As a corollary of the proof it is not difficult to show that if $M \equiv_A N$ then for any cardinal a, $X_a M \equiv_A X_a N$. Further, it follows that if M and N are $L_{\omega_1,\omega}$-equivalent then $\bigoplus_a M \equiv \bigoplus_a N$.

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A minimal model for strong analysis

by

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In [6] it is shown that axiomatic second order arithmetic does not possess a minimal ω-model. Here we extend that result to general models of the full second order theory of $\langle \omega, +, \cdot \rangle$ and show that various model theoretic concepts, e.g., the existence of prime models, minimal ω-models, etc., all coincide, but are independent of Zermelo Fraenkel set theory and some of its extensions. These results are then applied to the weak second order theory of real numbers.

Let $\mathfrak{F} = \langle F, \omega, +, \cdot \rangle$ where F is the set of all functions mapping ω into ω. Consider a two sorted language $\mathfrak{L}$ for $\mathfrak{F}$ which contains individual variables $v_0, v_1, \ldots$ and function variables $\alpha_0, \alpha_1, \ldots$ Under our intended interpretation the individual variables range over ω and the function variables range over F. This distinction between variables has been introduced for convenience. We can easily find an equivalent (though less suggestive) one sorted language for $\mathfrak{F}$. Thus we assume that all of the standard first order concepts suitably generalize to $\mathfrak{L}$. In particular we shall be interested in the notions of proof ($\vdash$), satisfaction ($\models$), subsystem ($\subseteq$), and elementary subsystem ($\prec$). Let $T = \mathrm{Th}(\mathfrak{F})$ be the $\mathfrak{L}$-theory of $\mathfrak{F}$. A model $\mathfrak{P}$ of T is said to be *prime in the sense of Vaught* (cf. [16]) if $\mathfrak{P}$ is isomorphic to an elementary subsystem of every model of T. Let A be the set of functions $f \in F$ which are definable in $\mathfrak{F}$ by some formula $\varphi(\alpha_0)$ of $\mathfrak{L}$ and let $\mathfrak{A} = \langle A, \omega, +, \cdot \rangle$. We characterize the prime models of T in

THEOREM 1. *$\mathfrak{P}$ is a prime model of T in the sense of Vaught if and only if $\mathfrak{P}$ is isomorphic to $\mathfrak{A}$ and $\mathfrak{A}$ is a model of T.*

Proof. We use theorem 3.4 of [16] that a model is prime if and only if it is a denumerable atomic model. See [16] for an explanation of our terminology. For $n < \omega$ let $\boldsymbol{n}(v_0)$ be a purely existential formula with v_0 as its free variable and containing no function variables which defines n in $\mathfrak{F}$. If $\mathfrak{P} = \langle P, N, \oplus, \odot \rangle$ is a prime model of T, we construct an iso-

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morphism H of $\mathfrak{P}$ onto $\mathfrak{A}$ as follows. Let $p \in N$. p satisfies some atom $\varphi(v_0)$ in $\mathfrak{P}$ so that $\varphi(v_0)$ is consistent with T. Then $\mathfrak{F} \models (\exists v_0)\varphi(v_0)$ and we can find an $n < \omega$ what satisfies $\varphi(v_0)$ in $\mathfrak{F}$. Hence $\mathfrak{F} \models (\exists v_0)\big(\varphi(v_0) \wedge \wedge \boldsymbol{n}(v_0)\big)$ showing that $\varphi(v_0) \wedge \boldsymbol{n}(v_0)$ is consistent with T. But $\varphi(v_0)$ is an atom so that $T \vdash \varphi(v_0) \rightarrow \boldsymbol{n}(v_0)$. Since p satisfies $\varphi(v_0)$ in $\mathfrak{P}$ it must also satisfy $\boldsymbol{n}(v_0)$ in $\mathfrak{P}$. n is uniquely determined by p since $T \vdash \boldsymbol{n}(v_0) \rightarrow \sim \boldsymbol{m}(v_0)$ for every $n \neq m < \omega$. Thus we can define a function $H\colon N \rightarrow \omega$ by $H(p) = n$. H is one one since $T \vdash \boldsymbol{n}(v_0) \wedge \boldsymbol{n}(v_1) \rightarrow v_0 = v_1$ and is onto since $T \vdash (\exists v_0)\boldsymbol{n}(v_0)$ for every $n < \omega$. Next we show that H preserves the arithmetical operations. Let $p_i \in N$ for $i < 3$ with $p_0 \oplus p_1 = p_2$ and let $H(p_i) = n_i$. Now $\langle p_0, p_1, p_2\rangle$ satisfies $v_0 + v_1 = v_2$ in $\mathfrak{P}$, likewise p_i satisfies $\boldsymbol{n}_i(v_i)$ in $\mathfrak{P}$ for $i < 3$. Thus T is consistent with $v_0 + v_1 = v_2 \wedge \boldsymbol{n}_0(v_0) \wedge \boldsymbol{n}_1(v_1) \wedge \boldsymbol{n}_2(v_2)$ which consequently must be satisfiable in $\mathfrak{F}$. But this can only happen if $n_0 + n_1 = n_2$. We show that H preserves $\odot$ in exactly the same way. Now let $f \in P$. f satisfies some atom $\varphi(a_0)$ in $\mathfrak{P}$ so that $\varphi(a_0)$ is consistent with T. Then $\mathfrak{F} \models (\exists a_0)\varphi(a_0)$ and hence there is a $g \in F$ which satisfies $\varphi(a_0)$ in $\mathfrak{F}$. Let $p_0, p_1 \in N$ with $H(p_i) = n_i$ for $i < 2$. If $f(p_0) = p_1$, then f satisfies

$$\varphi(a_0) \wedge (\exists v_0, v_1)\big(a_0(v_0) = v_1 \wedge \boldsymbol{n}_0(v_0) \wedge \boldsymbol{n}_1(v_1)\big)$$

in $\mathfrak{P}$ and consequently this formula is consistent with T. Since $\varphi(a_0)$ is an atom,

$$T \vdash \varphi(a_0) \rightarrow (\exists v_0, v_1)\big(a_0(v_0) = v_1 \wedge \boldsymbol{n}_0(v_0) \wedge \boldsymbol{n}_1(v_1)\big)\,.$$

Then $f'(p_0) = p_1$ for every $f' \in P$ satisfying $\varphi(a_0)$ in $\mathfrak{P}$ and $g'(n_0) = n_1$ for every $g' \in F$ satisfying $\varphi(a_0)$ in $\mathfrak{F}$. Thus $\varphi(a_0)$ is uniquely satisfied by f in $\mathfrak{P}$, and is uniquely satisfied by g in $\mathfrak{F}$. Since jointly consistent atoms are equivalent in T, g is uniquely determined by f and we may extend H to P by setting $H(f) = g$. The preceding argument also shows that H is one one on P and that H preserves equations of the form $f(p_0) = p_1$. Since g uniquely satisfies $\varphi(a_0)$ in $\mathfrak{F}$ it is a member of A. If $g \in A$, then it is definable in $\mathfrak{F}$ by some formula $\psi(a_0)$ and consequently $\psi(a_0)$ is consistent with T. Then $\mathfrak{P} \models (\exists a_0)\psi(a_0)$ and there is an $f \in P$ which satisfies $\psi(a_0)$ in $\mathfrak{P}$. Let $\varphi(a_0)$ be the atom that f satisfies in $\mathfrak{P}$ so that $\varphi(a_0) \wedge \psi(a_0)$ is consistent with T. Then $T \vdash \varphi(a_0) \rightarrow \psi(a_0)$ and consequently $H(f) = g$. Then H is onto and in fact is the required isomorphism. Conversely suppose that $\mathfrak{A}$ is a model of T. We show that $\mathfrak{A}$ is prime by proving that it is an atomic model of T. Let $f \in A$ and $n < \omega$. We will show that there is an atom of T which $\langle f, n\rangle$ satisfies in $\mathfrak{A}$. Our method is general and works equally well for any finite sequence $\langle f_0, \ldots, f_{k-1}, n_0, \ldots, n_{l-1}\rangle$ of elements of $\mathfrak{A}$. Let $\varphi(a_0)$ define f in $\mathfrak{F}$. We claim that the formula $\psi(a_0, v_0)$, which is $\varphi(a_0) \wedge \boldsymbol{n}(v_0)$, is an atom of T that $\langle f, n\rangle$ satisfies in $\mathfrak{A}$. First $\mathfrak{F} \models (\exists a_0, v_0)\psi(a_0, v_0)$ so that $\psi(a_0, v_0)$ is consistent with T. If $\psi(a_0, v_0) \wedge \lambda(a_0, v_0)$ is consistent with T, then $\mathfrak{F} \models (\exists a_0, v_0) \times \times \big(\psi(a_0, v_0) \wedge \lambda(a_0, v_0)\big)$. But $\mathfrak{F} \models (\exists!\, a_0, !v_0)\psi(a_0, v_0)$ so that

$$\mathfrak{F} \models (\forall a_0, v_0)\big(\psi(a_0, v_0) \rightarrow \lambda(a_0, v_0)\big)\,, \quad \text{i.e.,} \quad T \vdash \psi(a_0, v_0) \rightarrow \lambda(a_0, v_0)\,,$$

which implies that $\psi(a_0, v_0)$ is indeed an atom of T. Now $\langle f, n\rangle$ satisfies $\psi(a_0, v_0)$ in $\mathfrak{F}$. Since $\boldsymbol{n}(v_0)$ contains no function variables and n satisfies $\boldsymbol{n}(v_0)$ in $\mathfrak{F}$ it must do the same in $\mathfrak{A}$. Also f uniquely satisfies $\varphi(a_0)$ in $\mathfrak{F}$. Hence $\mathfrak{A} \models (\exists!\, a_0)\varphi(a_0)$. If $\varphi(a_0)$ is satisfied in $\mathfrak{A}$ by some $g \neq f$, then there are n_0, n_1 such that $g(n_0) = n_1 \neq f(n_0)$ and consequently $(\forall a_0, v_0, v_1) \times \big(\varphi(a_0) \wedge \boldsymbol{n}_0(v_0) \wedge \boldsymbol{n}_1(v_1) \rightarrow a_0(v_0) = v_1\big)$ holds in $\mathfrak{A}$, and therefore also holds in $\mathfrak{F}$ since $\mathfrak{A}$ is a model of T. But this implies that $f(n_0) = n_1$, a contradiction. Thus $g = f$, i.e., f satisfies $\varphi(a_0)$ in $\mathfrak{A}$, and $\langle f, n\rangle$ satisfies the atom $\psi(a_0, v_0)$ in $\mathfrak{A}$, q.e.d.

There is another notion of prime model in current usage. A model of $\mathfrak{P}$ of T is said to be *prime in the sense of Robinson* (cf. [11]) if $\mathfrak{P}$ is isomorphic to a subsystem of every model of T. $\mathfrak{S}$ is called an *ω-model* if it has the form $\mathfrak{S} = \langle S, \omega, +, \cdot\rangle$ where $S \subseteq F$, and a *minimal* ω-model (cf. [2]) if it is a subsystem of every other ω-model of T. We characterize this notion of prime model in

THEOREM 2. *$\mathfrak{P}$ is a prime model of T in the sense of Robinson if and only if $\mathfrak{P}$ is isomorphic to $\mathfrak{A}$ and $\mathfrak{A}$ is a model of T.*

Proof. If $\mathfrak{A}$ is a model of T, then by theorem 1 it is prime in the sense of Vaught, *a fortiori*, prime in the sense of Robinson. Conversely suppose that $\mathfrak{P}$ is a prime model of T in the sense of Robinson. Let $\mathfrak{M}$ be a subsystem of $\mathfrak{F}$ which is isomorphic to $\mathfrak{P}$. Since $\mathfrak{M}$ is also a model of T it must be an ω-model of the form $\mathfrak{M} = \langle M, \omega, +, \cdot\rangle$. Let $\mathfrak{S} = \langle S, \omega, +, \cdot\rangle$ be an arbitrary ω-model of T and let H be an embedding of $\mathfrak{M}$ onto a subsystem of $\mathfrak{S}$. Since $\boldsymbol{n}(v_0)$ contains no function variables it is uniquely satisfied in any ω-model by the number $n < \omega$. We will show that H is an identity function by using the fact that embeddings preserve purely existential formula. Each $n < \omega$ uniquely satisfies $\boldsymbol{n}(v_0)$ in $\mathfrak{M}$. Since $\boldsymbol{n}(v_0)$ is purely existential $H(n)$ satisfies $\boldsymbol{n}(v_0)$ in $\mathfrak{S}$ giving $H(n) = n$. Let $f \in M$ and $n_0, n_1 < \omega$ such that $f(n_0) = n_1$. $\langle f, n_0, n_1\rangle$ satisfies $a_0(v_0) = v_1$ in $\mathfrak{M}$ and consequently $\langle H(f), H(n_0), H(n_1)\rangle$ satisfies $a_0(v_0) = v_1$ in $\mathfrak{S}$. Since H is an identity on ω, $H(f)(n_0) = f(n_0)$ for every $n_0 < \omega$ giving $H(f) = f$. Thus $\mathfrak{M}$ is a subsystem of every ω-model of T, i.e., it is a minimal ω-model of T. We determine M as follows. Let $g \in A$ and let $\varphi(a_0)$ define g in $\mathfrak{F}$. Since $\mathfrak{M}$ is a model of T, $\mathfrak{M} \models (\exists!\, a_0)\varphi(a_0)$ so that $\varphi(a_0)$ uniquely determines some function $f \in M$. If $g(n_0) = n$, then

$$(\forall a_0, v_0, v_1)\big(\varphi(a_0) \wedge \boldsymbol{n}_0(v_0) \wedge \boldsymbol{n}_1(v_1) \rightarrow a_0(v_0) = v_1\big)$$

must hold in $\mathfrak{F}$ and consequently must also hold in $\mathfrak{M}$. But this can only happen if $f(n_0) = n_1$. Thus $f = g$ and $\mathfrak{A}$ is a subsystem of M. We show that $\mathfrak{A} = \mathfrak{M}$ by finding an ω-model $\mathfrak{S}$ of T which omits any given function $f \in F - A$. Although this could be done by the methods of [16], it is more convenient to use theorem 2.1 of [5]. This asserts that if T is a consistent theory in a countable logic and S is a finite or countable set of sets of formulas $\sigma(v_0)$ such that each $\Sigma \in S$ has the property $(*)$ for each formula $\varphi(v_0)$ which is consistent with T, there exists $\sigma(v_0) \in \Sigma$ such that $\varphi(v_0) \wedge \sim\sigma(v_0)$ is consistent with T, then T has a countable model which omits each $\Sigma \in S$. There is no difficulty in applying this result to the two sorted logic $\mathfrak{L}$. For Σ_0 take the set $\{\sim \boldsymbol{n}(v_0)\colon n < \omega\}$. If $\varphi(v_0)$ is a formula consistent with T, then $\mathfrak{F} \models (\exists v_0)\varphi(v_0)$ and we can find an $n < \omega$ which satisfies $\varphi(v_0)$ in $\mathfrak{F}$. Hence $\mathfrak{F} \models (\exists v_0)\big(\varphi(v_0) \wedge \boldsymbol{n}(v_0)\big)$ showing that $\varphi(v_0) \wedge \boldsymbol{n}(v_0)$ is consistent with T. Thus Σ_0 has the property $(*)$. If $f \in F - A$, then for Σ_1 take the set

$$\big\{(\forall v_0, v_1)\big(\boldsymbol{n}_0(v_0) \wedge \boldsymbol{n}_1(v_1) \to a_0(v_0) = v_1\big)\colon f(n_0) = n_1\big\}.$$

If $\varphi(a_0)$ is a formula consistent with T, then $\mathfrak{F} \models (\exists a_0)\varphi(a_0)$ so that some function $g \in F$ satisfies $\varphi(a_0)$ in $\mathfrak{F}$. Since f is not definable in $\mathfrak{F}$ we may take $g \neq f$, i.e., there are $n_0, n_1 < \omega$ such that $g(n_0) \neq n_1 = f(n_0)$. Hence

$$\mathfrak{F} \models (\exists a_0)\Big(\varphi(a_0) \wedge (\exists v_0, v_1)\big(\boldsymbol{n}_0(v_0) \wedge \boldsymbol{n}_1(v_1) \wedge a_0(v_0) \neq v_1\big)\Big)$$

so that $\varphi(a_0) \wedge \sim(\forall v_0, v_1)\big(\boldsymbol{n}_0(v_0) \wedge \boldsymbol{n}_1(v_1) \to a_0(v_0) = v_1\big)$ is consistent with T. But $f(n_0) = n_1$ and consequently Σ_1 has the property $(*)$. Let $\mathfrak{S}$ be a model of T which omits both Σ_i. Since $\mathfrak{S}$ omits Σ_0 we may take $\mathfrak{S}$ to be an ω-model, and since $\mathfrak{S}$ omits Σ_1, but f satisfies Σ_1, f will not belong to S. Thus $M = A$ and $\mathfrak{P}$ is isomorphic to $\mathfrak{A}$, q.e.d.

Thus the notions of prime models (in both senses) and minimal ω-models are coextensive for the theory T and are nonvacuous if and only if $\mathfrak{A}$ is a model of T. We say that $\mathfrak{F}$ satisfies an *analytic basis theorem* if whenever $\varphi(a_0) \in \mathfrak{L}$ is a formula with one free variable and $\mathfrak{F} \models (\exists a_0)\varphi(a_0)$, then $\varphi(a_0)$ is satisfied in $\mathfrak{F}$ by some function $f \in A$. We say that $\mathfrak{F}$ admits an *analytic well ordering* if there is a formula $\lambda(a_0, a_1)$ with two free variables such that $\{\langle f_0, f_1\rangle\colon \mathfrak{F} \models \lambda[f_0, f_1]\}$ is a well ordering of F. Then we have the well known

LEMMA. *$\mathfrak{A}$ is a model of T if and only if $\mathfrak{F}$ satisfies an analytic basis theorem.*

LEMMA. *If $\mathfrak{F}$ admits an analytic well ordering then $\mathfrak{F}$ satisfies an analytic basis theorem.*

Let ZF be Zermelo Fraenkel set theory including the axiom of choice, $V = L$ is the axiom of constructibility, CH is the continuum hypothesis, and MC asserts the existence of a measurable cardinal. Then we have the independence result

THEOREM 3. *The statement "$\mathfrak{A}$ is a prime model of T" is relatively consistent with* (1) $\mathrm{ZF}+V = L$, (2) $\mathrm{ZF}+V \neq L$, (3) $\mathrm{ZF}+\mathrm{CH}$, (4) $\mathrm{ZF}+\mathrm{MC}$, (5) $\mathrm{ZF}+\sim\mathrm{MC}$. *The statement "$\mathfrak{A}$ is not a prime model of T" is relatively consistent with* (6) $\mathrm{ZF}+V \neq L$, (7) $\mathrm{ZF}+\mathrm{CH}$, (8) $\mathrm{ZF}+\sim\mathrm{CH}$, (9) $\mathrm{ZF}+\mathrm{MC}$, (10) $\mathrm{ZF}+\sim\mathrm{MC}$.

Proof. Let $\mathfrak{M}$ be a countable transitive model of $\mathrm{ZF}+V = L$. We know from [4] that $\mathfrak{M}$ satisfies CH, from [12] that $\mathfrak{M}$ satisfies $\sim$MC, and from [1] that $\mathfrak{M}$ satisfies "$\mathfrak{F}$ admits an analytic well ordering (in fact a Δ^1_2 well ordering)". This proves (1), (3) and (5). Let $\mathfrak{N}$ be obtained from $\mathfrak{M}$ by adjoining a single generic function $f\colon \omega \to \omega$. From [3] we know that $\mathfrak{M}$ coincides with the constructible sets of $\mathfrak{N}$, however $f \notin \mathfrak{M}$, from [7] that $\mathfrak{N}$ has the property $(*)$ if $g\colon \omega \to \omega$, $g \in \mathfrak{N}$, and g is definable in $\mathfrak{N}$ from elements of $\mathfrak{M}$ then $g \in \mathfrak{M}$, and from [1] that the predicate "$a_0\colon \omega \to \omega$ is non-constructible" can be expressed in Π^1_2 form, say $\varphi(a_0) \in \mathfrak{L}$. Then in $\mathfrak{N}$, $\varphi(a_0)$ is a formula which is satisfiable in $\mathfrak{F}$ but is not satisfied by any element of A. This proves (6) and since $\mathfrak{N}$ satisfies CH (cf. [3]) we obtain (7) as well. The extension of $\mathfrak{N}$ of $\mathfrak{M}$ is mild in the sense of [8] so that $\mathfrak{N}$ satisfies MC if and only if $\mathfrak{M}$ satisfies MC (cf. [8]). Since $\mathfrak{M}$ does not, neither does $\mathfrak{N}$, and we have proved (10). By a result of Solovay (stated in [9]) a non-generic f may be chosen so that $\mathfrak{N}$ is a model of ZF, $f \notin \mathfrak{M}$, every element of $\mathfrak{N}$ is constructible from f, and f is Δ^1_3 in $\mathfrak{N}$. Then in $\mathfrak{N}$, $\mathfrak{F}$ admits a well ordering which is Δ^1_2 in a Δ^1_3 function, and hence a Δ^1_3 well ordering. This proves (2). We can prove (8) in exactly the same way that we proved (6) by constructing $\mathfrak{N}$ as in [3], to violate CH, and then noting that by [7] the property $(*)$ holds for this $\mathfrak{N}$. Now let $\mathfrak{M}$ be a countable transitive model of $\mathrm{ZF}+\mathrm{MC}$ containing an ordinal $\varkappa$ and a normal $\varkappa$-complete nonprincipal ultrafilter D on $\varkappa$ (in the sense of $\mathfrak{M}$) such that every element of $\mathfrak{M}$ is constructible relative to D. From [13] we know that $\mathfrak{M}$ satisfies "$\mathfrak{F}$ admits an analytic well ordering (in fact a Δ^1_3 well ordering)." This proves (4). Let $\mathfrak{N}$ be obtained from $\mathfrak{M}$ by adjoining a single generic function $f\colon \omega \to \omega$. Since this extension is mild, by [8] we know that D uniquely extends to a normal $\varkappa$-complete nonprincipal ultrafilter D' on $\varkappa$ (in the sense of $\mathfrak{N}$), $\mathfrak{M}$ coincides with the elements of $\mathfrak{N}$ constructible relative to D', and $f \notin \mathfrak{M}$. From [13] we know that the predicate "$a_0\colon \omega \to \omega$ is non-constructible relative to D'" can be expressed in Π^1_3 form, say $\varphi(a_0) \in \mathfrak{L}$, and from [7] we see that $\mathfrak{N}$ has the property $(*)$. Then in $\mathfrak{N}$, $\varphi(a_0)$ is a formula which is satisfiable in $\mathfrak{F}$ but is not satisfied by any element of A. This proves (9), q.e.d.

There is one asymmetry in the statement of our theorem. We have not shown that "$\mathfrak{A}$ is a prime model of T" is consistent with $\mathrm{ZF}+\sim\mathrm{CH}$. This seems to be related to the open problem (summer 1967, cf. [9]) as to whether $\sim$CH is consistent with the existence of a projective well ordering of $\mathfrak{F}$.

We apply our results to certain weak second order theories. Let $\mathfrak{R} = \langle R, +, \cdot \rangle$ where R is the set of real numbers and $+, \cdot$ are the usual arithmetic operations. Let $\mathfrak{L}^w$ be a weak second order language for $\mathfrak{R}$ and let T^w be its weak second order theory. The notion of "prime model in the sense of Robinson" has an immediate generalization to the case of T^w models, and so does "prime model in the sense of Vaught" once we have defined w-elementary subsystem to read exactly like its first order equivalent except that we require all parameters to be individual. There is a sentence in T^w which guarantees that each model of T^w admits an Archimedean ordering and therefore has a unique embedding into $\mathfrak{R}$. Thus it is meaningful to talk about minimal models of T^w. Let $B = \{x \in R\colon 0 < x < 1$ and x is irrational$\}$ and define a function θ from B onto F by letting $\theta(x) = f$ where $1 + f(n)$ is the nth denominator in the continued fraction expansion of x. For each subsystem $\mathfrak{S}^w = \langle S^w, +, \cdot \rangle$ of $\mathfrak{R}$ let $H(\mathfrak{S}^w) = \mathfrak{S} = \langle S, \omega, +, \cdot \rangle$ where $S = \{\theta(x)\colon x \in S^w\}$, and for each subsystem $\mathfrak{S}$ of $\mathfrak{F}$ let $H(\mathfrak{S}) = \mathfrak{S}^w$ where S^w is the closure under rational operations of $\{\theta^{-1}(f)\colon f \in S\}$. Then we have the

LEMMA. *H takes models into models, is self inverse there, and preserves the notion of proper elementary subsystem.*

We merely sketch a proof of this result. By [10] there is a formula $\varphi(v_0, v_1, v_2)$ in $\mathfrak{L}^w$ with three free variables, each individual, such that if $\mathfrak{S}^w$ is a model of T^w, $x \in S^w$, and $n, p < \omega$, then $\langle x, n, p \rangle$ satisfies φ in $\mathfrak{S}^w$ if and only if p is the nth denominator in the continued fraction expansion of x. From this we immediately see that H takes models of T^w into models of T preserving the notion of proper elementary subsystem. Conversely it is clear that given a family of functions, we can define the field operations which give rise to these functions as continued fraction expansions in a perfectly elementary way, i.e., in the language $\mathfrak{L}$. Thus H takes models of T into models of T^w preserving the notion of proper elementary subsystem. The self inverse property is immediate. Let $\mathfrak{A}^w = H(\mathfrak{A})$. Then granting our lemma all of the results which are mentioned in theorems 1–3 go over for models of T^w (by replacing T by T^w and $\mathfrak{A}$ by $\mathfrak{A}^w$ in their statements). This is in sharp distinction to the first order case where it is known (cf. [15]) that the algebraic reals is a minimal, and prime in both senses, model of the first order theory of $\mathfrak{R}$. We briefly compare these results with those concerning the weak second order theory of complex numbers. Let $C = \langle C, +, \cdot \rangle$ where C is the set of complex numbers and $+, \cdot$ are the usual arithmetic operations, and let T_C^w be its weak second order theory. $\mathfrak{S} = \langle S, \oplus, \odot \rangle$ is a model of T_C^w if and only if $\mathfrak{S}$ is an algebraically closed field of characteristic 0 and infinite degree of transcendence (cf. [14]). Thus every such $\mathfrak{S}$ has a proper subsystem $\mathfrak{S}'$ which is also a model of T_C^w, and consequently there is no minimal model. On the other hand, given models $\mathfrak{S}$, $\mathfrak{S}'$ of T_C^w, where $\mathfrak{S}$ has countable degree of transcendence, by purely algebraic methods, we can find an embedding H which maps $\mathfrak{S}$ isomorphically onto a subsystem $\mathfrak{S}''$ of $\mathfrak{S}'$. The methods of [15] then generalize so that $\mathfrak{S}''$ will be a w-elementary subsystem of $\mathfrak{S}'$. Thus T_C^w has prime models in both senses, just as in the first order case (cf. [15]).

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