# Equivalence of the Borsuk and the ANR-system approach to shapes

by

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In a previous paper [5] the authors have presented an alternate approach, based on ANR-systems, to Borsuk's theory of shapes of metric compacta. The purpose of this paper is to prove that this approach is actually equivalent to Borsuk's. For simplicity the proofs are presented only in the absolute case.

The basic notion in Borsuk's theory [2], [3] is that of a fundamental sequence of mappings. For two compacta $X$, $Y$ embedded in the Hilbert cube $I^\infty$ a fundamental sequence $\underline{\varphi}\colon X \to Y$ consists of a sequence of maps $\varphi_n\colon I^\infty \to I^\infty$ such that for every neighborhood $V$ of $Y$ there is an integer $n_0 \in N$ and a neighborhood $U$ of $X$ such that $\varphi_n|U \simeq \varphi_{n'}|U$ in $V$ for all $n$, $n' \geqslant n_0$. Two fundamental sequences $\underline{\varphi}, \underline{\psi}\colon X \to Y$ are considered to be homotopic, $\underline{\varphi} \simeq \underline{\psi}$, provided for every neighborhood $V$ of $Y$ there is an $n_0 \in N$ and a neighborhood $U$ of $X$ such that $\varphi_n|U \simeq \psi_n|U$ in $V$ for all $n \geqslant n_0$. The composite $\underline{\psi}\underline{\varphi}\colon X \to Z$ of $\underline{\varphi}\colon X \to Y$ and $\underline{\psi}\colon Y \to Z$ is the fundamental sequence $\underline{\chi}\colon X \to Z$, where $\chi_n = \psi_n \varphi_n$, $n \in N$. The identity sequence $\underline{1}_X\colon X \to X$ consists of the sequence of maps $1_{I^\infty}\colon I^\infty \to I^\infty$. $X$ is said to be of the same shape as $Y$ (in the sense of Borsuk), written as $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$, provided there exist fundamental sequences $\underline{\varphi}\colon X \to Y$ and $\underline{\psi}\colon Y \to X$ such that $\underline{\psi}\underline{\varphi} \simeq \underline{1}_X$ and $\underline{\varphi}\underline{\psi} \simeq \underline{1}_Y$.

In our approach [5] the basic notion (in the case of metric compacta) is that of a map of ANR-sequences. An ANR-sequence is an inverse sequence $\underline{X} = \{X_n, p_{n,n+1}\}$, where each $X_n$ is an ANR, i.e. a compact ANR for metric spaces. Along with the bonding maps $p_{nn'} = p_{n,n+1} \ldots \ldots p_{n'-1,n'}\colon X_{n'} \to X_n$, $n \leqslant n'$, one also considers projections $p_n\colon \mathrm{Inv\,lim}\,\underline{X} \to X_n$. A map of sequences $\underline{f}\colon \underline{X} \to \underline{Y} = \{Y_n, q_{n,n+1}\}$, where $\underline{X}$ and $\underline{Y}$ are ANR-sequences, consists of an increasing function $f\colon N \to N$ and of a collection of maps $f_n\colon X_{f(n)} \to Y_n$ such that $f_n p_{f(n)f(n')} \simeq q_{nn'} f_{n'}$ for $n \leqslant n'$. Two maps of sequences $\underline{f}, \underline{g}\colon \underline{X} \to \underline{Y}$ are considered to be homotopic, $\underline{f} \simeq \underline{g}$,

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provided for each $n \in N$ there is an $n' \in N$, $n' \geqslant f(n), g(n)$, such that $f_n p_{f(n)n'} \simeq g_n p_{g(n)n'}$. The composite $\underline{g}\underline{f}$: $\underline{X} \to \underline{Z}$ of $\underline{f}$: $\underline{X} \to \underline{Y}$ and $\underline{g}$: $\underline{Y} \to \underline{Z}$ is a map of sequences $\underline{h}$: $\underline{X} \to \underline{Z}$ where $h = fg$: $N \to N$ and $h_n = g_n f_{g(n)}$: $X_{fg(n)} \to Z_n$. The identity map of sequences $\underline{1}_{\underline{X}}$: $\underline{X} \to \underline{X}$ is given by the identity $1_N$: $N \to N$ and the maps $1_{X_n}$: $X_n \to X_n$.

An ANR-sequence $\underline{X}$ is said to be associated with $X$ provided $X = \operatorname{Inv}\lim \underline{X}$. Two metric compacta $X$ and $Y$ are said to be of the same shape (in the sense of ANR-systems), written as $[X] = [Y]$, provided there exist associated ANR-sequences $\underline{X}$ and $\underline{Y}$ and maps of sequences $\underline{f}$: $\underline{X} \to \underline{Y}$ and $\underline{g}$: $\underline{Y} \to \underline{X}$ such that $\underline{g}\underline{f} \simeq \underline{1}_{\underline{X}}$ and $\underline{f}\underline{g} \simeq \underline{1}_{\underline{Y}}$. It follows from ([5], Corollary 1) that if such maps exist for one pair of sequences $\underline{X}$ and $\underline{Y}$ associated with $X$ and $Y$ respectively, then such maps exist for any other pair $\underline{X}'$, $\underline{Y}'$ of sequences associated with $X$ and $Y$.

We can now state and prove the main result of this paper.

THEOREM. *Two metric compacta $X$ and $Y$ are of the same shape in the sense of Borsuk if and only if they are of the same shape in the sense of ANR-systems.*

For every compactum $X \subset I^\infty$ we can choose a decreasing sequence of ANR's $X_1 \supset X_2 \supset \ldots \supset X$ such that $\bigcap X_n = X$ and that each $X_n$ is a neighborhood of $X$. Clearly, if $i_{nn'}$: $X_{n'} \to X_n$, $n \leqslant n'$, denotes the inclusion, then $\underline{X} = \{X_n, i_{nn'}\}$ is an ANR-sequence associated with $X$. We call $\underline{X}$ the inclusion ANR-sequence for $X$. For given $X, Y \subset I^\infty$ we choose two fixed inclusion ANR-sequences $\underline{X} = \{X_n, i_{nn'}\}$ and $\underline{Y} = \{Y_n, j_{nn'}\}$ and use these throughout the remainder of the paper.

LEMMA 1. *Let $\varphi_n$: $I^\infty \to I^\infty$ be a sequence of maps, let $f$: $N \to N$ be an increasing function and let $f_n$: $X_{f(n)} \to Y_n$ be maps such that for each $n \in N$*

(i) $m, m' \geqslant f(n) \Rightarrow \varphi_m | X_{f(n)} \simeq \varphi_{m'} | X_{f(n)}$ *in* $Y_n$,

(ii) $f_n = \varphi_{f(n)} | X_{f(n)}$.

*Then the maps $\varphi_n$ form a fundamental sequence $\underline{\varphi}$: $X \to Y$ and the function $f$ and the maps $f_n$ form a map of sequences $\underline{f}$: $\underline{X} \to \underline{Y}$. If $\underline{\varphi}$ and $\underline{f}$ satisfy* (i) *and* (ii), *they are said to be related.*

Proof. For every neighborhood $V$ of $Y$ there is an $n_0 \in N$ such that $Y_{n_0} \subset V$, because $\bigcap Y_n = Y \subset V$ and the $Y_n$ form a decreasing sequence. Then $U = X_{f(n_0)}$ is a neighborhood of $X$, and by (i), for $m, m' \geqslant f(n_0)$,

$$\varphi_m | U \simeq \varphi_{m'} | U \quad \text{in } Y_{n_0} \subset V,$$

which proves that $\underline{\varphi}$ is a fundamental sequence.

On the other hand, for $n \leqslant n'$, $f(n) \leqslant f(n')$, so that (i) and (ii) imply

$$(1) \qquad f_n = \varphi_{f(n)} | X_{f(n)} \simeq \varphi_{f(n')} | X_{f(n)} \quad \text{in } Y_n .$$



Since $X_{f(n')} \subset X_{f(n)}$, we can restrict (1) to $X_{f(n')}$ and obtain

$$f_n | X_{f(n')} \simeq \varphi_{f(n')} | X_{f(n')} \quad \text{in } Y_n .$$

However, by (ii), $f_{n'} = \varphi_{f(n')} | X_{f(n')}$, so that $f_n | X_{f(n')} \simeq f_{n'}$ in $Y_n$. In other words,

$$f_n i_{f(n)f(n')} \simeq j_{nn'} f_{n'} \quad \text{in } Y_n ,$$

which proves that $\underline{f}$: $\underline{X} \to \underline{Y}$ is a map of sequences.

LEMMA 2. *Every fundamental sequence $\underline{\varphi}$: $X \to Y$ admits a related map of sequences $\underline{f}$: $\underline{X} \to \underline{Y}$.*

Proof. For every $n \in N$, $Y_n$ is a neighborhood of $Y$. Therefore, there is an index $g_1(n) \in N$ and a neighborhood $U_n$ of $X$ such that

$$(2) \qquad m, m' \geqslant g_1(n) \Rightarrow \varphi_m | U_n \simeq \varphi_{m'} | U_n \quad \text{in } Y_n .$$

Since $\bigcap X_k = X$ and the $X_k$ form a decreasing sequence, there is an index $g_2(n) \in N$ such that

$$(3) \qquad k \geqslant g_2(n) \Rightarrow X_k \subset U_n .$$

Choose an increasing function $f$: $N \to N$ such that for each $n \in N$

$$(4) \qquad f(n) \geqslant g_1(n), g_2(n) .$$

Then, by (2),

$$(5) \qquad m, m' \geqslant f(n) \Rightarrow \varphi_m | U_n \simeq \varphi_{m'} | U_n \quad \text{in } Y_n .$$

Since, by (3), $X_{f(n)} \subset U_n$, we can restrict (5) to $X_{f(n)}$ and obtain

$$(6) \qquad m, m' \geqslant f(n) \Rightarrow \varphi_m | X_{f(n)} \simeq \varphi_{m'} | X_{f(n)} \quad \text{in } Y_n ,$$

which is (i). It follows from (6) that $\varphi_{f(n)}(X_{f(n)}) \subset Y_n$ and we can define $f_n$: $X_{f(n)} \to Y_n$ as the map $f_n = \varphi_{f(n)} | X_{f(n)}$, which is (ii). By Lemma 1, $\underline{f}$ is indeed a map of sequences related to $\underline{\varphi}$.

A map of sequences $\underline{f}$: $\underline{X} \to \underline{Y}$ will be called regular, provided $f$: $N \to N$ is strictly increasing, i.e. if $n < n'$ implies $f(n) < f(n')$. Note that the composite $\underline{h} = \underline{g}\underline{f}$ of two regular maps of sequences $\underline{f}$ and $\underline{g}$ is regular.

LEMMA 3. *For every map $\underline{f}$: $\underline{X} \to \underline{Y}$ there is a regular map $\underline{g}$: $\underline{X} \to \underline{Y}$ such that $\underline{f} \simeq \underline{g}$.*

Proof. We first choose a strictly increasing function $g$: $N \to N$ such that $f(n) \leqslant g(n)$ for each $n \in N$, for instance we take $g(n) = f(n) + n$.

We then define $g_n\colon X_{g(n)}\to Y_n$ to be $g_n = f_n p_{f(n)g(n)}$. The function $g$ and the maps $g_n$ form a map of sequences $\underline{g}\colon \underline{X}\to\underline{Y}$ because for $n\leqslant n'$ we have

$$g_n p_{g(n)g(n')} = f_n p_{f(n)g(n)} p_{g(n)g(n')} = f_n p_{f(n)f(n')} p_{f(n')g(n')} \simeq q_{nn'} f_{n'} p_{f(n')g(n')} = q_{nn'} g_{n'}\,.$$

It immediately follows from the definition of $\underline{g}$ that $\underline{f}\simeq\underline{g}$.

Lemma 4. *Every regular map of sequences $\underline{f}\colon \underline{X}\to\underline{Y}$ admits a related fundamental sequence $\underline{\varphi}\colon X\to Y$.*

Proof. Let $\underline{f}\colon\underline{X}\to\underline{Y}$ be a regular map of sequences. We shall define maps $\varphi_n\colon I^\infty\to I^\infty$ by induction on $n$. If $f(1)=1$, we take for $\varphi_1$ any extension of $f_1$ to $I^\infty$ with values in $I^\infty$; if $f(1)>1$ we take for $\varphi_1$ any map of $I^\infty$ into $I^\infty$. Assume that we have already defined maps $\varphi_n\colon I^\infty\to I^\infty$ for all $n\leqslant k$ in such a way that the following holds:

$(\mathrm{i})_k$ $k\geqslant m,\ m'\geqslant f(n)\Rightarrow \varphi_m|X_{f(n)}\simeq\varphi_{m'}|X_{f(n)}$ in $Y_n$,

$(\mathrm{ii})_k$ $f(n)\leqslant k\Rightarrow f_n=\varphi_{f(n)}|X_{f(n)}$.

Note that $(\mathrm{i})_1$ and $(\mathrm{ii})_1$ hold because $f(n)\leqslant 1$ implies $n=1$ and $f(1)=1$. We shall now define $\varphi_{k+1}\colon I^\infty\to I^\infty$, $k\geqslant 1$, in such a way that $(\mathrm{i})_{k+1}$ and $(\mathrm{ii})_{k+1}$ hold.

If $f(1)>k+1$, we can take for $\varphi_{k+1}$ any map from $I^\infty$ to $I^\infty$ for the conditions $(\mathrm{i})_{k+1}$ and $(\mathrm{ii})_{k+1}$ are vacuously fulfilled.

If $f(1)=k+1$, we define $\varphi_{k+1}$ as any extension of $f_1\colon X_{k+1}\to Y_1\subset I^\infty$ to $I^\infty$ with values in $I^\infty$. In this case $(\mathrm{ii})_{k+1}$ holds because $f(n)\leqslant k+1 = f(1)\leqslant f(n)$ implies $f(n)=f(1)$ and $n=1$ so that the assertion of $(\mathrm{ii})_{k+1}$ reduces to $\varphi_{k+1}|X_{k+1}=f_1$. Condition $(\mathrm{i})_{k+1}$ holds in this case because $m=m'=k+1$.

Finally, we consider the case $f(1)<k+1$ and we denote by $l$ the largest integer for which $f(l)\leqslant k$. We shall define, by induction on $j=0,1,\ldots,l$, maps $\varphi^j_{k+1}\colon X_{f(l-j)}\to Y_{l-j}$, each extending the preceding one (here by convention we consider that $f(0)=0$ and $X_0=Y_0=I^\infty$). The map $\varphi_{k+1}$ will be obtained as $\varphi^l_{k+1}\colon I^\infty\to I^\infty$. We start the induction by putting $\varphi^0_{k+1}=\varphi_k|X_{f(l)}$ if $f(l+1)>k+1$ and in the case $f(l+1)=k+1$ we choose for $\varphi^0_{k+1}$ an extension of $f_{l+1}\colon X_{f(l+1)}\to Y_{l+1}\subset Y_l$ to $X_{f(l)}$ with values in $Y_l$ such that

$$\varphi^0_{k+1}\simeq f_l \quad\text{in } Y_l\,. \tag{7}$$

Such an extension exists because by definition $f_{l+1}\simeq f_l|X_{f(l+1)}$ in $Y_l$ and $f_l$ extends $f_l|X_{f(l+1)}$ to $X_{f(l)}$, so we can apply Borsuk's extension theorem (see [1], (8.1), p. 94 or [4], Theorem 2.2, p. 117). Note that in this case

$$\varphi^0_{k+1}|X_{f(l+1)}=f_{l+1}\,. \tag{8}$$



Now assume that we have already defined $\varphi^0_{k+1},\ldots,\varphi^j_{k+1}$, $0\leqslant j<l$, in such a way that

$(\mathrm{iii})_j$ $\varphi^{i+1}_{k+1}|X_{f(l-i)}=\varphi^i_{k+1}$, $0\leqslant i<j$,

$(\mathrm{iv})_j$ $\varphi^i_{k+1}\simeq\varphi_k|X_{f(l-i)}$ in $Y_{l-i}$, $0\leqslant i\leqslant j<l$.

Note that $(\mathrm{iii})_0$ is vacuously fulfilled. If $f(l+1)>k+1$, $(\mathrm{iv})_0$ obviously holds. If $f(l+1)=k+1$, then by (7), $\varphi^0_{k+1}\simeq f_l$ in $Y_l$. Moreover, since $f(l)\leqslant k$, we conclude from $(\mathrm{ii})_k$ and $(\mathrm{i})_k$ that

$$f_l=\varphi_{f(l)}|X_{f(l)}\simeq\varphi_k|X_{f(l)} \quad\text{in } Y_l\,,$$

so that $\varphi^0_{k+1}\simeq\varphi_k|X_{f(l)}$ in $Y_l$, which is $(\mathrm{iv})_0$.

In order to define $\varphi^{j+1}_{k+1}$, observe that by $(\mathrm{iv})_j$ $\varphi^j_{k+1}\simeq\varphi_k|X_{f(l-j)}$ in $Y_{l-j}$ and that $\varphi_k|X_{f(l-j-1)}$ is an extension of $\varphi_k|X_{f(l-j)}$ to $X_{f(l-j-1)}$. Since $k\geqslant f(l)\geqslant f(l-j-1)$, it follows from $(\mathrm{i})_k$ that $\varphi_k|X_{f(l-j-1)}$ takes values in $Y_{l-j-1}$. Applying Borsuk's extension theorem, we can extend $\varphi^j_{k+1}$ to a map $\varphi^{j+1}_{k+1}\colon X_{f(l-j-1)}\to Y_{l-j-1}$ in such a way that $\varphi^{j+1}_{k+1}\simeq\varphi_k|X_{f(l-j-1)}$ in $Y_{l-j-1}$. This completes the induction step.

We now put $\varphi_{k+1}=\varphi^l_{k+1}$. In order to verify $(\mathrm{i})_{k+1}$ and $(\mathrm{ii})_{k+1}$ assume that $f(n)\leqslant m,\ m'\leqslant k+1$. If $f(n)=k+1$, then $m=m'=k+1$ and $(\mathrm{i})_{k+1}$ obviously holds. By definition of $l$, $f(l+1)\geqslant k+1=f(n)$ so that $l+1\geqslant n$. On the other hand, $f(l)\leqslant k<k+1=f(n)$ implies $l<n$, so that $l+1=n$. Since $\varphi_{f(l+1)}=\varphi^l_{k+1}$ is an extension of $\varphi^0_{k+1}$, it follows from (8) that

$$\varphi_{f(l+1)}|X_{f(l+1)}=\varphi^0_{k+1}|X_{f(l+1)}=f_{l+1}\,,$$

which shows that in this case $(\mathrm{ii})_{k+1}$ holds.

We now assume that $f(n)\leqslant k$ and therefore $n\leqslant l$. In this case $(\mathrm{ii})_{k+1}$ is true by $(\mathrm{ii})_k$. Since $\varphi_{k+1}=\varphi^l_{k+1}$ extends $\varphi^{l-n}_{k+1}$, we have

$$\varphi_{k+1}|X_{f(n)}=\varphi^{l-n}_{k+1}\,. \tag{9}$$

By $(\mathrm{iv})_{l-n}$ we also have

$$\varphi^{l-n}_{k+1}\simeq\varphi_k|X_{f(n)} \quad\text{in } Y_n\,, \tag{10}$$

so that

$$\varphi_{k+1}|X_{f(n)}\simeq\varphi_k|X_{f(n)} \quad\text{in } Y_n\,. \tag{11}$$

This shows that $(\mathrm{i})_{k+1}$ holds for $m=k+1$, $m'=k$. All other cases follow easily from $(\mathrm{i})_k$ and (11).

We have thus defined, by induction, a sequence of maps $\varphi_n\colon I^\infty\to I^\infty$ which satisfies $(\mathrm{i})_k$ and $(\mathrm{ii})_k$ for all $k\in N$, i.e. it satisfies (i) and (ii). By Lemma 1, the maps $\varphi_n$ form a fundamental sequence $\underline{\varphi}$ related to $\underline{f}$.

Remark 1. The assertion of Lemma 4 is false if one omits the assumption that $\underline{f}$ is regular.

LEMMA 5. *$\varphi$: $X \to Y$ be related to $\underline{f}$: $\underline{X} \to \underline{Y}$ and $\underline{\psi}$: $X \to Y$ to $\underline{g}$: $\underline{X} \to \underline{Y}$. Then $\underline{\varphi} \simeq \underline{\psi}$ is equivalent to $\underline{f} \simeq \underline{g}$.*

Proof. First assume that $\underline{\varphi} \simeq \underline{\psi}$. Then for each $n \in N$ there is a neighborhood $U_n$ of $X$ and an integer $h_1(n) \in N$ such that

$$m \geqslant h_1(n) \Rightarrow \varphi_m|U_n \simeq \psi_m|U_n \quad \text{in } Y_n. \tag{12}$$

Let $h_2(n) \in N$ be such that $X_{h_2(n)} \subset U_n$, and choose an integer $h(n) \geqslant h_1(n),\ h_2(n),\ f(n),\ g(n)$. Then $X_{h(n)} \subset U_n$ and

$$m \geqslant h(n) \Rightarrow \varphi_m|X_{h(n)} \simeq \psi_m|X_{h(n)} \quad \text{in } Y_n. \tag{13}$$

Moreover, by (i) and (ii),

$$m \geqslant h(n) \geqslant f(n) \Rightarrow \varphi_m|X_{f(n)} \simeq \varphi_{f(n)}|X_{f(n)} = f_n \quad \text{in } Y_n, \tag{14}$$

$$m \geqslant h(n) \geqslant g(n) \Rightarrow \varphi_m|X_{g(n)} \simeq \varphi_{g(n)}|X_{g(n)} = g_n \quad \text{in } Y_n. \tag{15}$$

Restricting (14) and (15) to $X_{h(n)} \subset X_{f(n)}, X_{g(n)}$, we obtain

$$m \geqslant h(n) \Rightarrow \varphi_m|X_{h(n)} \simeq f_n|X_{h(n)} \quad \text{in } Y_n, \tag{16}$$

$$m \geqslant h(n) \Rightarrow \psi_m|X_{h(n)} \simeq g_n|X_{h(n)} \quad \text{in } Y_n. \tag{17}$$

This together with (13) yields

$$f_n|X_{h(n)} \simeq g_n|X_{h(n)} \quad \text{in } Y_n, \tag{18}$$

which proves that $\underline{f} \simeq \underline{g}$.

Now assume that $\underline{f} \simeq \underline{g}$. Then for each $n \in N$ there is an $h(n) \geqslant f(n)$, $g(n)$, such that (18) holds. By (i) and (ii) we have again (16) and (17), which together with (18) yields (13). Moreover, for every neighborhood $V$ of $Y$ there is an $n \in N$ such that $Y_n \subset V$. Thus, by (13), for $m \geqslant h(n)$ and $U = X_{h(n)}$ we have

$$\varphi_m|U \simeq \psi_m|U \quad \text{in } V,$$

which proves that $\underline{\varphi} \simeq \underline{\psi}$.

LEMMA 6. *Let $\underline{\varphi}$: $X \to Y$ and $\underline{f}$: $\underline{X} \to \underline{Y}$, $\underline{\psi}$: $Y \to Z$ and $\underline{g}$: $\underline{Y} \to \underline{Z}$, $\underline{\chi}$: $X \to Z$ and $\underline{h}$: $\underline{X} \to \underline{Z}$ be related in pairs. Then $\underline{\chi} \simeq \underline{\psi}\underline{\varphi}$ is equivalent to $\underline{h} \simeq \underline{g}\underline{f}$. The fundamental sequence $\underline{1}_X$: $X \to X$ and the map of sequences $\underline{1}_X$: $\underline{X} \to \underline{X}$ are related.*

Proof. Since $\underline{\varphi}$ and $\underline{f}$ are related we have, by (i) and (ii),

$$m \geqslant fg(n) \Rightarrow \varphi_m|X_{fg(n)} \simeq \varphi_{fg(n)}|X_{fg(n)} = f_{g(n)} \quad \text{in } Y_{g(n)}. \tag{19}$$

Similarly, since $\underline{\psi}$ and $\underline{g}$ are related, we have

$$m \geqslant g(n) \Rightarrow \psi_m|Y_{g(n)} \simeq \psi_{g(n)}|Y_{g(n)} = g_n \quad \text{in } Z_n. \tag{20}$$



Thus, if $k$: $N \to N$ is an increasing function and $k(n) \geqslant fg(n)$, $g(n)$ for each $n \in N$, then

$$m \geqslant k(n) \Rightarrow \psi_m \varphi_m|X_{fg(n)} \simeq g_n f_{g(n)} \quad \text{in } Z_n. \tag{21}$$

Restricting (21) to $X_{k(n)} \subset X_{fg(n)}$ we obtain

$$m \geqslant k(n) \Rightarrow \psi_m \varphi_m|X_{k(n)} \simeq g_n f_{g(n)}|X_{k(n)} \quad \text{in } Z_n. \tag{22}$$

It follows from (22) that the function $k$ and the maps

$$k_n = \psi_{k(n)}\varphi_{k(n)}|X_{k(n)}$$

form a map of sequences $\underline{k}$: $\underline{X} \to \underline{Z}$ related to the fundamental sequence $\underline{\psi}\underline{\varphi}$. It also follows from (22) that

$$k_n \simeq g_n f_{g(n)}|X_{k(n)} \quad \text{in } Z_n,$$

which proves that $\underline{k} \simeq \underline{g}\underline{f}$.

Since $\underline{k}$ is related to $\underline{\psi}\underline{\varphi}$ and $\underline{h}$ is related to $\underline{\chi}$, we conclude from Lemma 5 that $\underline{\chi} \simeq \underline{\psi}\underline{\varphi}$ is equivalent to $\underline{h} \simeq \underline{k}$. However, $\underline{k} \simeq \underline{g}\underline{f}$, so that $\underline{h} \simeq \underline{k}$ is equivalent to $\underline{h} \simeq \underline{g}\underline{f}$. Consequently, $\underline{\chi} \simeq \underline{\psi}\underline{\varphi}$ is equivalent to $\underline{h} \simeq \underline{g}\underline{f}$.

Remark 2. Note that the composites $\underline{\psi}\underline{\varphi}$ and $\underline{g}\underline{f}$ need not be related when $\underline{\varphi}, \underline{f}$ and $\underline{\psi}, \underline{g}$ are related in pairs.

Proof of the Theorem. $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ means that there are fundamental sequences $\underline{\varphi}$: $X \to Y$ and $\underline{\psi}$: $Y \to X$ such that $\underline{\psi}\underline{\varphi} \simeq \underline{1}_X$ and $\underline{\varphi}\underline{\psi} \simeq \underline{1}_Y$. By Lemma 2 we can find maps of sequences $\underline{f}$: $\underline{X} \to \underline{Y}$ and $\underline{g}$: $\underline{Y} \to \underline{X}$ related to $\underline{\varphi}$ and $\underline{\psi}$ respectively. Since $\underline{1}_{\underline{X}}$: $\underline{X} \to \underline{X}$ is related to $\underline{1}_X$: $X \to X$, we conclude from Lemma 6 that $\underline{g}\underline{f} \simeq \underline{1}_{\underline{X}}$ and similarly, $\underline{f}\underline{g} \simeq \underline{1}_{\underline{Y}}$, which shows that $[\underline{X}] = [\underline{Y}]$.

Conversely, $[\underline{X}] = [\underline{Y}]$ implies the existence of maps of sequences $\underline{f}$: $\underline{X} \to \underline{Y}$ and $\underline{g}$: $\underline{Y} \to \underline{X}$ such that $\underline{g}\underline{f} \simeq \underline{1}_{\underline{X}}$ and $\underline{f}\underline{g} \simeq \underline{1}_{\underline{Y}}$. By Lemma 3, we can assume without loss of generality that $\underline{f}$ and $\underline{g}$ are regular maps of sequences. Then we can find, by Lemma 4, fundamental sequences $\underline{\varphi}$: $X \to Y$ and $\underline{\psi}$: $Y \to X$ related to $\underline{f}$ and $\underline{g}$ respectively. It follows from Lemma 6 that $\underline{\psi}\underline{\varphi} \simeq \underline{1}_X$ and $\underline{\varphi}\underline{\psi} \simeq \underline{1}_Y$, which shows that $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$, and the proof is completed.

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*Reçu par la Rédaction le 27. 1. 1970*

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# Concerning the unions of absolute neighborhood retracts having brick decompositions*

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**1. Introduction.** In the study of retracts, one is interested in determining those properties of polyhedra that are also possessed by compact metric absolute neighborhood retracts. A basic property of polyhedra is that they can be decomposed into simplexes in such a way that if any number of them meet, their intersection is a face of each of them, and hence is a simplex. This property of polyhedra leads to the notion of a brick decomposition of a space.

If $X$ is a topological space, then a *brick decomposition* of $X$ is a finite collection $\{X_1, X_2, \ldots, X_n\}$ of compact metric absolute retracts in $X$ such that (1) $X = X_1 \cup X_2 \cup \ldots \cup X_n$ and (2) if any number of the sets $X_1, X_2, \ldots,$ and $X_n$ intersect, their intersection is an absolute retract.

Clearly, every polyhedron admits a brick decomposition. Further, any metric continuum admitting a brick decomposition is an absolute neighborhood retract [4, page 178]. However, not every compact metric absolute neighborhood retract has a brick decomposition [4, page 178]. The existence of compact metric absolute neighborhood retracts with no brick decomposition is related to the existence of such retracts with the singularity of Mazurkiewicz [4, page 152; 3].

In [4, page 179], Borsuk mentions the following open question: If $X$ and $Y$ are spaces such that $X$, $Y$, and $X \cap Y$ have brick decompositions, then does $X \cup Y$ have a brick decomposition? The purpose of this paper is to give a negative answer to this question.

The example that we describe here is obtained by an easy modification of the construction of [3]. A similar construction could be made using toroidal upper semicontinuous decompositions and the techniques of [2].

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* Research supported in part by the National Science Foundation under Grant No. GP-9641, and in part by the Institute for Advanced Study.