that the shape classification of 0-dimensional compacta agrees with their topological classification. The theory is presented in detail only in the absolute case, while for the relative case, i.e. the case of pairs of spaces, we content ourselves with indicating the appropriate changes.

2. Category of ANR-systems. A directed set $(A, \leqslant)$ is said to be closure-finite provided for every $\alpha \in A$ the set of all predecessors of α is finite. Note that the natural numbers N with the usual ordering form a closure-finite directed set. Another example is the set $F(\Omega)$ of all (non-empty) finite subsets of a given set Ω ordered by inclusion ($\alpha \leqslant \alpha'$ if and only if $\alpha \subset \alpha'$). For $\alpha \in (A, \leqslant)$ we define the rank $r(\alpha)$ as the maximal cardinal of a chain (linearly ordered set) in A having α for its terminal point. If $(A, \leqslant)$ is closure-finite, each α has a finite number of predecessors and therefore $r(\alpha)$ is a well-defined natural number. Note that in the case of $F(\Omega)$ the rank $r(\alpha)$ is just the cardinal of α and in the case of the integers N the rank $r(n) = n$.

By an ANR in this paper we mean a compact absolute neighborhood retract for metric spaces (see [2], p. 100). We shall now introduce

* During this research J. Segal was visiting the University of Zagreb on a Fulbright grant.