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On Kan extensions of cohomology theories and Serre classes of groups

by

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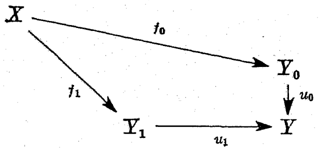
1. Introduction. This paper constitutes a continuation of the investigation initiated in [4], [5]. In those papers we introduced a process, involving the Kan extension of a functor, for extending a cohomology theory from a category J_0 of based topological spaces to a larger category J_1. This process generalizes a characterization of Čech cohomology which has been noted by Eilenberg and Steenrod and studied by Dold. However, the process partakes far more of the spirit of Kan's work on extending functors than of the original description of Čech cohomology, so that the examples of cohomology theories expressible as Kan extensions take one very far from Čech cohomology, while retaining a certain generalized continuity property. We should mention here that Lee and Raymond [11] have studied generalized Čech theories in a somewhat different sense, more strongly motivated by the classical description of Čech theory. There is some small overlap with the present authors' work, and a comparison of the two approaches will form the subject of a later paper (1).

A principal concern in [4], [5] is that of deciding under what conditions the Kan extension (2), h_1, to J_1 of a cohomology theory h on J_0 (or *maximal* extension in the terminology of [2]) is itself a cohomology theory. We always require that the categories J considered suitable for supporting a cohomology theory be admissible; that is, they should be non-empty full subcategories of the category of based spaces and based maps, they should admit mapping cones, and should contain entire homotopy types. We can state the axioms for h or, more precisely, (h, σ), where σ is the suspension transformation $h^n \to h^{n+1}\Sigma$, to be a cohomology theory in any admissible category; but the Kan extension of a cohomology theory from an admissible category J_0 to an admissible category J_1 may well fail to be a cohomology theory. After some preliminary algebraic argument in Section 2, we formulate a criterion for the Kan extension

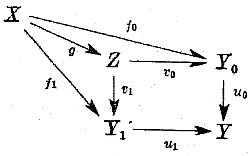
(1) *Remark on Čech extensions of cohomology functors*, Proc. Adv. St. Inst. Aarhus (1970), pp. 44–66.
(2) We use here the notation h_1 in preference to the ${}_0h$ notation of [4], [5].

of a cohomology theory to satisfy the axioms in Sections 2 and 3. This criterion is much more general than that of [5] and presumably comes reasonably close to constituting a set of necessary and sufficient conditions for *any* cohomology theory on J_0 to extend to a cohomology theory on J_1. It divides itself into two parts; there is a condition for the extended theory to satisfy the exactness axiom, which is expressed by means of a local pull-back property (Theorem 2.14), and a condition for the extended theory to satisfy the suspension axiom, which is expressed by means of a local right-adjointness property (Theorem 3.5). This latter notion is closely related to that of a locally adjunctable functor due to Kaput [10]. Indeed both Kaput's definition and ours involve an existence and a universal statement for a factorization of a morphism f; the existence statements are identical, but our universal statement is less restrictive than Kaput's. In addition, our concept is related to a pair of categories (J_1, J_0) and so only requires the factorization if $f\colon \Sigma X \to Y$, $X \in |J_1|$, $Y \in |J_0|$. The main advantage of the new criterion over that of [5] is that the sufficient conditions of [5] are *localized*. For example the weak local pull-back property asserts that a diagram of homotopy classes of maps,



$X \in |J_1|$, $Y_0, Y_1, Y \in |J_0|$

may be embedded in a diagram



$Z \in |J_0|$;

that is, the morphisms g, v_0, v_1 may be chosen *after* f_0 and f_1 are given, unlike the situation prevailing for a weak (global) pull-back of u_0 and u_1. We indicate by two examples at the end of Section 3 how the new criterion enables us to guarantee that certain Kan extensions, in situations not covered by [5], are indeed cohomology theories.

The second of these two examples constitutes essentially the subject matter of Section 4. We consider the *acyclic ring*, or *Serre class* [13], [14], C_P, of torsion abelian groups A whose elements have orders which are multiples of primes in a certain family P (which could, in particular, consist of a single prime p or of all primes). If J_0 is the category of 1-connected finite CW-complexes whose homology groups belong to C_P and if J_1 is the category of 1-connected finite dimensional CW-complexes, then the Kan extension, h_1, to J_1, of any cohomology theory h defined on J_0 is again a cohomology theory and we identify it in the case that h is representable with finitely-generated coefficients. It turns out that, in fact, (see (4.2)),

(1.1) $$h_1^n(X) \cong h^{n-1}(X; Z_{P^\infty}), \quad X \in |J_1|.$$

Here $h(X; Z_{P^\infty})$ refers to the cohomology theory obtained from h by putting in the coefficient group

$$Z_{P^\infty} = \bigoplus_{p \in P} Z_{p^\infty},$$

in the sense of [9]; it is proved in that reference that the groups $h(X; Z_{P^\infty})$ are determined by X and P up to natural isomorphism if $2 \notin P$ and up to 'quasi-natural' isomorphism if $2 \in P$. It is worth remarking that if $h = H$, ordinary reduced cohomology with integer coefficients, then h_1 is a theory which does not even satisfy the restricted wedge axiom,

$$h_1(\bigvee S^n) \neq \prod h_1(S^n)$$

over any infinite wedge of spheres.

There is also established in [9] a universal coefficient theorem for $h(\ ; G)$ which plays a crucial role in the proof of our main result, Theorem 4.1. The proof is, in fact, broken up into a sequence of lemmas in order to display the role of the various elements in the hypotheses of the theorem and to enunciate certain features of the Kan extension process which should play a role in any future attempt to identify the nature of a particular extended theory. The paper ends with two variants of Theorem 4.1, in which we alter the hypotheses but obtain essentially the same conclusion. Thus, in all cases considered in Section 4, (1.1) holds. This relation serves to illustrate the violence which can be done to a cohomology theory defined on J_1 by replacing it by the Kan extension of its restriction to J_0. If we start with ordinary cohomology with integer coefficients, and carry out the process described, then (taking P, for the purpose of this remark, to be the set of all primes), the groups of an n-sphere are given by

(1.2) $$h_1^{n+1}(S^n) = Q_1, \quad h_1^i(S^n) = 0, \quad i \neq n+1.$$

This result illustrates a fact to which we hope to give attention in a subsequent paper, namely, that there exists a close connection between the Kan extension process for cohomology theories and Adams' notion of completing a space with respect to a homology theory (3).

(3) *Localization, homology and a construction of Adams,* Battelle Institute Report 47 (1971).

We remark that the arguments used in the proof of Theorem 4.1 involve a study of colimits for functors to sets; since this theory is so much more elementary than that involved in a study of functors to groups, we have felt it to be adequate simply to refer to the arguments at the beginning of Section 2 and ask the reader, in considering functors to sets, to ignore anything in those arguments referring specifically to groups rather than sets.

It is hoped to devote a subsequent paper to applications of the lemmas of Section 4 to other examples of the Kan extension of a cohomology theory, different from those treated in this paper.

It is a pleasure to acknowledge the benefit of very helpful conversations with Guido Mislin in connection with specific aspects of this work.

2. The construction of colimits of functors to groups. In order to motivate the abstract study of colimits undertaken in this section, we first recall explicitly the definition of the Kan extension of a cohomology theory h, defined on J_0, a category of based spaces and based maps, to a larger category J_1 of based spaces and based maps, as given in [5]. Let $\tilde{J}_0, \tilde{J}_1$ be the *homotopy* categories associated with J_0, J_1 (i.e., the morphisms of $\tilde{J}_0(\tilde{J}_1)$ are based homotopy classes of based maps in $J_0(J_1)$). Let $X \in |J_1|$ and form the category $\tilde{J}_{10}(X)$ of $\tilde{J}_0$-*objects under* X as follows. An object of $\tilde{J}_{10}(X)$ is a morphism $f\colon X \to Y$ in $\tilde{J}_1$ with $Y \in |J_0|$. A morphism $u\colon f_1 \to f_2$ in $\tilde{J}_{10}(X)$ is a morphism $u\colon Y_1 \to Y_2$ in $\tilde{J}_0$ such that the $\tilde{J}_1$-diagram

$$\begin{array}{ccc} & & Y_1 \\ & \overset{f_1}{\nearrow} & \\ X & & \big\downarrow u \\ & \underset{f_2}{\searrow} & \\ & & Y_2 \end{array}$$

commutes. With the evident definition of composition, $\tilde{J}_{10}(X)$ is a category. Moreover, the contravariant functor h from J_0 to $\mathrm{Ab}^{\mathbf{Z}}$, the category of graded abelian groups, induces a functor $h_X\colon \tilde{J}_{10}(X) \to \mathrm{Ab}^{\mathbf{Z}}$ by the rule

$$(2.1) \qquad h_X(f) = h(Y), \quad h_X(u) = h(u).$$

We then define the functor $h_1\colon J_1 \to \mathrm{Ab}^{\mathbf{Z}}$ by $h_1(X) = \varinjlim h_X$; the universal property of the colimit then implies the definition of $h_1(g)$ for $g\colon X' \to X$. Since we only consider cohomology theories defined on full subcategories of the category of all based spaces and based maps, it follows that h_1 does indeed extend h; but it may, of course, fail to be a cohomology functor, although it is evidently homotopy invariant.

Thus, replacing $\tilde{J}_{10}(X)^{\mathrm{opp}}$ by I and h_X by F, we wish to study the colimit $\varinjlim F$, of a (covariant) functor $F\colon I \to \mathfrak{G}$ where $\mathfrak{G}$ is the category of groups (there is no gain in simplicity in supposing the values of F to be abelian).

We wish to study conditions on I which guarantee that the standard construction of the direct limit when I is a directed set still apply to our case; plainly, however, I is *not* a directed set and, indeed, it may well fail to be even *quasi-filtered* (see [8]). Thus we require a substantial generalization of the usual theory although this generalization may be known in the folklore. In addition we are, of course, hoping that the colimit of F, viewed as a functor on J_1, will inherit from h the exactness property required of a cohomology theory.

We suppose now that I is a connected category such that

(a) given $\cdot\!\!\begin{smallmatrix}\nearrow\\ \searrow\end{smallmatrix}$ in I, we may construct a commutative diagram $\cdot\!\!\begin{smallmatrix}\nearrow\\ \searrow\end{smallmatrix}\!\!\begin{smallmatrix}\searrow\\ \nearrow\end{smallmatrix}\!\cdot$ in I; and

(b) given $\cdot \rightrightarrows \cdot$ in I, we may construct a commutative diagram $\cdot \rightrightarrows \cdot \to$ in I.

We will then say that I is *adapted* (for colimits); notice that conditions (a) and (b) together yield the definition (in a suitable 'universe') of a quasi-filtered category. We remark that condition (a) implies that any two objects of I may be connected by a path $\cdot \to \cdot \leftarrow \cdot$. We also remark that if I satisfies (a) and

(b′) I admits finite coproducts,

then I satisfies (b) and hence is adapted.

Now let $F\colon I \to \mathfrak{G}$ be a functor. We will write F_i for $F(i)$, $i \in |I|$, and $\varphi\colon F_i \to F_j$ for $F(\varphi)$, where $\varphi\colon i \to j$ in I. We introduce a relation in $\bigcup_i F_i$ by declaring

$$(2.2) \qquad x_i \sim x_j, \quad x_i \in F_i, \quad x_j \in F_j$$

if there is $i \xrightarrow{\varphi} k \xleftarrow{\psi} j$ in I with $\varphi(x_i) = \psi(x_j)$.

THEOREM 2.3. *If I is a connected category satisfying condition* (a), *then* (2.2) *is an equivalence relation.*

Proof. Only transitivity is in question, so we suppose $x_i \sim x_j$, $x_j \sim x_l$, thus,

$$i \xrightarrow{\varphi} k \xleftarrow{\psi} j, \quad \varphi(x_i) = \psi(x_j),$$

$$j \xrightarrow{\theta} m \xleftarrow{\chi} l, \quad \theta(x_j) = \chi(x_l).$$

By condition (a) we have a commutative square in I

$$\begin{array}{ccc} j & \xrightarrow{\psi} & k \\ {\scriptstyle\theta}\big\downarrow & & \big\downarrow{\scriptstyle\varrho} \\ m & \xrightarrow[\sigma]{} & n \end{array}$$

Then $i \xrightarrow{\varrho\varphi} n \xleftarrow{\sigma\chi} l$ in I and

$$\varrho\varphi(x_i) = \varrho\psi(x_j) = \sigma\theta(x_j) = \sigma\chi(x_l),$$

so $x_i \sim x_l$.

We write $[x_i]$ for the equivalence class of x_i. We define a product operation among the equivalence classes by the rule

$$[x_i][x_j] = [\varphi(x_i)\cdot\psi(x_j)], \tag{2.4}$$

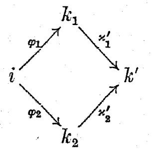
where $i \xrightarrow{\varphi} k \xleftarrow{\psi} j$ in I.

THEOREM 2.5. *If I is adapted then* (2.4) *is well-defined.*

Proof. We first show that the function $(x_i, x_j) \to [\varphi(x_i)\cdot\psi(x_j)]$ is well-defined. Thus suppose

$$i \xrightarrow{\varphi_\lambda} k_\lambda \xleftarrow{\psi_\lambda} j \text{ in } I, \quad \lambda = 1, 2.$$

We construct, by (a), the commutative square

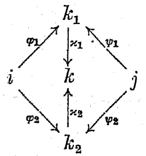


and then use (b) to construct the commutative diagram

$$j \underset{\varkappa_2'\psi_2}{\overset{\varkappa_1'\psi_1}{\rightrightarrows}} k' \xrightarrow{\varkappa} k.$$

If we set $\varkappa_\lambda = \varkappa\varkappa_\lambda'$, $\lambda = 1, 2$, we have the commutative diagram

(2.6)



Thus

$$[\varphi_1(x_i)\psi_1(x_j)] = [\varkappa_1\varphi_1(x_i)\varkappa_1\psi_1(x_j)] = [\varkappa_2\varphi_2(x_i)\varkappa_2\psi_2(x_j)] = [\varphi_2(x_i)\psi_2(x_j)],$$

so that the function $(x_i, x_j) \mapsto [\varphi(x_i)\psi(x_j)]$ is well-defined.

Second we show that $[\varphi(x_i)\psi(x_j)]$ depends only on $[x_i]$ and $[x_j]$. It is plainly sufficient to show that it is independent of the choice of x_i from $[x_i]$; and it is then obvious that we need only consider the effect of replacing x_i by $\mu(x_i)$ where $\mu\colon i \to l$ in I. There is a diagram

$$l \xrightarrow{\nu} m \xleftarrow{\varrho} j$$

in I and by what we have already shown, since we have $i \xrightarrow{\nu\mu} m \xleftarrow{\varrho} j$ in I,

$$[\varphi(x_i)\cdot\psi(x_j)] = [\nu\mu(x_i)\cdot\varrho(x_j)]. \tag{2.7}$$

However the right-hand side is the image of $(\mu(x_i), x_j)$ under the given function, so that our claim is established and the theorem proved.

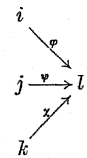
Let us write G for the collection of equivalence classes under (2.2) and $\theta_i\colon F_i \to G$ for the function $\theta_i(x_i) = [x_i]$. Plainly θ_i is homomorphic with respect to (2.4) and if $\varphi\colon i \to j$ in I then

$$\theta_i = \theta_j\varphi. \tag{2.8}$$

THEOREM 2.9. *If I is adapted, then G is a group under* (2.4) *and*

$$\varinjlim F = (G;\ \theta_i).$$

Proof. If e_i is the identity of F_i then plainly $e_i \sim e_j$ for any $i, j \in |I|$ and $e = [e_i]$ is the identity of G. Equally plainly $[x_i^{-1}]$ is the inverse of $[x_i]$ and associativity follows immediately from the observation that, given $i, j, k \in |I|$, we can find



in I. For $([x_i][x_j])[x_k] = [x_i]([x_j][x_k]) = [\varphi(x_i)\psi(x_j)\chi(x_k)]$.

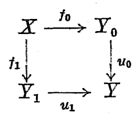
Now suppose given a group H and homomorphisms $\varrho_i\colon F_i \to H$ such that if $\varphi\colon i \to j$ in I then $\varrho_i = \varrho_j\varphi$. We define a homomorphism $\varrho\colon G \to H$ by $\varrho[x_i] = \varrho_i(x_i)$. If $i \xrightarrow{\varphi} k \xleftarrow{\psi} j$ in I and if $\varphi(x_i) = \psi(x_j)$, then $\varrho_i(x_i) = \varrho_k\varphi(x_i) = \varrho_k\psi(x_j) = \varrho_j(x_j)$, so that ϱ is well-defined. It is easy to see that ϱ is a homomorphism and that

$$\varrho\theta_i = \varrho_i. \tag{2.10}$$

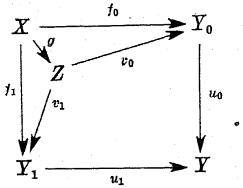
Moreover, ϱ is obviously uniquely determined by (2.10) so that the theorem is completely proved.

We now consider how to apply Theorem 2.9 to our example (2.1), so that $I = I(X) = \tilde{\mathcal{J}}_{10}(X)^{\mathrm{opp}}$, $F = h_X$.

DEFINITION 2.11. $\tilde{\mathcal{J}}_0$ *has weak local pull-backs relative to* $\tilde{\mathcal{J}}_1$ if and only if, given the diagram



commutative in $\widetilde{J}_1$ with $Y_0, Y_1, Y \in |J_0|$, there exists a diagram



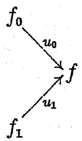
commutative in $\widetilde{J}_1$ with $Z \in |J_0|$.

THEOREM 2.12. (i) *$I(X)$ is connected for all $X \in |J_1|$;*

(ii) *$I(X)$ satisfies condition* (a) *for all $X \in |J_1|$ if and only if $\widetilde{J}_0$ has weak local pull-backs relative to $\widetilde{J}_1$;*

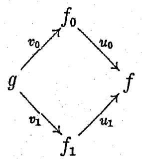
(iii) *$I(X)$ satisfies condition* (b′) *for all $X \in |J_1|$ if J_0 is closed under (finite) products.*

Proof. (i) is trivial since J_0 contains singletons so that $\widetilde{J}_{10}(X)$ has a terminal object. (ii) is obvious since Definition 2.11 asserts that, for each X, the diagram



$(f = u_0 f_0 = u_1 f_1)$

in $\widetilde{J}_{10}(X)$ gives rise to a commutative diagram



in $\widetilde{J}_{10}(X)$. As to (iii), we simply show that if $Y_1, Y_2 \in |J_0|$ and $(Y_1 \times Y_2; p_1, p_2)$ is their product, and if $f_i\colon X \to Y_i$, $i = 1, 2$, in $\widetilde{J}_{10}(X)$, then $\{f_1, f_2\}\colon X \to Y_1 \times Y_2$, together with p_1 and p_2, constitutes the product of f_1 and f_2 in $\widetilde{J}_{10}(X)$. For given $u_i\colon f \to f_i$ in $\widetilde{J}_{10}(X)$, $i = 1, 2$, then $u_i f = f_i$, $\{u_1, u_2\} f = \{f_1, f_2\}$ and $\{u_1, u_2\}\colon f \to \{f_1, f_2\}$ is the unique morphism in $\widetilde{J}_{10}(X)$ such that $p_i\{u_1, u_2\} = u_i$, $i = 1, 2$.

We say J_0 is J_1*-adapted* if J_0 is closed under (finite) products and $\widetilde{J}_0$ has weak local pull-backs relative to $\widetilde{J}_1$.

COROLLARY 2.13. *If J_0 is J_1-adapted, $I(X)$ is adapted for all $X \in |J_1|$.*

Now define $h_1(X) = \varinjlim h_X$. This clearly defines a contravariant functor $h_1\colon \widetilde{J}_1 \to \mathrm{Ab}$, called the *Kan extension* of h. We prove (compare Theorems 3.9, 3.10 of [5]):

THEOREM 2.14. *If J_0 is J_1-adapted, h_1 satisfies the exactness axiom.*

Proof. Let $X' \xrightarrow{g'} X \xrightarrow{g''} X''$ be a mapping cone sequence in J_1; we must show

$$h_1(X') \xleftarrow{h_1(g')} h_1(X) \xleftarrow{h_1(g'')} h_1(X'')$$

exact. Since $I(X)$ is adapted for all $X \in |J_1|$, we may represent an element ξ of $h_1(X)$ by means of a pair (a, f) where $f\colon X \to Y$ in $\widetilde{J}_1$ and $a \in h(Y)$. Moreover (4), $h_1(g')[a, f] = [a, fg']$. Thus if $h_1(g')[a, f] = 0$, $[a, fg'] = 0$ so that, in view of (2.2), there exists $u'\colon f' \to fg'$ in $\widetilde{J}_{10}(X')$ such that $h(u')a = 0$. Consider the diagram

$$\begin{array}{ccccc} X' & \xrightarrow{g'} & X & \xrightarrow{g''} & X'' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} \\ Y' & \xrightarrow{u'} & Y & \xrightarrow{u''} & Y'' \end{array} \tag{2.15}$$

where the bottom row is also a mapping cone sequence (and so in J_0). Since the top row is a mapping cone sequence there exists $f''\colon X'' \to Y''$ such that the right-hand square in (2.15) is homotopy-commutative; and since $h(u')a = 0$, there exists $a'' \in h(Y'')$ with $h(u'')a'' = a$. Then $u''\colon f \to u''f$ in $\widetilde{J}_{10}(X)$, so that

$$h_1(g'')[a'', f''] = [a'', f''g''] = [a'', u''f] = [h(u'')a'', f] = [a, f]$$

and the theorem is proved. We draw particular attention to the representation of an element of $h_1(X)$ as

$$[a, f], \quad f\colon X \to Y \text{ in } \widetilde{J}_1, \quad a \in h(Y). \tag{2.16}$$

Since the homotopy axiom is automatically satisfied by h_1, it remains only to consider the suspension axiom. This will be handled in the next section, always under the hypothesis that J_0 is J_1-adapted.

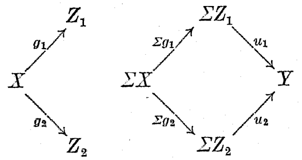
3. Invariance under suspension. Let J_0 be J_1-adapted and let (h, σ) be a cohomology theory on J_0. We extend h to $h_1\colon J_1 \to \mathrm{Ab}$ as in Section 2. Then the natural equivalence $\sigma\colon h^n \to h^{n+1}\Sigma\colon J_0 \to \mathrm{Ab}$ extends to a natural transformation $\sigma_1\colon h_1^n \to h_1^{n+1}\Sigma\colon J_1 \to \mathrm{Ab}$ and we seek conditions in this section under which σ_1 is an equivalence. We remark first of all that σ_1 is given by

$$\sigma_1[a, f] = [\sigma a, \Sigma f] \tag{3.1}$$

(4) As in [5], we write $[a, f]$ for the equivalence class containing (a, f).

We now introduce a condition on the triple (J_1, J_0, Σ) which will guarantee the result we seek to achieve. Recall that $\widetilde{J}$ is the category obtained from J by replacing maps by homotopy classes.

DEFINITION 3.2. We say that $\Sigma\colon \widetilde{J}_1 \to \widetilde{J}_1$ is *locally right-$\widetilde{J}_0$-adjunctable* if (i) given $f\colon \Sigma X \to Y$ in $\widetilde{J}_1$, $Y \in |J_0|$, there exist $Z \in |J_0|$, $g\colon X \to Z$ in $\widetilde{J}_1$, $u\colon \Sigma Z \to Y$ in $\widetilde{J}_0$ such that $f = u \circ \Sigma g$; and (ii) given



$$Z_1, Z_2, Y \in |J_0|, \qquad u_1 \circ \Sigma g_1 = u_2 \circ \Sigma g_2,$$

there exist $v_1\colon Z_1 \to Z$, $v_2\colon Z_2 \to Z$, $u\colon \Sigma Z \to Y$ in $\widetilde{J}_0$ with $v_1 g_1 = v_2 g_2$, $u \circ \Sigma v_1 = u_1$, $u \circ \Sigma v_2 = u_2$.

This definition may be domesticated by the following observation.

PROPOSITION 3.3. *Suppose $\Sigma\colon \widetilde{J}_1 \to \widetilde{J}_1$ has a right $\widetilde{J}_0$-adjoint $\Omega\colon \widetilde{J}_0 \to \widetilde{J}_0$, that is, there is a natural equivalence*

$$\widetilde{J}_1(\Sigma X, Y) \cong \widetilde{J}_1(X, \Omega Y), \quad X \in |J_1|, \ Y \in |J_0|.$$

Then Σ is locally right-$\widetilde{J}_0$-adjunctable.

Proof. Let $e\colon \Sigma\Omega \to 1$ be the counit of the adjunction, and let $f'\colon X \to \Omega Y$ be the adjoint of $f\colon \Sigma X \to Y$, so that

(3.4) $$f = e \circ \Sigma f'.$$

We fulfil condition (i) of Definition 3.2 by taking $Z = \Omega Y$, $g = f'$, $u = e$. As to condition (ii), we first remark that if $u_1 \circ \Sigma g_1 = u_2 \circ \Sigma g_2$, then $u_1' g_1 = u_2' g_2$. Thus we fulfil condition (ii) by taking $Z = \Omega Y$, $v_1 = u_1'$, $v_2 = u_2'$, $u = e$.

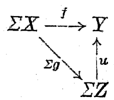
On the other hand we will produce examples below where Σ is locally right-$\widetilde{J}_0$-adjunctable but does not have a right $\widetilde{J}_0$-adjoint (let alone a right $\widetilde{J}_0$-adjoint extendable to a right $\widetilde{J}_1$-adjoint).

We now prove the main result of this section.

THEOREM 3.5. *If J_0 is J_1-adapted and Σ is locally right-$\widetilde{J}_0$-adjunctable, then $\sigma_1\colon h_1^n \to h_1^{n+1}\Sigma\colon J_1 \to \mathrm{Ab}$ is a natural equivalence.*

Proof. We must show that σ_1 given by (3.1) is onto and one-one.

σ_1 is onto. We construct

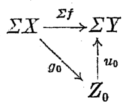


as condition (i) of Definition 3.2 permits. Given $[\alpha, f] \in h_1(\Sigma X)$, let $\beta \in h_1(Z)$ be such that $\sigma\beta = h(u)\alpha$. Then

$$[\alpha, f] = [\sigma\beta, \Sigma g] = \sigma_1[\beta, g],$$

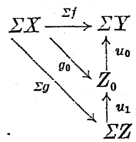
so σ_1 is onto.

σ_1 is one-one. Let $[\alpha, f] \in h_1(X)$ and suppose $\sigma_1[\alpha, f] = 0$. This implies a commutative diagram



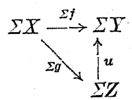
$\alpha \in h(Y)$,

with $h(u_0)\sigma\alpha = 0$. By condition (i) we may augment this diagram to



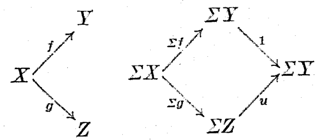
so we may assume

(3.6)



with $h(u)\sigma\alpha = 0$.

Now consider



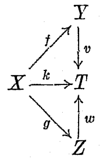
By condition (ii) of Definition 3.2, we may find

(3.7) $$v\colon Y \to T, \quad w\colon Z \to T, \quad s\colon \Sigma T \to \Sigma Y \quad \text{in} \quad \widetilde{J}_0$$

with $vf = wg$, $s \circ \Sigma v = 1$, $s \circ \Sigma w = u$. Set

(3.8) $$k = vf = wg$$

and consider



Define $\beta \in h(T)$ by $h(s)\sigma\alpha = \sigma\beta$. Then $h(\Sigma v)\sigma\beta = \sigma\alpha$, since $s \circ \Sigma v = 1$, so that

$$h(v)\beta = \alpha$$

and $[\alpha, f] = [\beta, k]$. Also $h(\Sigma w)\sigma\beta = h(u)\sigma\alpha = 0$, since $s \circ \Sigma w = u$, so that

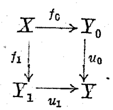
$$h(w)\beta = 0 .$$

Thus $[\alpha, f] = [\beta, k] = [h(w)\beta, g] = 0$, and the theorem is proved.

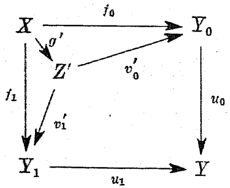
COROLLARY 3.9. *If J_0 is J_1-adapted and Σ is locally right-$\widetilde{J}_0$-adjunctable, then the Kan extension of a cohomology theory (h, σ) is a cohomology theory (h_1, σ_1).*

Since the hypotheses of Corollary 3.9 are substantially weaker than those of Theorems 3.9 and 3.10 cf [5] all the examples given in that paper (see especially section 4 of [5]) are applications of Corollary 3.9. We now give two further examples (among many) to indicate the wider scope of Corollary 3.9.

EXAMPLE 3.10. Let J_0 be the category of finite CW-complexes [5] and let J_1 be the category of compact spaces. Then J_0 is certainly admissible and closed under finite products. We verify that $\widetilde{J}_0$ has weak local pull-backs relative to $\widetilde{J}_1$. For, given



we can construct a *first approximation*



([5]) Recall that we always insist that our categories J are *admissible* for a cohomology theory. Thus J_0 is really the category of spaces of the (based) homotopy type of finite CW-complexes. We will, however, always permit ourselves this abuse of language

where Z' is the weak (homotopy-) pull-back (see Proposition 3.2 of [5]) of u_0 and u_1. Z' is not in J_0 in general but, by Milnor's Theorem [12] it may be assumed to be a CW-complex and hence $g'X$ lies in some finite subcomplex Z of Z',

$$g'X \subseteq Z \subseteq Z' .$$

If $g\colon X \to Z$ is just g' with restricted range, and $v_i\colon Z \to Y_i$, $i = 1, 2$, is just v_i' with restricted domain, then we obtain the diagram verifying the condition given in Definition 2.11.

Finally we show that $\Sigma\colon \widetilde{J}_1 \to \widetilde{J}_1$ is locally right-$\widetilde{J}_0$-adjunctable. To obtain the factorization

$$\Sigma X \xrightarrow{\Sigma g} \Sigma Z \xrightarrow{u} Y$$

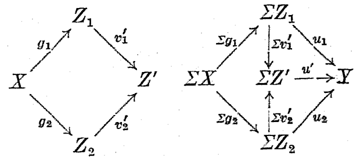
of $f\colon \Sigma X \to Y$, we take as our first approximation

$$\Sigma X \xrightarrow{\Sigma f'} \Sigma Z' \xrightarrow{e} Y,$$

$Z' = \Omega Y$, the loop-space on Y; f' adjoint to f.

Again by Milnor's Theorem we may assume Z' to be a CW-complex and then we find $Z \subseteq Z'$, a finite subcomplex containing $f'X$. Then we define $g\colon X \to Z$ by restricting the range of f' and $u\colon \Sigma Z \to Y$ by restricting the domain of e. As to part (ii) of Definition 3.2 we take $Z' = \Omega Y$, $v_i' = u_i'$, adjoint to u_i, $i = 1, 2$, $u' = e\colon \Sigma\Omega \to Y$ as our first approximation, yielding

(3.11)

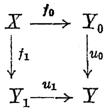


Recall that our arrows are homotopy classes, so that the left hand diagram of (3.11) represents a homotopy-commutative diagram $v_1'g_1 \simeq v_2'g_2$. The homotopy is a map $F\colon X \times I \to Z'$ so $F(X \times I)$ lies in some finite subcomplex Z of Z', chosen so that v_1', v_2' have their images in Z. Then we verify part (ii) of Definition 3.2 by taking v_1, v_2 to be v_1', v_2' with range restricted to Z, and $u\colon \Sigma Z \to Y$ to be the restriction of u'.

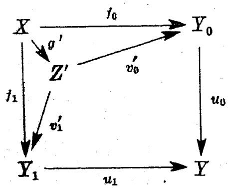
We now give an example which will figure prominently in the sequel. Let $\boldsymbol{C}$ be an acyclic ring (Serre class [6] [13]) of abelian groups; we say that a CW-complex Y *belongs to* $\boldsymbol{C}$ if it is 1-connected and all its (reduced) integral homology groups or, equivalently, all its homotopy groups belong to $\boldsymbol{C}$.

([6]) We always assume that our Serre classes $\boldsymbol{C}$ satisfy conditions IIA and III of [13].

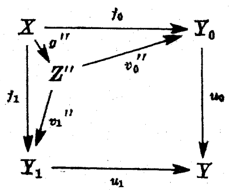
EXAMPLE 3.12. Let J_0 be the category of finite CW-complexes belonging to a Serre class $\boldsymbol{C}$ and let J_1 be the category of 1-connected finite-dimensional CW-complexes. Then J_0 is admissible since the mapping cone C_f of a map $f\colon A \to X$ of finite 1-connected complexes is finite and 1-connected, and the homology groups of C_f belong to $\boldsymbol{C}$ by a simple application of the exact homology sequence. J_0 is plainly closed under finite products and we verify that $\widetilde{J}_0$ has weak local pull-backs relative to $\widetilde{J}_1$. As in Example 3.10, we start from



and construct the first approximation



where Z' is the weak (homotopy-) pull-back of u_0 and u_1. An easy argument using the homotopy sequence of a fibration shows that the homotopy groups of Z' belong to $\boldsymbol{C}$; and, as already argued, we may assume Z' to be a CW-complex. As a second approximation we take Z'' to be the universal cover of Z'. Since X is 1-connected, g' lifts to $g''\colon X \to Z''$ and we obtain



Since Y_0, Y, Y_1 are all 1-connected finite complexes, their homotopy groups are finitely-generated. So therefore are the homotopy groups of Z' and Z'' and hence the homology groups of Z''. Thus $Z'' \in \boldsymbol{C}$ and the skeleta of Z'' may be taken to be finite complexes. Now Z'' admits a *homology decomposition* (see [6]). By varying Z'' within its homotopy type (but retaining the property that its skeleta are finite complexes) we may

suppose that, for each k, there is a subcomplex $Z''^{(k)}$ of Z'' such that

(i) $Z''_k \subseteq Z''^{(k)} \subseteq Z''_{k+1}$, where Z''_k is the k-skeleton of Z'';

(ii) $H_i(Z''^{(k)}) = 0$, $i > k$;

(iii) the inclusion $Z''^{(k)} \subseteq Z''$ induces an isomorphism in homology in dimensions $\leqslant k$.

Conditions (ii) and (iii) guarantee that $Z''^{(k)} \in \boldsymbol{C}$ and condition (i) then guarantees that $Z''^{(k)} \in |J_0|$. If $\dim X \leqslant k$, then g'' may be deformed into Z''_k and thus we verify the condition given in Definition 2.11 by taking $Z = Z''^{(k)}$ and defining $g\colon X \to Z$ by restricting the range of g'' (after submitting g'' to the necessary deformation), and $v_i\colon Z \to Y_i$, $i = 1, 2$, by restricting the domain of v''_i.

The argument showing that $\Sigma\colon \widetilde{J}_1 \to \widetilde{J}_1$ is locally right-$\widetilde{J}_0$-adjunctable now follows very similar lines, replacing the weak homotopy-pull-back by the loop-space. Thus, without giving all the details, we obtain the required factorization

$$\Sigma X \xrightarrow{\Sigma g} \Sigma Z \xrightarrow{u} Y$$

of $f\colon \Sigma X \to Y$ by first setting $Z' = \Omega Y$, then setting (7) $Z'' = \widetilde{\Omega} Y$, the universal cover of ΩY, and finally setting Z = suitable homology section $Z''^{(k)}$. Again, part (ii) of Definition 3.2 is verified by first setting $Z' = \Omega Y$, $v'_i = u'_i$, adjoint to $u_i\colon \Sigma Z_i \to Y$, $i = 1, 2$, then setting $Z'' = \widetilde{\Omega} Y$, and finally setting $Z = Z''^{(l)}$ if X is k-dimensional and $l = \max\{\dim Z_1, \dim Z_2, k+1\}$. With this choice of Z we may suppose v''_i maps Z_i into Z and $v''_1 g_1 \simeq v''_2 g_2$ as maps $X \to Z$. The reader should easily be able to fill in the gaps in our description of this argument.

This example figures very prominently in the next section. We recall from [5] the remark that the restriction to 1-connected complexes in this, and other, examples has no restricting effect on the scope of the extended theory h_1, since any theory defined on 1-connected complexes extends to all complexes, together with the full apparatus of additional algebraic structure, by passing to the double suspension.

4. The main theorem. Let P be a family of prime numbers and let $\boldsymbol{C}_P$ be the Serre class consisting of torsion abelian groups A such that the order of any $a \in A$ is a product of members of P. Let $\boldsymbol{Q}_1$ be the group of rationals mod 1 and let $\boldsymbol{Z}_{P^\infty}$ be the subgroup of $\boldsymbol{Q}_1$ which is the direct sum of its p-components, $p \in P$. Let h be a cohomology theory and let $h(\ ;\boldsymbol{Z}_{P^\infty})$ be the theory obtained from h by introducing the coefficient group $\boldsymbol{Z}_{P^\infty}$ as in [9]. We specialize Example 3.12 by taking $\boldsymbol{C} = \boldsymbol{C}_P$ and prove

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(7) Of course, $\widetilde{\Omega}$ is right adjoint to Σ on the category of 1-connected CW-complexes.

THEOREM 4.1. *Let h be a representable cohomology theory with finitely-generated coefficients* (8), *and let h_0 be its restriction to the category J_0 of 1-connected finite CW-complexes belonging to C_P. Then the Kan extension h_1 of h_0 to the category J_1 of 1-connected finite-dimensional CW-complexes is given by*

$$h_1^n(X) \cong h^{n-1}(X; Z_{P^\infty}) . \tag{4.2}$$

The proof of this theorem will be achieved with the help of a series of lemmas; the lemmas themselves may well have an independent interest.

LEMMA 4.3. *Let Y be a finite CW-complex belonging to C_P and let h be a cohomology theory. Then $h^n(Y) \in C_P$.*

Proof. We apply the generalized Atiyah–Hirzebruch spectral sequence [1], [7]. Then

$$E_1^{p,q} \cong H^p(Y; \breve{h}^{q-p}) \cong \mathrm{Hom}(H_p(Y), \breve{h}^{q-p}) \oplus \mathrm{Ext}(H_{p-1}(Y), \breve{h}^{q-p}) ,$$

so that $E_1^{p,q} \in C_P$, since $H_r(Y)$ is of finite type. (Notice that $E_1^{p,q} = 0$ if $p = 0$ since we are using reduced cohomology and Y is connected.)

It follows that $E_\infty^{p,q} \in C_P$ and since $h^n(Y)$ has a finite filtration with quotients belonging to C_P, we deduce that $h^n(Y)$ itself belongs to C_P.

LEMMA 4.4. *Let $G \in C_P$. Then $G \otimes Z_{P^\infty} = 0$, and there is a natural isomorphism*

$$\mathrm{Tor}(G, Z_{P^\infty}) \cong G .$$

Proof. Since G is a torsion group and Z_{P^∞} is divisible, it follows that $G \otimes Z_{P^\infty} = 0$. To prove the second assertion it is plainly legitimate to replace Z_{P^∞} by Q_1. Then consider the sequence

$$0 \to Z \to Q \to Q_1 \to 0 ;$$

this gives rise to the exact sequence, natural in G,

$$\mathrm{Tor}(G, Q) \to \mathrm{Tor}(G, Q_1) \to G \otimes Z \to G \otimes Q .$$

Now $G \otimes Q = 0$ by the previous argument and $\mathrm{Tor}(G, Q) = 0$ since Q is torsion-free. Since $G \otimes Z \cong G$, the lemma is proved.

LEMMA 4.5. *Let Y be a 1-connected finite CW-complex belonging to C_P and let h be a cohomology theory. There is then a natural equivalence of cohomology theories on the category J_0 of such complexes Y, given by*

$$h^n(Y) \cong h^{n-1}(Y; Z_{P^\infty}) .$$

(8) We must distinguish between the *coefficients* of a theory h, that is, the graded group $h(S^0)$, and the introduction of a group G as a coefficient group into a theory h, as in [9].

Proof. We have (see [9]) a short exact sequence

$$0 \to h^{n-1}(Y) \otimes Z_{P^\infty} \to h^{n-1}(Y; Z_{P^\infty}) \to \mathrm{Tor}\big(h^n(Y), Z_{P^\infty}\big) \to 0 , \tag{4.6}$$

which is natural in Y and commutes with suspension. By Lemma 4.3, $h^{n-1}(Y)$ and $h^n(Y)$ belong to C_P; thus by Lemma 4.4 and (4.6) we have isomorphisms

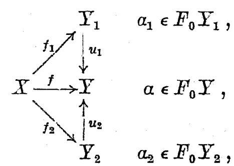
$$h^{n-1}(Y; Z_{P^\infty}) \cong \mathrm{Tor}\big(h^n(Y), Z_{P^\infty}\big) \cong h^n(Y) ,$$

which are natural in Y and commute with suspension.

LEMMA 4.7. *Let J_0 be J_1-adapted, let $F_0\colon \tilde{J}_0^{\mathrm{opp}} \to \mathfrak{S}$ be a contravariant functor from $\tilde{J}_0$ to the category of sets $\mathfrak{S}$, and let $F_1\colon \tilde{J}_1^{\mathrm{opp}} \to \mathfrak{S}$ be an extension of F_0 to $\tilde{J}_1^{\mathrm{opp}}$ such that*

(i) *every $\xi \in F_1 X$ is expressible as $\xi = (F_1 f)(a)$, $a \in F_0 Y$, $Y \in |J_0|$, $f\colon X \to Y$ in $\tilde{J}_1$;*

(ii) *if $(F_1 f_1)(a_1) = (F_1 f_2)(a_2)$, there exists a commutative diagram*



with $(F_0 u_1)(a) = a_1$, $(F_0 u_2)(a) = a_2$.

Then F_1 is the Kan extension of F_0.

(Compare the Proposition on p. 431 of [11]).

Proof. If F is the Kan extension of F_0, there is a canonical natural transformation $\omega\colon F \to F_1$ which is the identity on $\tilde{J}_0$. Now the theory of Section 2 applies to functors to sets as well as to functors to groups — and is, indeed, more elementary in this case (9). Thus we have the representation of FX given by (2.16), and condition (i) asserts then that ω is surjective, while condition (ii) asserts that ω is injective.

LEMMA 4.8. *Let J_0 be J_1-adapted. If $F\colon \tilde{J}_1^{\mathrm{opp}} \to \mathfrak{S}$ is a direct limit of functors representable in $\tilde{J}_0$, then F is the Kan extension of $F|\tilde{J}_0^{\mathrm{opp}}$.*

Proof. We have $F(X) = \varinjlim_\beta [X, T_\beta]$, $T_\beta \in |J_0|$. We proceed to verify (i) and (ii) of Lemma 4.7 for $F_1 = F$, $F_0 = F|\tilde{J}_0^{\mathrm{opp}}$. Set $F_\beta = [\ , T_\beta]$ and

(9) The index category I need only satisfy condition (a) at the beginning of Section 2 in order to be adapted for colimits of functors to sets. The construction of the colimit set is then exactly as in Section 2, except that we do not have to worry about group structure.

let $\eta_\beta \in F_\beta(T_\beta)$ be the class of the identity map of T_β. Let $\xi \in F(X)$ be represented by $f\colon X \to T_\beta$. Then $f = (F_\beta f)(\eta_\beta)$, so that $\xi = (Ff)[\eta_\beta]$ where $[\eta_\beta] \in FT_\beta$ is represented by η_β. This proves (i).

To verify (ii), we have

$$\begin{array}{ccc} & \overset{f_1}{\nearrow} & Y_1 \xrightarrow{g_1} T_{\beta_1} \\ X & & \\ & \underset{f_2}{\searrow} & Y_2 \xrightarrow[g_2]{} T_{\beta_2} \end{array}$$

where $a_i \in F(Y_i)$ is represented by $g_i \in F_{\beta_i}(Y_i)$, $i = 1, 2$. We may find T_β and maps $v_i\colon T_{\beta_i} \to T_\beta$, $i = 1, 2$, such that $v_1 g_1 f_1 = v_2 g_2 f_2$, since $(Ff_1)(a_1) = (Ff_2)(a_2)$. Then $F(v_i g_i)[\eta_\beta] = a_i$, $i = 1, 2$, and the lemma is proved. Such a functor F we may describe as $\tilde{J}_0$-*prorepresentable* [3].

Lemma 4.9. *If h is a representable cohomology theory with finitely-generated coefficients, then the functor $h^n(\ ;\boldsymbol{Z}_{p^k})\colon \tilde{J}_1^{\rm opp} \to \mathfrak{S}$ is $\tilde{J}_0$-prorepresentable, where J_0, J_1 are as in Theorem 4.1 and $p \in P$.*

Proof. Let h^n be represented by M_n, $h^n(X) = [X, M_n]$. Now (see [9])

$$h^n(X; \boldsymbol{Z}_{p^k}) = h^{n+4}(X \wedge L_k), \qquad L_k = L\boldsymbol{Z}_{p^k}$$

so [10] that $h^n(\ ;\boldsymbol{Z}_{p^k})$ is represented by $M_{n+4}^{L_k} = Q_n$. We may assume Q_n 1-connected (replacing it by its universal cover if necessary) since J_1 consists of 1-connected spaces. Then

$$\pi_i(Q_n) = \pi_{i+4}(M_{n+4}; \boldsymbol{Z}_{p^k})$$

and we have the universal coefficient sequence

$$(4.10) \qquad 0 \to \pi_{i+4}(M_{n+4}) \otimes \boldsymbol{Z}_{p^k} \to \pi_i(Q_n) \to \mathrm{Tor}\big(\pi_{i+3}(M_{n+4}), \boldsymbol{Z}_{p^k}\big) \to 0\,.$$

Since the homotopy groups of M_{n+4} are finitely-generated, it follows that $\pi_i(Q_n)$ is a finite group belonging to $\boldsymbol{C}_p$ and hence to $\boldsymbol{C}_P$. It follows as for Example 3.12 that we may find a homology decomposition

$$\ldots \subseteq Q_n^{(m)} \subseteq Q_n^{(m+1)} \subseteq \ldots$$

of Q_n with each $Q_n^{(m)}$ a finite complex belonging to $\boldsymbol{C}_P$. Thus, X being finite-dimensional,

$$h^n(X; \boldsymbol{Z}_{p^k}) = [X, Q_n] = \varinjlim_m [X, Q_n^{(m)}], \qquad Q_n^{(m)} \in |J_0|,$$

and the lemma is proved.

---

[10] If G is a finitely-generated abelian group, then the polyhedron LG is 1-connected, $H^4(LG) = G$, $H^i(LG) = 0$, $i \neq 4$.

By H we understand, here and throughout this section, ordinary (cellular) reduced cohomology with integer coefficients.

Lemma 4.11. *Let J_0, J_1, h be as in Theorem 4.1, and let*

$$F = h(\ ;\boldsymbol{Z}_{P^\infty})|J_0\,.$$

Then the Kan extension of F to J_1 is $h(\ ;\boldsymbol{Z}_{P^\infty})$.

Proof. Given a functor $S\colon \tilde{J}_1^{\rm opp} \to \mathfrak{S}$, let us also write S for its restriction S_0 to $\tilde{J}_0^{\rm opp}$ and $\bar{S} = \mathrm{Kan}\, S$ for the Kan extension of S_0 to $\tilde{J}_1^{\rm opp}$. (We could also write $\bar{S}(X) = \varinjlim_f S(Y)$, for $f\colon X \to Y$, $Y \in |J_0|$.)

Set $\boldsymbol{Z}_{P^k} = \bigoplus_{p \in P} \boldsymbol{Z}_{p^k}$; it follows from the universal coefficient theorem, (see [9]), that

$$(4.12) \qquad h(\ ;\boldsymbol{Z}_{P^k}) = \bigoplus_{p \in P} h(\ ;\boldsymbol{Z}_{p^k})\,.$$

Put $F_k = h(\ ;\boldsymbol{Z}_{P^k})$, $F_{kp} = h(\ ;\boldsymbol{Z}_{p^k})$, so that

$$F_k = \bigoplus_p F_{kp}\,,$$

and

$$\bar{F}_k = \mathrm{Kan}(F_k) = \mathrm{Kan}\big(\bigoplus_p F_{kp}\big) = \bigoplus_p \mathrm{Kan}(F_{kp}) = \bigoplus_p F_{kp}\,, \quad \text{by Lemmas 4.8 and 4.9,}$$
$$= F_k\,.$$

Finally represent $\boldsymbol{Z}_{P^\infty}$ as the direct limit (union) of the inclusions

$$(4.13) \qquad \boldsymbol{Z}_P \rightarrowtail \boldsymbol{Z}_{P^2} \rightarrowtail \ldots \rightarrowtail \boldsymbol{Z}_{P^k} \rightarrowtail \boldsymbol{Z}_{P^{k+1}} \rightarrowtail \ldots$$

Then, (Theorem 4.18 of [2]), $h(X; \boldsymbol{Z}_{P^\infty}) = \varinjlim_k h(X; \boldsymbol{Z}_{P^k})$, so that

$$\bar{F}(X) = \varinjlim_f F(Y) = \varinjlim_f \varinjlim_k F_k(Y) = \varinjlim_k \varinjlim_f F_k(Y) = \varinjlim_k F_k(X) = F(X)\,,$$

and the lemma is proved.

Proof of Theorem 4.1. The proof is now virtually immediate. By Lemma 4.5, there is a natural equivalence

$$h^n \cong h^{n-1}(\ ;\boldsymbol{Z}_{P^\infty})$$

of cohomology theories defined on J_0. The Kan extensions of these theories are therefore also naturally equivalent. But Lemma 4.11 asserts that the Kan extension of $h^{n-1}(\ ;\boldsymbol{Z}_{P^\infty})$ to J_1 is again $h^{n-1}(\ ;\boldsymbol{Z}_{P^\infty})$. This proves (4.2).

Remark. The requirement that h be representable is not, in fact, a limitation on the scope of Theorem 4.1. For if h is a cohomology theory with finitely-generated coefficients, then its restriction to the category of 1-connected finite CW-complexes is certainly representable. If this representable theory is called h', then h and h' may, of course, not coincide over the whole of J_1; and Theorem 4.1 enables us to infer that $h_1^n(X) = h'^{(n-1)}(X;\ \boldsymbol{Z}_{P^\infty})$.

We now introduce two variants of Theorem 4.1.

THEOREM 4.14. *Let h be a representable cohomology theory, let h_0 be its restriction to the category of 1-connected finite CW-complexes belonging to $\boldsymbol{C}_P$ and let h_1 be the Kan extension of h_0 to the category of 1-connected finite complexes. Then* (4.2) *holds.*

Notice that, compared with Theorem 4.1, we have removed a restriction from h, but imposed a restriction on J_1. Let us write J_2 for this new version of J_1; thus J_2 is the category of 1-connected finite CW-complexes. It is plain from a scrutiny of the proof of Theorem 4.1 that the only lemma requiring amendment in order to prove Theorem 4.14 is Lemma 4.9. Thus we must prove

LEMMA 4.9'. *If h is a representable cohomology theory, then the functor $h^n(\ ;\boldsymbol{Z}_{p^k})$: $\tilde{J}_2^{\rm opp} \to \mathfrak{S}$ is $\tilde{J}_0$-prorepresentable, where J_0 is as in Theorem* 4.1 *and J_2 is as above, $p \in P$.*

We base the proof of this lemma on the following general proposition concerning Serre classes.

PROPOSITION 4.15. *Let $\boldsymbol{C}$ be a Serre class and let $f\colon X \to Q$ be a map of a finite 1-connected CW-complex X into a CW-complex Q belonging to $\boldsymbol{C}$. Then f may be factored up to homotopy as*

$$X \xrightarrow{g} Y \xrightarrow{v} Q$$

where Y is a finite CW-complex belonging to $\boldsymbol{C}$.

Proof. Suppose $\pi_i(X) \in \boldsymbol{C}$, $i < m$, $m \geqslant 2$. Then if $A = \ker f_*\colon \pi_m(X) \to \pi_m(Q)$, A is a finitely-generated abelian group and we may attach $(m+1)$-cells e_γ to X, corresponding to each generator γ of a given generating set for A, by a map in the class γ. Let $X_m = X \cup (e_\gamma)$ result. Then X_m is finite and f extends to $f_m\colon X_m \to Q$. Moreover $\pi_m(X_m) = \pi_m(X)/A$ and $f_{m*}\colon \pi_m(X_m) \to \pi_m(Q)$ is a monomorphism. Thus $\pi_m(X_m) \in \boldsymbol{C}$. We may proceed inductively in this way and eventually arrive at a factorization

$$X \xrightarrow{i} X_\infty \xrightarrow{f_\infty} Q \tag{4.16}$$

where X_∞ belongs to $\boldsymbol{C}$ and is finite in each dimension. Thus the homology sections of X_∞ are finite complexes belonging to $\boldsymbol{C}$ and, since X is finite, i factors (up to homotopy) through some homology section Y of X_∞. Thus we obtain the required factorization by restricting the domain of f_∞ and the range of i in (4.16) to this homology section Y.

Proof of Lemma 4.9'. We proceed as in the proof of Lemma 4.9, obtaining a 1-connected CW-complex Q_n in $\boldsymbol{C}_p$ such that

$$h^n(X;\boldsymbol{Z}_{p^k}) = [X, Q_n]\,.$$

Thus to prove the lemma we must show that every homotopy class $X \xrightarrow{f} Q_n$ may be factored as

$$X \xrightarrow{g} Y \xrightarrow{v} Q_n\,, \qquad Y \in |J_0|\,,$$

and, second, that given two factorizations

$$\begin{array}{ccccc} & & Y_1 & & \\ & {}^{g_1}\nearrow & & \searrow^{v_1} & \\ X & & & & Q_n \\ & {}_{g_2}\searrow & & \nearrow_{v_2} & \\ & & Y_2 & & \end{array} \tag{4.17}$$

we may embed (4.17) in a commutative diagram

$$\begin{array}{ccccc} & & Y_1 & & \\ & {}^{g_1}\nearrow & \downarrow{\scriptstyle u_1} & \searrow^{v_1} & \\ X & & Y & \xrightarrow{w} & Q_n\,, \qquad Y \in |J_0|\,. \\ & {}_{g_2}\searrow & \uparrow{\scriptstyle u_2} & \nearrow_{v_2} & \\ & & Y_2 & & \end{array} \tag{4.18}$$

Now the first assertion follows immediately from Proposition 4.15, so it remains to prove the second. We may take the double mapping cylinder of g_1 and g_2; this amounts to replacing g_1 and g_2 by cofibration-inclusions and then taking the union Z of Y_1 and Y_2 with X amalgamated. We may then further suppose that $v_1g_1 = v_2g_2$ as maps (and not merely as homotopy classes) so that we have a commutative diagram

$$\begin{array}{ccccc} & & Y_1 & & \\ & {}^{g_1}\nearrow & \downarrow{\scriptstyle u_1'} & \searrow^{v_1} & \\ X & & Z & \xrightarrow{w'} & Q_n \\ & {}_{g_2}\searrow & \uparrow{\scriptstyle u_2'} & \nearrow_{v_2} & \\ & & Y_2 & & \end{array}$$

By Van Kampen's Theorem Z is 1-connected and it is certainly finite. Thus we may apply Proposition 4.15 to $w'\colon Z \to Q_n$ and immediately infer the existence of a diagram (4.18). This completes the proof of Lemma 4.9' and with it Theorem 4.14.

The second variant of Theorem 4.1 is concerned with ordinary cohomology H. We now take J_0 to be the category of 1-connected CW-complexes whose homology groups are finite groups in $\boldsymbol{C}_P$ and J_1 to be the category of 1-connected CW-complexes. It was already observed in [5] that the Kan extension, from J_0 to J_1, of any cohomology theory is again a cohomology theory. We prove

THEOREM 4.19. *If h_1 is the Kan extension of H from J_0 to J_1, where J_0, J_1 are as in the paragraph above, then*

$$h_1^n(X) = H^{n-1}(X; \boldsymbol{Z}_{P^\infty}),$$

the latter group being taken in the sense of [9].

Notice that, compared with Theorem 4.1 we have greatly restricted h, but we have also greatly enlarged J_0 and J_1 by allowing infinite-dimensional complexes.

Proof of Theorem 4.19. The argument is substantially easier than that of Theorem 4.1. $H^n(Y) \in \boldsymbol{C}_P$ if $Y \in |J_0|$ and the analogue of Lemma 4.5 continues to hold,

$$H^n(Y) \cong H^{n-1}(Y; \boldsymbol{Z}_{P^\infty}), \qquad Y \in |J_0| .$$

The analogue of Lemma 4.9 is the stronger, but trivial, statement that $H^n(\ ; \boldsymbol{Z}_{p^k})$ is $\tilde{J}_0$-representable, since the Eilenberg–MacLane complex $K(Z_{p^k}, n)$ is in J_0. The rest of the argument holds and the theorem follows.

Remarks. (i) Notice that although $H^n(\ ; \boldsymbol{Z}_{P^\infty})$ has its usual meaning in J_0, it does not have its usual meaning in J_1. For if the homology groups of X are not of finite type, then $C^\cdot(X) \otimes \boldsymbol{Z}_{P^\infty}$ and $\mathrm{Hom}\big(C_\cdot(X), \boldsymbol{Z}_{P^\infty}\big)$ are not cochain-equivalent. The cohomology groups of the former cochain complex enter into the statement of Theorem 4.19; those of the latter are the usual cohomology groups of X with values in $\boldsymbol{Z}_{P^\infty}$.

(ii) We may elaborate Theorem 4.19 by looking to see what happens if we replace H by $H(\ ; G)$. If G is a torsion group and $G = G_P \oplus G_{P'}$ where $G_P \in \boldsymbol{C}_P$ and P' is the set of primes complementary to P, then it is not difficult to see that the Kan extension of $H(\ ; G)$ is $H(\ ; G_P)$, where again the latter has to be understood in the sense of [9].

(iii) With regard to Theorem 4.1, 4.14 or 4.19, the result implies, of course, a natural transformation

$$\omega\colon h^{n-1}(X; \boldsymbol{Z}_{P^\infty}) \to h^n(X)$$

of cohomology theories on J_1 which is an equivalence on J_0. This natural transformation ω is nothing other than the composite

$$(4.20) \quad h^{n-1}(X; \boldsymbol{Z}_{P^\infty}) \to \mathrm{Tor}\big(h^n(X), \boldsymbol{Z}_{P^\infty}\big) \to \mathrm{Tor}\big(h^n(X), \boldsymbol{Q}_1\big) \to h^n(X) \otimes \boldsymbol{Z} \cong h^n(X) .$$

The intermediate terms in (4.20) are not cohomology theories, but they admit suspension isomorphisms. The arrows are all compatible with suspension and hence the composite is a natural transformation of cohomology theories. Of course, if $X \in |J_0|$, then each arrow in (4.20) is an isomorphism.

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