ω_μ-metric spaces and ω_μ-proximities

by

Frederick W. Stevenson (Tucson, Arizona)

§ 1. Introduction. Let ω_μ denote the μth infinite initial ordinal and let $\aleph_\mu$ denote this ordinal when considered as a cardinal number. An ω_μ-metric ϱ on a set X is a generalization of the ordinary concept of a metric in which the range of ϱ rather than being the non-negative reals, consists of the non-negative elements of an arbitrary linearly ordered abelian group of character ω_μ. Shu-Tang [6] established necessary and sufficient conditions for the ω_μ-metrizability of a topological space and Stevenson and Thron [8] gave necessary and sufficient conditions for the ω_μ-metrizability of a uniform space. In § 2 we provide necessary and sufficient conditions for the ω_μ-metrizability of a proximity space.

An ω_μ-metric ϱ on X naturally induces a uniformity, $\mathfrak{U}_\varrho$ a topology, $\mathfrak{T}_\varrho$, and a proximity relation δ_ϱ, on X. The uniformity $\mathfrak{U}_\varrho$ satisfies the property that the intersection of a family of entourages of cardinality $< \aleph_\mu$ is an entourage. Such uniformities we shall call ω_μ-uniformities. Similarly the topology $\mathfrak{T}_\varrho$ satisfies the property that the intersection of a family of open sets of cardinality $< \aleph_\mu$ is an open set. Such topological spaces are called ω_μ-additive spaces and have been studied by Sikorski [7] and Shu-Tang [6]. The proximity relation δ_ϱ satisfies the property that if the union of a family of sets of cardinality $< \aleph_\mu$ is near a given set A then at least on member of the family is near A. Such proximities, called ω_μ-proximities, are examined in § 3.

§ 2. ω_μ-metrizability of a proximity space. The metrizability of a proximity space has been characterized by Efremovic [2], Efremovic and Svarc [3], and Leader [4]. It is Leader's characterization which we shall generalize here.

THEOREM (Leader [4]). *For any proximity space (X, δ) the following three conditions are equivalent:*

(i) *(X, δ) is metrizable.*

(ii) *There exists a sequence $\{\mathcal{C}_n\}$ of admissible coverings with $\mathcal{C}_{n+1}$ a refinement of $\mathcal{C}_n$ such that $A\delta B$ iff for every n there exists $D \in \mathcal{C}_n$ such that $D \cap A \neq \emptyset$ and $D \cap B \neq \emptyset$.*

(iii) *There exists a sequence* $\{U_n\}$ *of admissible symmetric entourages with* $U_{n+1} \subseteq U_n$ *such that* $A\delta B$ *iff* $A \times B \cap U_n \neq \emptyset$ *for all* n.

The following definitions explain the terms above. A set $P \subset X \times X$ is *compressed* iff P is infinite and $A\delta B$ for every pair of subsets A and B of X where $A \times B \cap P$ is infinite. A covering $\mathcal{C}$ of X is *admissible* iff for every compressed set P there exists $(x, y) \in P$ and $C \in \mathcal{C}$ such that $x, y \in C$. An *entourage* U is a subset of $X \times X$ containing the diagonal; an entourage U is *admissible* iff $U \cap P \neq \emptyset$ for every compressed set P.

As might be expected the generalization of this theorem extends to an arbitrary regular cardinal number those concepts above which are based on countability. The proof that we offer here is not, however, a generalization of Leader's proof. The extended theorem is preceded by some auxiliary definitions and three lemmata.

Definition 2.1. Let ω_μ be a regular cardinal number, let $\{x_\alpha\colon \alpha < \omega_\mu\}$ and $\{y_\alpha\colon \alpha < \omega_\mu\}$ be ω_μ-sequences, let $X = \{x_\alpha, y_\alpha\colon \alpha < \omega_\mu\}$ and let $\mathcal{C} = \{C_k\colon k \in K\}$ be a cover of X. The cover is said to be *permissible* with respect to x_α, y_α iff $x_\alpha, y_\alpha \notin C_k$ for all $\alpha < \omega_\mu$ and $k \in K$ and for every ω_μ-subsequence $\{x_{i_\beta}\colon \beta < \omega_\mu\}$ of $\{x_\alpha\}$ and $\{y_{i_\beta}\colon \beta < \omega_\mu\}$ of $\{y_\alpha\}$ there exists at least one $k \in K$, $\beta < \omega_\mu$ and $\gamma < \omega_\mu$ such that $x_{i_\beta}, y_{i_\gamma} \in C_k$.

Lemma 2.2. *No pair of* ω_μ*-sequences has a finite permissible cover.*

Proof. We proceed by induction on the cardinality of the cover. Clearly if $\mathcal{C}$ has one element it could not be permissible with respect to any sequences $\{x_\alpha\}$ and $\{y_\alpha\}$. Suppose the lemma is true for covers with n members. Suppose also that there exists a pair of ω_μ-sequences $\{x_\alpha\}$ and $\{y_\alpha\}$ and a permissible cover $\mathcal{C} = \{C_1, \ldots, C_{n+1}\}$. We complete the proof by deriving a contradiction.

First we note that for at least one of the members C_{i_0} the set $A = \{\alpha\colon x_\alpha \in C_{i_0}\}$ has a complement of cardinality $\aleph_\mu$. If this were not true it would follow that for all $k = 1, \ldots, n+1$ there would exist $\gamma_k < \omega_\mu$ such that if $\gamma \geqslant \gamma_k$ then $x_\gamma \in C_k$. Thus it would be impossible for y_β to be in any member of the cover for $\beta > \max\{\gamma_1, \ldots, \gamma_{n+1}\}$.

Secondly we note that C_{i_0} contains elements y_α also, but those y_α which occur in only C_{i_0} and no other member of the cover are of cardinality $< \aleph_\mu$. To see this suppose that there exists a subsequence $\{y_{i_\alpha}\colon \alpha < \omega_\mu\} \subset C_{i_0}$ such that each y_{i_α} occurs only in C_{i_0}. Then there exists a C_j such that x_{i_β} and $y_{i_\gamma} \in C_j$ for some $\beta, \gamma < \omega_\mu$. Now since $y_{i_\beta} \in C_{i_0}$ it follows that $x_{i_\beta} \notin C_{i_0}$ so $j \neq i_0$. Thus we have $y_{i_\gamma} \in C_j \neq C_{i_0}$, a contradiction.

Now we delete from ω_μ those ordinals in A along with those subscripts α such that $y_\alpha \in C_{i_0}$ and $y_\alpha \notin C_j$ for $j \neq i_0$. Using the two facts noted above we see that the set of ordinals remaining, call it I, is of cardinality $\aleph_\mu$.

Consider the ω_μ-sequences $\{x_i\colon i \in I\}$ and $\{y_i\colon i \in I\}$. Clearly they are covered by $\mathcal{C}' = \{C_1, \ldots, C_{i_0-1}, C_{i_0+1}, \ldots, C_{n+1}\}$. Furthermore this cover is permissible. To see this let $\{x_{i_\alpha}\}$ and $\{y_{i_\alpha}\}$ be ω_μ-subsequences of $\{x_i\}$ and $\{y_i\}$ respectively. Now they are also ω_μ-subsequences of the original ω_μ-sequences $\{x_\alpha\}$ and $\{y_\alpha\}$ and hence there exists k, β, and γ such that $x_{i_\beta}, y_{i_\gamma} \in C_k$. Now $C_k \neq C_{i_0}$ because $x_{i_\beta} \notin C_{i_0}$. Also it is clear that x_i and $y_i \notin C_j$ for any $i \in I$ and $j = 1, \ldots, n+1$. Thus the cover $\mathcal{C}'$ is permissible and has n elements contradicting the induction hypothesis as promised.

Lemma 2.3. *If* $\{x_\alpha\}$ *and* $\{y_\alpha\}$ *are* ω_μ*-sequences and* $\mathcal{C} = \{C_1, \ldots, C_n\}$ *is a cover of* $\{x_\alpha, y_\alpha\colon \alpha < \omega_\mu\}$ *such that* $x_\alpha, y_\alpha \notin C_j$ *for all* $\alpha < \omega_\mu$ *and* $1 \leqslant j \leqslant n$, *then there exists an* ω_μ*-subsequence* $\{x_{i_\alpha}\}$ *of* $\{x_\alpha\}$ *and* $\{y_{i_\alpha}\}$ *of* $\{y_\alpha\}$ *such that no member of the cover contains elements of both subsequences.*

Proof. This follows directly from lemma 2.2.

Definition 2.4. Let (X, δ) be a proximity space and let be a regular cardinal number.

(i) A set $P \subset X \times X$ is ω_μ*-compressed* iff P has cardinality $\geqslant \aleph_\mu$ and $A\delta B$ for every pair of subsets A and B of X such that $A \times B \cap P$ has cardinality $\geqslant \aleph_\mu$.

(ii) A covering $\mathcal{C}$ of X is an ω_μ*-admissible covering* iff for every ω_μ-compressed set P there exists $(x, y) \in P$ and $C \in \mathcal{C}$ such that x and $y \in C$.

(iii) A *diagonal set* K is a subset of $X \times X$ containing the diagonal. (We save "entourage" for its familiar usage as a member of a uniformity on X.)

(iv) A diagonal set K is an ω_μ*-admissible diagonal set* iff $K \cap P \neq \emptyset$ for every ω_μ-compressed set P.

Lemma 2.5. *Let* (X, δ) *be a proximity space and let* $\{K_\alpha\colon \alpha < \omega_\mu\}$ *be a family of* ω_μ*-admissible diagonal sets such that* $K_\beta \subset K_\alpha$ *for* $\alpha < \beta$ *and such that* $A\delta B$ *iff* $A \times B \cap K_\alpha \neq \emptyset$ *for all* $\alpha < \omega_\mu$. *Then for each* $\alpha < \omega_\mu$ *there exists* $\beta < \omega_\mu$ *such that* $K_\beta \circ K_\beta \subseteq K_\alpha$.

Proof. Suppose that this theorem is not true. Then there exists $\alpha_0 < \omega_\mu$ such that $K_\alpha \circ K_\alpha \not\subseteq K_{\alpha_0}$ for all $\alpha < \omega_\mu$. It follows that there exists ω_μ-sequences $\{x_\alpha\}$, $\{y_\alpha\}$, and $\{z_\alpha\}$ such that $(x_\alpha, y_\alpha) \in K_\alpha$, $(y_\alpha, z_\alpha) \in K_\alpha$, and $(x_\alpha, z_\alpha) \notin K_{\alpha_0}$ for all $\alpha < \omega_\mu$.

Let $\mathfrak{U}$ be the totally bounded uniformity on X generated by δ. Now a member $U \in \mathfrak{U}$ is generated by a finite cover $\mathcal{C} = \{C_1, \ldots, C_n\}$ where C_i is such that $C_i \Supset B_i$ and $\{B_1, \ldots, B_n\}$ is a cover of X. (Here $\Supset$ is the order relation associated with δ defined as follows: $A \Supset B$ iff $X \sim A\,\bar{\delta}\,B$). Now $U = \{(x, y)\colon x, y \in C_i$ for some $i = 1, \ldots, n\}$.

First we show that if $U \in \mathfrak{U}$ then there exists $\beta_0 < \omega_\mu$ such that if $\alpha > \beta_0$ then $(x_\alpha, y_\alpha) \in U$. Suppose that this is not true. Then we have $U \in \mathfrak{U}$ such that for each $\alpha < \omega_\mu$ there exists $\beta > \alpha$ such that $(x_\beta, y_\beta) \notin U$.

It follows that there exists an ω_μ-subsequence $\{x_{i_\beta}\}$ of $\{x_a\}$ and $\{y_{i_\beta}\}$ of $\{y_a\}$ such that $(x_{i_\beta}, y_{i_\beta}) \notin U$. Thus by lemma 2.3 there exists an ω_μ-subsequence $\{x_\gamma\}$ of $\{x_{i_\beta}\}$ and $\{y_\gamma\}$ of $\{y_{i_\beta}\}$ such that $(x_\gamma, y_{\gamma'}) \notin U$ for all $\gamma, \gamma' < \omega_\mu$. But this statement is contradicted by the following argument. Clearly $\{x_\gamma\}$ and $\{y_\gamma\}$ are also ω_μ-subsequences of $\{x_a\}$ and $\{y_a\}$ respectively. Now $(x_\gamma, y_\gamma) \in K_a$ for all $a < \omega_\mu$ since for every a there exists $\gamma > a$ and we have $(x_\gamma, y_\gamma) \in K_\gamma \subset K_a$. Thus $A \delta B$ where $A = \{x_\gamma\colon \gamma < \omega_\mu\}$, and $B = \{y_\gamma\colon \gamma < \omega_\mu\}$. It follows that there exists γ and $\gamma' < \omega_\mu$ such that $(x_\gamma, y_{\gamma'}) \in U$. This contradiction establishes our original statement.

Secondly we show that $P = \{(x_a, z_a)\colon a < \omega_\mu\}$ is an ω_μ-compressed set. Let C and D be subsets of X such that $C \times D \cap P$ is of cardinality $\aleph_\mu$. Thus there exists ω_μ-subsequences $\{x_{i_\beta}\}$ and $\{z_{i_\beta}\}$ of $\{x_a\}$ and $\{z_a\}$ respectively such that $(x_{i_\beta}, z_{i_\beta}) \in C \times D$ for all $\beta < \omega_\mu$. Let $U \in \mathfrak{U}$ be given and let $V \in \mathfrak{U}$ such that $V \circ V \subset U$. Then by the statement shown above there exists β_0 such that if $a > \beta_0$ then $(x_a, y_a) \in V$. Similarly there exists γ_0 such that if $a > \gamma_0$ then $(y_a, z_a) \in V$. Now for $i_\beta > \max\{\beta_0, \gamma_0\}$ we have $(x_{i_\beta}, y_{i_\beta}) \in V$ and $(y_{i_\beta}, z_{i_\beta}) \in V$ and hence $(x_{i_\beta}, z_{i_\beta}) \in U$. Thus $U(C) \cap D \neq \emptyset$. Since U was arbitrary it follows that $C \delta D$.

Finally we may conclude that $K_{a_0} \cap P \neq \emptyset$ because K_{a_0} is an ω_μ-admissible diagonal set. Thus there exists $\gamma < \omega_\mu$ such that $(x_\gamma, z_\gamma) \in K_{a_0}$. This contradicts the fact that $(x_a, z_a) \notin K_{a_0}$ for all $a < \omega_\mu$ which had been derived from the negation of the theorem. This contradiction establishes the theorem.

THEOREM 2.6. *For any proximity space (X, δ) the following three conditions are equivalent:*

(i) (X, δ) *is ω_μ-metrizable.*

(ii) *There exists an ω_μ-sequence $\{\mathfrak{C}_a\}$ of ω_μ-admissible coverings with $\mathfrak{C}_a$ a refinement of $\mathfrak{C}_\beta$ for $\beta < a$ such that $A \delta B$ iff for every a there exists $D \in \mathfrak{C}_a$ such that $D \cap A \neq \emptyset$ and $D \cap B \neq \emptyset$.*

(iii) *There exists an ω_μ-sequence $\{K_a\}$ of ω_μ-admissible symmetric diagonal sets with $K_a \subseteq K_\beta$ for $\beta < a$ such that $A \delta B$ iff $A \times B \cap K_a \neq \emptyset$ for all $a < \omega_\mu$.*

Proof. The equivalence of (ii) and (iii) is straight forward so we shall concentrate on the equivalence of (i) and (iii) here.

Suppose (iii) is true. Then it follows from lemma 2.5 that $\{K_a\}$ is a base for a uniformity. Forthermore this base is linearly ordered and of cardinality $\aleph_\mu$. Clearly this uniformity is compactible with δ. Now every uniform space with a linearly ordered base of cardinality $\aleph_\mu$ is ω_μ-metrizable (theorem 3, Stevenson and Thron [8]) so (X, δ) is ω_μ-metrizable.

Suppose that (X, δ) is ω_μ-metrizable. Letting ϱ be an ω_μ-metric which induces δ on X we have $A \delta B$ iff $\operatorname{lub}\{\varrho(a, b)\colon a \in A, b \in B\} = 0$.

We may assume that the range of ϱ is $\mathfrak{D}_\mu$ (see [8]) the set of ω_μ-sequences of zeros and ones. Let 1_a be the ω_μ-sequence which is zeros for all terms except the ath term (which is 1). Letting $U_{1_a} = \{(x, y)\colon \varrho(x, y) < 1_a\}$ it is easily shown that U_{1_a} is a symmetric diagonal set. Furthermore $A \delta B$ iff $A \times B \cap U_{1_a} \neq \emptyset$ for all $a < \omega_\mu$. It remains to show that U_{1_a} is an ω_μ-admissible set, i.e. $U_{1_a} \cap P \neq \emptyset$ for all ω_μ-compressed sets P. Suppose that $U_{1_{a_0}} \cap P = \emptyset$ for some $a_0 < \omega_\mu$ and some set $P \subset X \times X$. Now let

$A = \{x\colon \varrho(x, y) \geqslant 1_{a_0}$ for some $y\}$ and $B = \{y\colon \varrho(x, y) \geqslant 1_{a_0}$ for some $x\}$.

Now $A \times B = X \times X \sim U_{1_{a_0}} \supseteq P$. If P has cardinality $< \aleph_\mu$ then P is not ω_μ-compressed. If P has cardinality $\geqslant \aleph_\mu$ then $A \times B \cap P$ is of cardinality $\geqslant \aleph_\mu$ also. But $A \bar{\delta} B$ because $A \times B \cap U_{1_{a_0}} = \emptyset$. Therefore P is not ω_μ-compressed. Therefore we conclude that $U_{1_a} \cap P \neq \emptyset$ for all ω_μ-compressed sets P.

COROLLARY 2.7. *Let δ be a proximity relation on X. If there exists an ω_μ-sequence $\{K_a\}$ of ω_μ-admissible symmetric diagonal sets with $K_a \subset K_\beta$ for $\beta < a$ such that $A \delta B$ iff $A \times B \cap K_a \neq \emptyset$ for all a then δ admits a largest (finest) compatible uniformity on X.*

Proof. This follows from theorem 2.6; the fact that an ω_μ-metric space is a uniform space with a linearly ordered base; and the result of Alfsen and Njastad [1] that a uniform space with linearly ordered base admits a largest uniformity.

§ 3. ω_μ-proximities.

An ω_μ-metric on X induces a special type of uniformity, proximity, and topology on X.

DEFINITION 3.1. (i) A topology $\mathcal{T}$ is called *ω_μ-additive* iff $\bigcap\{G_i\colon G_i \in \mathcal{T}, i \in I\} \in \mathcal{T}$ where $|I| < \aleph_\mu$ (we use $|I|$ to denote the cardinality of the set I).

(ii) A uniformity $\mathfrak{U}$ on X is called an *ω_μ-uniformity* iff $\bigcap\{U_i\colon U_i \in \mathfrak{U}, i \in I\} \in \mathfrak{U}$ where $|I| < \aleph_\mu$.

(iii) A proximity relation δ on X is called an *ω_μ-proximity* iff $A \delta \bigcup\{B_i\colon i \in I\}$ implies that $A \delta B_i$ for at least one $i \in I$ where $|I| < \aleph_\mu$.

THEOREM 3.2. *If δ is an ω_μ-proximity relation on X then the associated relation $\Subset$ satisfies the following property: if $A \Subset B_i$ for all $i \in I$ then $A \Subset \bigcap\{B_i\colon i \in I\}$ where $|I| < \aleph_\mu$.*

THEOREM 3.3. (i) *If ϱ is an ω_μ-metric on X then $\mathfrak{U}_\varrho$ is an ω_μ-uniformity, δ_ϱ is an ω_μ-proximity, and $\mathcal{T}_\varrho$ is an ω_μ-additive topology.*

(ii) *If $\mathfrak{U}$ is an ω_μ-uniformity on X then $\delta_{\mathfrak{U}}$ is an ω_μ-proximity relation.*

(iii) *If δ is an ω_μ-proximity relation on X then $\mathcal{T}_\delta$ is an ω_μ-additive topology.*

The proofs of Theorems 3.2 and 3.3 are straight forward.

A major theorem in the theory of proximity spaces assures the existence of a unique totally bounded uniformity compatible with a given proximity relation. This theorem can be generalized as follows: for every ω_μ-proximity relation there exists a unique ω_μ-bounded, ω_μ-uniformity compatible with δ. The proof may be accomplished by simply generalizing the proof of the original theorem (i.e. $\omega_\mu = \omega_0$) as given, for example, in Thron [9]. Therefore we shall merely supply the appropriate definition and theorems.

DEFINITION 3.4. A uniform space $(X, \mathfrak{U})$ is *ω_μ-bounded* iff for every $U \in \mathfrak{U}$ there exists a set A of cardinality $< \aleph_\mu$ such that $U(A) = X$. $(X, \mathfrak{U})$ is *strictly ω_μ-bounded* iff μ is the least ordinal number for which $(X, \mathfrak{U})$ is ω_μ-bounded.

THEOREM 3.5. *Let (X, δ) be an ω_μ-proximity space. Then the family of all sets of the form $\bigcup \{A_i \times A_i\colon i \in I\}$ where $|I| < \aleph_\mu$, $A_i \supseteq B_i$, and $\bigcup \{B_i\colon i \in I\} = X$ is a base for a strictly ω_μ-bounded, ω_μ-uniformity $\mathfrak{U}$ on X. Furthermore $\delta_{\mathfrak{U}} = \delta$.*

THEOREM 3.6. *If $(X, \mathfrak{U})$ is a strictly ω_μ-bounded, ω_μ-uniform space then $\mathfrak{U}_{\delta_{\mathfrak{U}}} \subset \mathfrak{U}$.*

THEOREM 3.7. *If δ is an ω_μ-proximity relation on X then there exists one and only one strictly ω_μ-bounded, ω_μ-uniformity on X compatible with δ.*

Now if δ is an ω_μ-proximity relation it is necessarily an ω_ν-proximity relation for $0 \leqslant \nu \leqslant \mu$. Thus we may generalize theorems 3.5, 3.6, and 3.7 as follows:

THEOREM 3.8. *If δ is an ω_μ-proximity relation on X then there exists one and only one strictly ω_ν-bounded, ω_ν-uniformity on X compatible with δ for $0 \leqslant \nu \leqslant \mu$. This uniformity has as its base the family of all sets of the form $\bigcup \{A_i \times A_i\colon i \in I\}$ where $|I| < \aleph_\nu$, $A_i \supseteq B_i$, and $\bigcup \{B_i\colon i \in I\} = X$.*

We shall denote these strictly ω_ν-bounded, ω_ν-uniformities by $\mathfrak{U}_\nu$.

The question naturally arises whether there is ever more than one strictly ω_μ-bounded uniformity on X compatible with δ. Reed and Thron [5] have shown that if there exists one strictly ω_μ-bounded uniformity compatible with δ then there exists infinitely many uniformities in every class of strictly ω_ν-bounded uniformities on (X, δ) for $0 < \nu \leqslant \mu$. If δ is an ω_μ-proximity then more can be said.

THEOREM 3.9. *If δ is an ω_μ-proximity relation X then there exists a strictly decreasing sequence of uniformities between $\mathfrak{U}_\nu$ and $\mathfrak{U}_\sigma$ each of which is a strictly ω_ν-bounded uniformity compatible with δ where $0 \leqslant \sigma < \nu \leqslant \mu$.*

Proof. Corollary 2.1.1 of [5] says that if $\mathfrak{U}_0$ and $\mathfrak{U}_1$ are uniformities on X with different strict bounds and $\mathfrak{U}_0 \subset \mathfrak{U}_1$ then there is a strictly decreasing sequence of uniformities between $\mathfrak{U}_0$ and $\mathfrak{U}_1$ and these uniformities can all be chosen to have the same strict bound as $\mathfrak{U}_1$. This combined with theorem 3.7 and the fact that if $\nu < \nu'$ then $\mathfrak{U}_\nu \subset \mathfrak{U}_{\nu'}$ establishes the result.

The totally bounded uniformity compatible with a proximity relation δ is unique and hence naturally it is the largest and smallest member of the class of ω_0-bounded uniformities compatible with δ. In [5] it is shown that if $\mu > 0$ there is no minimal (and therefore no least) member of the class of strictly ω_μ-bounded uniformities on δ (corollary 2.1.5). However if δ admits an ω_μ-bounded uniformity then for all $\nu < \mu$ the class of strictly ω_ν-bounded uniformities admits a largest member (corollary 2.1.4). If δ is an ω_μ-proximity relation then we may characterize these largest elements; they are, in fact, the strictly ω_ν-bounded, ω_ν-uniformities $\mathfrak{U}_\nu$ described in theorem 3.8.

THEOREM 3.10. *If δ is an ω_μ-proximity relation on X and $\nu \leqslant \mu$ then $\mathfrak{U}_\nu \supseteq \mathfrak{V}$ for any strictly ω_ν-bounded uniformity $\mathfrak{V}$ on (X, δ).*

Proof. Let $\mathfrak{V}$ be a strictly ω_ν-bounded uniformity on (X, δ), let V be an arbitrary entourage of $\mathfrak{V}$ and let W be a symmetric entourage of $\mathfrak{V}$ such that $W \circ W \circ W \circ W \subset V$. Since $\mathfrak{V}$ is ω_ν-bounded there exists a set A of cardinality $< \aleph_\nu$ such that $W(A) = X$, i.e. $\bigcup \{W(x)\colon x \in A\} = X$. Now clearly $W(x) \subset W(W(x))$ so $\bigcup \{W(W(x)) \times W(W(x))\colon x \in A\} \in \mathfrak{U}_\nu$ since A is of cardinality $< \aleph_\nu$. But $\bigcup \{W(W(x)) \times W(W(x))\colon x \in A\} \subset V$ by the following argument. If $(a, b) \in \bigcup \{W(W(x)) \times W(W(x))\}$ then $a \in W(W(x))$ and $b \in W(W(x))$ for some $x \in A$. Thus $(a, x) \in W \circ W$ and $(b, x) \in W \circ W$. Therefore since W (and hence $W \circ W$) is symmetric we have $(a, b) \in W \circ W \circ W \circ W \subset V$.

We conclude that $V \in \mathfrak{U}_\nu$ and since V was an arbitrary member of $\mathfrak{V}$ it follows that $\mathfrak{V} \subseteq \mathfrak{U}_\nu$.

If δ is an ω_μ-proximity relation on X and X admits no non ω_μ-bounded uniformities then $\mathfrak{U}_\mu$ is naturally the largest uniformity compatible with δ. This occurs, for example, if $\mathfrak{U}_\mu$ has a linearly ordered base of cardinality $\aleph_\mu$.

There are, of course, classes of strictly ω_μ-bounded uniformities whose largest members are not ω_μ-uniformities. For example, the proximity relation δ induced by an ordinary metric d on the real line admits a largest uniformity. This uniformity is the metric uniformity and it is the largest member of the class of ω_1-bounded uniformities, however δ is an ω_0-proximity relation.

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A normal space X for which $X\times I$ is not normal

by

Mary Ellen Rudin (Madison, Wisc.)

The purpose of this paper is to construct (without using any set theoretic conditions beyond the axiom of choice) a normal Hausdorff space X whose Cartesian product with the closed unit interval I is not normal. Such a space is often called a *Dowker space*. The question of the existence of such a space is an old and natural one [3].

In 1951, C. H. Dowker [4] proved that a normal Hausdorff space is not countably paracompact if and only if its Cartesian product with I is not normal. Other interesting equivalences are given by C. H. Dowker and M. Katětov in [4] and [8], and one is a useful tool for constructing a Dowker space. M. Katětov [8] proved there is no perfectly normal Dowker space and B. J. Ball [1] proved there is no linear Dowker space.

In [10] I proved that the existence of a Souslin line implies the existence of a Dowker space. And, more recently, I observed that almost the same proof yields: if $\varkappa$ is a regular cardinal which is not the successor of a singular cardinal, then the existence of a Souslin tree of cardinality $\varkappa$ implies the existence of a Dowker space. The existence of a Souslin line and Souslin trees of these cardinalities has been proved consistent with the usual axioms of set theory ([13], [11], [7]).

I am indebted to N. Howes [6] for the idea that a *singular* cardinal might be useful in constructing a Dowker space. Howes also introduced me to the example of A. Miščenko given in [9] which I was able to prove is not normal. But successive modification of this example led me to the Dowker space X described below.

I. The definition of X and some notation will be given. We use the usual convention that an ordinal λ *is* the set of all ordinals less than λ. An ordinal γ is *cofinal* [5] with λ if there is a subset Γ of λ order isomorphic with γ such that $\alpha<\lambda$ implies there is a $\beta\in\Gamma$ such that $\alpha\leqslant\beta$. Let $\mathrm{cf}(\lambda)$ denote the smallest ordinal cofinal with λ.

Let N denote the set of all positive integers.

Let $F=\{f\colon N\to\omega_\omega \mid f(n)\leqslant\omega_n \text{ for all } n\in N\}$.

Let $X=\{f\in F \mid \exists i\in N \text{ such that } \omega_0<\mathrm{cf}(f(n))<\omega_i \text{ for all } n\in N\}$.