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KATEDRA METOD MATEMATYCZNYCH FIZYKI UNIWERSYTETU WARSZAWSKIEGO
DEPARTMENT OF MATHEMATICAL METHODS IN PHYSICS, WARSAW UNIWERSITY

Reçu par la Rédaction le 1. 7. 1970

'UDIA MATHEMATICA, T. XXXIX. (1971)

Continuous tensor products of Hilbert spaces and product operators

by

KAZIMIERZ NAPIÓRKOWSKI (Warszawa)

Contents

INTRODUCTION

The paper consists of two parts. In the first part the problem of the construction of a continuous tensor product of Hilbert spaces is considered. There are two natural ways to approach this problem.

The first way is by defining in a given Hilbert space $\mathfrak{H}$ the so-called *tensor structure*, i. e., by assigning to every partition of a certain Boolean algebra a unitary mapping from $\mathfrak{H}$ onto an infinite (incomplete) tensor product of Hilbert spaces [8]. The notion of tensor structure appears in a different form in a paper by Araki and Woods [1]. The authors have found the general model for tensor structures. They show that in the most interesting case of a non-atomic Boolean algebra (the "continuous" case) the Hilbert space is in a natural way isomorphic to an exponential

Hilbert space in such a way that the tensor structure of $\mathfrak{H}$ corresponds to a certain natural tensor structure in the exponential Hilbert space.

The other way is to construct a continuous tensor product of Hilbert spaces when a family of Hilbert spaces $(\mathfrak{H}_x)_{x\in X}$ labelled by elements of a space X with measure μ is given. The construction is similar to the one applied by von Neumann in the case of an infinite tensor product of Hilbert spaces [8].

One chooses a set of families $(\mathfrak{h}_x)_{x\in X}$, $\mathfrak{h}_x \in H_x$, and defines the scalar product as a continuous product of numbers of the form $(\mathfrak{h}_x^1 \mid \mathfrak{h}_x^2)_{\mathfrak{H}_x}$ (the continuous product of values of a function is defined by the integration of the logarithm of that function; some assumptions about the function are made to ensure the correctness of the definition).

Constructions of the above type have been developed by Guichardet [3] and Streater [9]. A problem that arises at the very beginning is the positive-definiteness of the scalar product in the continuous tensor product. This problem is considered in the first part of the present paper.

It is easy to show an example where the form used for the definition of the scalar product in the continuous tensor product of Hilbert spaces is not positive-definite. The example given in the present paper is probably well-known but has not been published. Dubin and Streater [2] show an example but in a very complicated form.

The converse example, i. e., where the scalar product is positive-definite, is shown, by Streater [9] and Guichardet [3]. The spaces $\mathfrak{H}_x$ used in this case are exponential Hilbert spaces and the families $(\mathfrak{h}_x)_{x\in X}$ are chosen according to that structure. This turns out to be the only possible example. This is the point of Theorem 1. If the set of families used for the construction is wide enough, then one can define in the continuous tensor product a tensor structure. This makes it possible to apply the Araki-Woods theorem.

We see that the two approaches give the same result. An analogous situation arises when the direct integral is considered. The first approach corresponds to assigning the decomposition of a given Hilbert space into a direct sum to any partition of a Boolean algebra. This leads to the choice of a commutative family of projectors. The other approach corresponds to the construction of a direct integral of Hilbert spaces (measurable vector fields are chosen, one defines a scalar product and so on).

The two approaches are connected by the spectral Theorem of von Neumann.

The second part of the present paper is devoted to product operators. Araki and Woods have found the general form of product operators of a given tensor structure. From the theorem which they have obtained it follows that a product operator can be decomposed in a certain sense into a continuous product of operators. The problem arises how to construct eigenfunctions of a product operator. In the case of a product of two Hilbert spaces a similar problem (for operators of the form $A_1 \otimes I_2 + I_1 \otimes A_2$) has been considered by K. Maurin and L. Maurin [7]. The authors show that the generalized eigenfunctions of such operators can be constructed as tensor products of the generalized eigenfunctions of the operators A_1 and A_2. In the case of the continuous tensor product a result of that type cannot be expected even if all the operators considered have discrete spectra. The role of vectors of the form $u \otimes v$ is played in the case of a non-atomic tensor structure by so-called *product vectors*. It is quite easy to check that two product vectors are never orthogonal, and thus they cannot be eigenvectors of the same Hermitian operator.

Theorem 2 of the present paper shows a method of constructing the eigenfunctionals of a product operator A from the eigenfunctionals of the operators A_x which are factors in the decomposition of A into a continuous product. It is not known, however, what connection there is between the functionals obtained in this way with the functionals giving the decomposition of the Hilbert space into a direct integral [6].

The construction uses the fact that in the case of the continuous tensor product $\mathfrak{H} = \bigotimes_X \mathfrak{H}_x$ the space $\mathfrak{H}$ and all the spaces $\mathfrak{H}_x$ are exponential Hilbert spaces ($\mathfrak{H} = e^H$, $\mathfrak{H}_x = e^{H_x}$). It is assumed that in the space H a nuclear space Φ is embedded and that Φ satisfies some conditions that allow us to define spaces $\Phi_x \subset H_x$ which are also nuclear. The next step is to construct a space $e^\Phi \subset e^H$ which is also a nuclear space. The eigenfunctionals mentioned above belong to the spaces $(e^\Phi)'$ and $(e^{\Phi_x})'$, respectively. The spaces $(e^{\Phi_x})'$ can be embedded in a natural way into $(e^\Phi)'$, and thus to the eigenfunctionals of A_x correspond some functionals on e^Φ which turn out to be eigenfunctionals of A. Their products with respect to a certain bilinear operation defined in $(e^\Phi)'$ are also eigenfunctionals of A. This bilinear operation is generated in $(e^\Phi)' \supset e^{\Phi'}$ by the operation of addition in Φ'.

The set of functionals obtained in this way is quite large. It is shown that it separates the vectors of e^Φ (Theorem 3).

The assumption of nuclearity of Φ is not used in the proofs of Theorems 2 and 3; it would be sufficient to assume that the topology of Φ is generated by a family of scalar products (Φ is a projective limit of Hilbert spaces). The nuclearity of Φ is assumed because some conclusions implied by it (e. g., the nuclearity of e^Φ and e^{Φ_x}) can be useful if one wants to find eigenfunctionals giving the decomposition of a Hilbert space into a direct integral.

The author would like to thank to Prof. K. Maurin for his kind interest in this work.

I. Continuous tensor product of Hilbert spaces

1. DEFINITIONS AND FORMULATION OF THE PROBLEM

1.1. Continuous family of Hilbert spaces. Let X be a topological space, $(\mathfrak{H}_x)_{x\in\Gamma}$ — a family of Hilbert spaces, and Γ — a family of mappings $\mathfrak{h}(\cdot)\colon X\ni x\to\mathfrak{h}(x)\in\mathfrak{H}_x$ such that:

1° for each $\mathfrak{h}_1,\mathfrak{h}_2\in\Gamma$ $\big(\mathfrak{h}_1(\cdot)\mid\mathfrak{h}_2(\cdot)\big)_{\mathfrak{H}(\cdot)}$ is a continuous function on X.

2° for each $x\in X$ the set $\{\mathfrak{h}(x):\mathfrak{h}\in\Gamma\}$ spans the space $\mathfrak{H}_x$.

Definition. A continuous family of Hilbert spaces is a triplet satisfying conditions 1° and 2°.

1.2. Continuous tensor product of Hilbert spaces. Let X be a locally compact space countable at infinity, and μ a Radon measure on X. We assume that the following conditions are fulfilled:

(a) Each compact connected component of X is simply connected, and arcwise connected, and its measure is an integer.

(b) The one-point compactification of each non-compact connected component of X is simply connected and arcwise connected.

(c) Each connected component of X is open and closed.

Let us define on X a continuous family of Hilbert spaces Γ and an element $\Omega\in\Gamma$ such that

3° $\big(\Omega(x)\mid\Omega(x)\big)=1$ for each $x\in X$,

4° for each $\mathfrak{h}\in\Gamma$, $\mathfrak{h}(x)=\Omega(x)$ except for a compact set,

5° $\bigwedge\limits_{\mathfrak{h}_1,\mathfrak{h}_2\in\Gamma}\ \bigwedge\limits_{x\in X}\big(\mathfrak{h}_1(x)|\mathfrak{h}_2(x)\big)\neq 0.$

On the set $\Gamma\times\Gamma$ we define the following function:

$$(\mathfrak{h}_1\mid\mathfrak{h}_2):=\exp\Big(\int\limits_X\ln\big(\mathfrak{h}_1(x)|\mathfrak{h}_2(x)\big)\,d\mu\Big).$$

The conditions satisfied by X, μ and Γ imply the correctness of this definition if one takes that branch of the logarithm for which $\lim\limits_{X\to\infty}\ln\big(\mathfrak{h}_1(x)\mid\mathfrak{h}_2(x)\big)=0$ in the Alexandrov compactification of X.

The function defined above has a Hermitian symmetry. It can be extended to a Hermitian form defined on the space of formal linear combinations of elements of Γ with complex coefficients. If that form is positive, then by using a standard procedure one can construct a Hilbert space from the space of linear combinations of elements of Γ. We shall denote that space by $\overset{\Gamma\Omega}{\underset{X}{\otimes}}\mathfrak{H}_x$.

Definition. We say that a continuous tensor product is *positive* if the Hermitian form defined above is positive.

A continuous tensor product is not always positive. One can see that from the following counterexample.

Let $X=[0,1]$, let μ be Lebesgue measure on X, $\mathfrak{H}_x=\boldsymbol{C}^2$, and Γ the set of functions of the form $\big(1,f(\cdot)\big)$, where $\big(1,f(x)\big)\in\boldsymbol{C}^2$, $f\geqslant 0$ and continuous.

The positivity of the continuous tensor product is the same as the positivity of the Hermitian matrix

$$a_{ij}=(\mathfrak{h}_i\mid\mathfrak{h}_j)$$

for any choice of $h_i\in\Gamma$, $i=1,2,\ldots,n$.

Let $\mathfrak{h}_1=(1,0)$ and let $\mathfrak{h}_2$ and $\mathfrak{h}_3$ be functions from Γ which approximate sufficiently enough the functions:

$$\mathfrak{h}_2'(x)=\begin{cases}(1,0), & x<\frac12,\\ (1,1), & x\geqslant\frac12,\end{cases}$$

$$\mathfrak{h}_3'(x)=\begin{cases}(1,0), & x<\frac12,\\ (1,2), & x\geqslant\frac12.\end{cases}$$

Then

$$a_{ij}\approx\begin{pmatrix}1, & 1, & 1\\ 1, & \sqrt2, & \sqrt3\\ 1, & \sqrt3, & \sqrt5\end{pmatrix}.$$

But $\det(a_{ij})<-0.04<0$, whence (a_{ij}) is not positive.

The aim of the first part of the present paper is to find sufficient conditions for the positivity of a continuous tensor product. We shall begin with an important example.

1.3. Exponential Hilbert space.

Definition. An exponential Hilbert space is a *triple* $(H,e^H,e^{(\cdot)})$, where H and e^H are separable Hilbert spaces and $e^{(\cdot)}$ is a mapping $H\to e^H$ that has the following properties:

(a) $(e^{h_1}\mid e^{h_2})e^H=\exp(h_1\mid h_2)_H$,

(b) $\{e^h:h\in H\}$ spans the space e^H.

The following lemmas state some elementary properties of exponential Hilbert spaces which will be useful later.

LEMMA 1.3.1. *The mapping $e^{(\cdot)}$ is a homeomorphism between H and its image in e^H.*

Proof. Let $h_n\to h_0$; then $(e^{h_n}|e^h)\to(e^{h_0}|e^h)$ for each $h\in H$ and $\|e^{h_n}\|\to\|e^{h_0}\|$; hence $e^{h_n}\to e^{h_0}$. Conversely, let $e^{h_n}\to e^{h_0}$; then, for any $h\in H$, $(e^{h_n}|e^h)=\exp(h_n|h)\to\exp(h_0|h)$, and because the sequence h_n is bounded

$(\|e^{h_n}\| = \exp(\frac{1}{2}\|h_n\|))$, we can choose a subsequence h_{n_l} such that

$$(h_{n_l}|h) \to (h_0|h)+2k\pi i \quad (k\in N \text{ depends on } h).$$

But $(h_{n_l}|\lambda h) \to (h_0|\lambda h)+2\lambda k\pi i$.

If one takes a non-real or irrational λ, λk is an integer only if $k=0$. We see that from each subsequence of h_n one can choose a subsequence convergent to h_0, so $h_n \to h_0$.

Lemma 1.3.2 (see [1]). *The space e^H is isomorphic to the direct sum*

$$H_0 \oplus H_1 \oplus H_2 \oplus \ldots,$$

where $H_n = (\otimes H)^n_s$ (the subspace of $(\otimes H)^n$ spanned by the vectors $h\otimes h\otimes \ldots \otimes h$, $h\in H$; $H_0 = C^1$).

The mapping $e^{(\cdot)}$ takes the form

$$Ue^h = 1+h+\frac{h\otimes h}{\sqrt{2!}}+\frac{(\otimes h)^n}{\sqrt{n!}}$$

(U denotes the unitary operator giving the above isomorphism).

Proof. It is easy to see that $(Ue^{h_1}|Ue^{h_2}) = \exp(h_1|h_2)$ and that the vectors Ue^h, $h\in H$, span the space $\bigoplus_{n=0}^{\infty} H_n$. We see that U preserves the scalar product and transforms a total set onto a total set, and so U is unitary.

Lemma 1.3.3. *Let H, H_1 be two Hilbert spaces; then a continuous linear operator $A: H_1 \to H$; $\|A\|\leqslant 1$ generates a unique continuous mapping $e^A: e^{H_1} \to e^H$ such that $e^A e^{h_1} = e^{Ah_1}$.*

Proof. If e^A exists, then it is unique, because it is defined on a total set in e^{H_1}. From Lemma 1.3.2 it follows that $e^A = \bigoplus_{n=0}^{\infty}(\otimes A)^n$, and so we get $\|e^A\| = \sup_{n=0,1,2,\ldots} \|A\|^n = 1$.

Now we shall construct an example of a continuous tensor product $\overset{\Gamma,\Omega}{\underset{X}{\otimes}} \mathfrak{H}_x$. Let $\mathfrak{H} = e^{L^2(X,\mu)}$, and let μ be a Radon measure on X. We take $\Gamma = \{e^{h(\cdot)}: h(\cdot)\in C_0(X)\}$ and identify the space $\mathfrak{H}_x$ with the space e^{C^1} spanned by the vectors $e^{h(x)}$; $e^h\in\Gamma$, $h(x)\in C^1$. As Ω we take e^0.

It is easy to see that conditions 1°-5° are fulfilled and the scalar product in $\overset{\Gamma,\Omega}{\underset{X}{\otimes}} \mathfrak{H}_x$ is the same as the scalar product in $\mathfrak{H}$ (Γ can be considered as a subset of $\mathfrak{H}$), and so it is positive. Moreover, Γ is a total set in $\mathfrak{H}$, and so we have a natural isomorphism $\mathfrak{H} \cong \overset{\Gamma,\Omega}{\underset{X}{\otimes}} \mathfrak{H}_x$.

We shall show that under some additional assumptions this is the only possible situation when a continuous tensor product $\overset{\Gamma,\Omega}{\underset{X}{\otimes}} \mathfrak{H}_x$ is positive. In the proof we make use of a theorem of Araki and Woods [1] in a modified form, using the notion of a tensor structure.

1.4. Tensor structures.

Definition. We say that in a Hilbert space $\mathfrak{H}$ a tensor structure over a Boolean σ-algebra $\mathfrak{B}$ is defined if to each countable partition $B = \mathrm{V}_i B_i$ (B being the maximal element of $\mathfrak{B}$) a unitary operator $U_{(B_i)}$ is assigned:

$$U_{(B_i)}: \mathfrak{H} \to \overset{(h_{B_i})}{\underset{i}{\otimes}} \mathfrak{H}_{B_i}, \quad \dim \mathfrak{H}_{B_i} > 1.$$

The tensor product denotes here the incomplete tensor product space of v. Neumann determined by the sequence of vectors (h_{B_i}), [8].

We assume the associativity of the mapping $(B_i) \to U_{(B_i)}$ in the following form:

For each subpartition $B_i = \underset{j}{\mathrm{V}} B_{ij}$ there exist (unique) unitary operators

$$U_{B_i,(B_{ij})}: \mathfrak{H}_{B_i} \to \overset{(h_{B_i,B_{ij}})}{\underset{j}{\otimes}} \mathfrak{H}_{B_{ij}}$$

such that

$$U_{(B_{ij})} = (\underset{i}{\otimes} U_{B_i,(B_{ij})}) \circ U_{(B_i)};$$

in particular $(h_{B_i}) \approx (\underset{j}{\otimes} h_{B_i,B_{ij}})$ in the sense of v. Neumann.

Example. We shall begin with a lemma about exponential Hilbert spaces.

Lemma 1.4.1. *Let $\mathfrak{H} = e^H$ be an exponential Hilbert space and $H = \bigotimes_{i=1}^{\infty} H_i$; then the mapping*

$$\mathfrak{H} \ni e^h \to \underset{i}{\otimes} e^{h_i} \in \overset{(e^{O_i})}{\underset{i}{\otimes}} e^{H_i} \quad (O_i \textit{ denoting zero in } H_i),$$

where $h = \sum_{i=1}^{\infty} h_i$, $h_i\in H_i$, can be extended to a unitary operator

$$U: \mathfrak{H} \to \overset{(e^{O_i})}{\underset{i}{\otimes}} e^{H_i}.$$

Proof. The mapping $e^h \to \underset{i}{\otimes} e^{h_i}$ preserves the scalar product and transforms a total set onto a total set.

Now if $\mathfrak{H} = e^H$ is an exponential Hilbert space and $H = \int_X H_x d\mu$, then to each partition $X = \bigcup_{i=1}^{\infty} X_i$ (disjoint union of measurable sets) corresponds the isomorphism $H \underset{(e^{O}{}_i)}{\cong} \bigoplus_{i=1} H_{X_i}, H_{X_i} = \int_{X_i} H_x d\mu$, and, by Lemma 1.4.1, the isomorphism $\mathfrak{H} \cong \bigotimes_i e^{H_i}$. These isomorphisms define in $\mathfrak{H}$ a tensor structure over the Boolean algebra of (measure classes of) measurable subsets of X.

Definition. A vector $\mathfrak{h} \in \mathfrak{H}$ is called a *product vector of a given tensor structure in* $\mathfrak{H}$ if under each of the isomorphisms $U_{(B_i)}$ of that structure it becomes a vector of the form $\bigotimes_i \mathfrak{h}_i$, $\mathfrak{h}_i \in \mathfrak{H}_{Bi}$.

EXAMPLE. The product vectors of the tensor structure defined above are vectors of the form $\mathfrak{h} = e^h$, $h \in H$.

THEOREM (Araki–Woods). *Suppose we are given in the Hilbert space $\mathfrak{H}$ a tensor structure over a non-atomic Boolean σ-algebra $\mathfrak{B}$ such that its product vectors span $\mathfrak{H}$. Than there exists an isomorphism*

$$U\colon \mathfrak{H} \to e^H, \quad \text{where } H = \int_X H_x d\mu$$

and $\mathfrak{B}$ is isomorphic to the Boolean algebra of classes of measurable sets in X, the tensor structure in $\mathfrak{H}$ is isomorphic to the one in e^H defined in the Example. The product vectors become vectors of the form ce^h, $h \in H$.

2. A NECESSARY CONDITION FOR THE POSITIVITY OF A CONTINUOUS TENSOR PRODUCT

Definition. A family Γ which defines a continuous family of Hilbert spaces satisfying conditions 1°-5° is called *equi-contractible* if for each pair $\mathfrak{h}_0, \mathfrak{h}_1$ of elements of Γ there exists a mapping:

$$g\colon [0,1] \times X \to \bigcup_X \mathfrak{H}_x$$

with the properties:

(a) $\bigwedge_t g(t, x) \in \mathfrak{H}_x$;

(b) $g(1, x) \underset{x}{\equiv} \mathfrak{h}_1(x)$, $g(0, x) \underset{x}{\equiv} \mathfrak{h}_0(x)$;

(c) the function $[0,1] \times X \ni (t, x) \mapsto \big(g(t,x) \,|\, \mathfrak{h}(x)\big)_{\mathfrak{H}_x}$ is continuous for each $\mathfrak{h} \in \Gamma$ and different from zero;

(d) for each pair $K \subset \mathcal{O}$, K being compact and $\mathcal{O}$ open sets in X, there exists a function $\gamma\colon X \to [0,1]$;

$$\gamma|_K = 1, \quad \gamma|_{X-\mathcal{O}} = 0$$

such that the mapping $X \mapsto g\big(\gamma(x), x\big)$ is an element of Γ;

(e) if $\mathfrak{h}_0(x) = \mathfrak{h}_1(x)$ for some x, then

$$g(t, x) = \mathfrak{h}_0(x) = \mathfrak{h}_1(x) \quad \text{for all } t \in [0,1].$$

Remark. The existence of an equi-contractible family Γ on X implies some restrictions for the topology of X.

We shall assume that X is normal.

LEMMA 2.1. *Let Γ be equi-contractible and let $\mathfrak{h} \in \Gamma$. Then the choice of the branch of $\ln\big(\mathfrak{h}_1(\cdot) \,|\, \mathfrak{h}(\cdot)\big)_{\mathfrak{H}(\cdot)}$ determines the choice of the branch of $\ln\big(g(\cdot,\cdot) \,|\, \mathfrak{h}(\cdot)\big)_{\mathfrak{H}(\cdot)}$ on $[0,1[\times X$, ond $\big|\ln g|)t, x)\big(\mathfrak{h}(x)\big)_{\mathfrak{H}(x)}\big|$ is bounded on $[0,1] \times X$.*

Proof. The connected components of X are arcwise and simply connected, and so are the components of $[0,1] \times X$. From condition (c) it follows that $\ln\big(g(t,x) \,|\, \mathfrak{h}(x)\big)_{\mathfrak{H}_x}$ can be defined on $[0,1] \times X$ as a continuous function of (t, x). Condition (b) allows us to fix the branch. The boundedness follows from conditions (d) and (e) because $\mathfrak{h}_1(x) = \mathfrak{h}_0(x) = g(t,x) = \mathfrak{h}(x) = \Omega(x)$ except for a compact set.

Now we are going to formulate the condition for the positivity of the continuous tensor product $\bigotimes_X^{\Gamma,\Omega} \mathfrak{H}_x$. To simplify the calculations we may assume without loss of generality that $\big(\Omega(x) \,|\, \mathfrak{h}(x)\big) \equiv 1$ for each $\mathfrak{h} \in \Gamma$.

THEOREM 1. *Let X be a metrisable locally compact space and μ a non-atomic Radon measure on X, and let X, μ fulfil conditions* (a)-(c) *of page* 310 *Let $\big(X, (\mathfrak{H}_x), \Gamma\big)$ be a continuous family of Hilbert spaces such that Γ is equi-contractible. Let the continuous tensor product $\bigotimes_X^{\Gamma,\Omega} \mathfrak{H}_x$ be positive.*

Then

1°. *There exists an isomorphism*

$$U\colon \bigotimes_X^{\Gamma,\Omega} \mathfrak{H}_x \to e^{\int_X H_x d\mu}$$

such that the image of Γ contains only vectors of the form e^h, $h \in \int_X H_x d\mu$.

2° *For almost every $x \in X$ there exist isomorphisms*

$$U_x\colon \mathfrak{H}_x \to e^{H_x}$$

such that for $\mathfrak{h} \in \Gamma$ $U_x \mathfrak{h}(x)$ is a vector of the form e^{h_x}, where $U\mathfrak{h} = e^h$.

Proof. Let Z be a measurable subset of X. Let K_ε be compact and $\mathcal{O}_\varepsilon$ — open sets such that $K_\varepsilon \subset Z$, $K_\varepsilon \subset \mathcal{O}_\varepsilon$ and $\mu(Z - K_\varepsilon) < \varepsilon$, $\mu(\mathcal{O}_\varepsilon - K_\varepsilon) < \varepsilon$. To any $\mathfrak{h} \in \Gamma$ we assign an element $\mathfrak{h}_\varepsilon \in \Gamma$ such that $\mathfrak{h}_\varepsilon|_{K_\varepsilon} = \mathfrak{h}|_{K_\varepsilon}$ and $h_\varepsilon|_{X-\mathcal{O}_\varepsilon} = \Omega|_{X-\mathcal{O}_\varepsilon}$. The assumption of equi-contractibility of Γ implies the existence of such $\mathfrak{h}_\varepsilon$ (one should take $\mathfrak{h}_1 = \mathfrak{h}, \mathfrak{h}_0 = \Omega$) and permits such a choice of $\mathfrak{h}_\varepsilon$ and of a branch of the logarithm that $\ln(\mathfrak{h}_\varepsilon(x)|\mathfrak{h}'(x)) = 0$ for $x \in X - \mathcal{O}_\varepsilon$ and $|\ln(\mathfrak{h}_\varepsilon(x)|\mathfrak{h}'(x))|$ is uniformly bounded (in ε) for each $\mathfrak{h}' \in \Gamma$.

LEMMA 2.2. *There exists a strong limit $\lim\limits_{\varepsilon\to 0}\mathfrak{h}_\varepsilon$ if $\mathfrak{h}_\varepsilon$ are chosen as described above.*

Proof. Let $\mathfrak{h}' \varepsilon \Gamma$. Then

$$\begin{aligned}(\mathfrak{h}_\varepsilon|\mathfrak{h}') &= \exp\Big(\int_{\mathcal{O}_\varepsilon} \ln(\mathfrak{h}_\varepsilon(x)|\mathfrak{h}'(x))\,d\mu\Big)\\ &= \exp\Big(\int_{K_\varepsilon} \ln(\mathfrak{h}_\varepsilon(x)|\mathfrak{h}'(x))\,d\mu + \int_{\mathcal{O}_\varepsilon - K_\varepsilon} \ln(\mathfrak{h}_\varepsilon(x)|\mathfrak{h}'(x))\,d\mu\Big)\\ &= \exp\Big(\int_{Z} \ln(\mathfrak{h}(x)|\mathfrak{h}'(x))\,d\mu - \int_{Z-K_\varepsilon} \ln(\mathfrak{h}(x)|\mathfrak{h}'(x))\,d\mu +\\ &\qquad + \int_{\mathcal{O}_\varepsilon - K_\varepsilon} \ln(\mathfrak{h}_\varepsilon(x)|\mathfrak{h}'(x))\,d\mu\Big).\end{aligned}$$

But $|\ln(\mathfrak{h}_\varepsilon(x)|\mathfrak{h}'(x))|$ is bounded (independently on ε); hence

$$\int_{\mathcal{O}_\varepsilon - K_\varepsilon} \ln(\mathfrak{h}_\varepsilon(x)|\mathfrak{h}'(x))\,d\mu \underset{\varepsilon\to 0}{\to} 0 \quad \text{and} \quad \int_{Z-K_\varepsilon} \ln(\mathfrak{h}(x)|\mathfrak{h}'(x))\,d\mu \underset{\varepsilon\to 0}{\to} 0.$$

The strong convergence follows from the fact that $\lim\|\mathfrak{h}_\varepsilon\| = \|\lim \mathfrak{h}_\varepsilon\|$ (analogous calculations).

Let us denote $\lim\limits_{\varepsilon\to 0}\mathfrak{h}_\varepsilon$ by $\mathfrak{h}_Z$ and the subspace of $\overset{\Gamma,\Omega}{\underset{X}{\otimes}} \mathfrak{H}_x$ spanned by the vectors $\mathfrak{h}_Z$, $\mathfrak{h} \in \Gamma$, by $\mathfrak{H}_Z$. We are now going to define a tensor structure in $\overset{\Gamma,\Omega}{\underset{X}{\otimes}} \mathfrak{H}_x$.

Let $X = \bigcup\limits_{i=1}^{\infty} X_i$ be a countable partition of X.

LEMMA 2.3. *Let $\mathfrak{h}$, $\mathfrak{h}' \in \Gamma$. Then the sequences of vectors $(\mathfrak{h}_{X_i})_{i=1}^{\infty}$ and $(\mathfrak{h}'_{X_i})_{i=1}^{\infty}$ satisfy the condition*

$$\sum_{i=1}^{\infty} |(\mathfrak{h}_{X_i}|\mathfrak{h}'_{X_i}) - 1| < \infty.$$

Proof. Let us compute the scalar product

$$\begin{aligned}(\mathfrak{h}_{X_i}|\mathfrak{h}'_{X_i}) &= \lim_{\varepsilon\to 0}(\mathfrak{h}_{X_i}|\mathfrak{h}'_{\varepsilon(X_i)}) = \lim_{\varepsilon\to 0}\exp\Big(\int_{X_i} \ln(\mathfrak{h}(x)|\mathfrak{h}'_\varepsilon(x))\,d\mu\Big)\\ &= \lim_{\varepsilon\to 0}\exp\Big(\int_{X_i} \ln(\mathfrak{h}(x)|\mathfrak{h}'(x))\,d\mu -\\ &\qquad - \int_{X_i - K_\varepsilon} \ln(\mathfrak{h}(x)|\mathfrak{h}'(x))\,d\mu + \int_{X_i - K_\varepsilon} \ln(\mathfrak{h}(x)|\mathfrak{h}'_\varepsilon(x))\,d\mu\Big).\end{aligned}$$

But the last two integrals converge to zero as $\varepsilon \to 0$; hence

$$(\mathfrak{h}_{X_i}|\mathfrak{h}'_{X_i}) = \exp\Big(\int_{X_i} \ln(\mathfrak{h}(x)|\mathfrak{h}'(x))\Big)\,d\mu$$

and

$$\prod_{i=1}^{\infty}(\mathfrak{h}_{X_i}|\mathfrak{h}'_{X_i}) = \exp\Big(\int_{\bigcup_i X_i} \ln(\mathfrak{h}(x)|\mathfrak{h}'(x))\,d\mu\Big) = (\mathfrak{h}|\mathfrak{h}').$$

Since the infinite product is convergent, the sum $\sum\limits_{i=1}^{\infty}|(\mathfrak{h}_{X_i}|\mathfrak{h}'_{X_i}) - 1|$ is also convergent.

COROLLARY. *Vectors of the form $\underset{i}{\otimes}\mathfrak{h}_{X_i}$, $\mathfrak{h} \in \Gamma$, belong to the incomplete tensor product space $\overset{(\Omega_{X_i})}{\underset{i}{\otimes}} \mathfrak{H}_{X_i}$.*

LEMMA 2.4. *Vectors $\underset{i}{\otimes}\mathfrak{h}_{X_i}$, $\mathfrak{h} \in \Gamma$, span the space $\overset{(\Omega_{X_i})}{\underset{i}{\otimes}} \mathfrak{H}_{X_i}$.*

Proof. It suffices to show that each vector $(\overset{n}{\underset{i=1}{\otimes}} \mathfrak{h}^i_{X_i}) \otimes (\overset{\infty}{\underset{i=n+1}{\otimes}} \Omega_{X_i})$, $\mathfrak{h}^i \in \Gamma$, can be approximated by vectors of the form $\underset{i}{\otimes}\mathfrak{h}_{X_i}$, $\mathfrak{h} \in \Gamma$. Let us take compact sets $(K_i)_{i=1}^n$ and open sets $(\mathcal{O}_i)_{i=1}^n$ such that $K_i \subset X_i$, $K_i \subset \mathcal{O}_i$, $\mathcal{O}_i \cap \mathcal{O}_j = \emptyset$ for $i \neq j$ (we have assumed the space X to be normal) and $\mu(X_i - K_i) < \varepsilon$, $\mu(\mathcal{O}_i - K_i) < \varepsilon$.

We shall construct such a function $\mathfrak{h}_\varepsilon \in \Gamma$ that

$$\mathfrak{h}_\varepsilon|_{K_i} = \mathfrak{h}^i|_{K_i} \quad \text{and} \quad \mathfrak{h}_\varepsilon|_{X-\bigcup_i \mathcal{O}_i} = \Omega|_{X-\bigcup_i \mathcal{O}_i}.$$

We take, in the condition of equi-contractibility of Γ, $\mathfrak{h}_0 = \Omega$, $\mathfrak{h}_1 = \mathfrak{h}^1$, $K = K_1$, $\mathcal{O} = \mathcal{O}_1$ and we get a function $\mathfrak{h}^{1'}$ such that $\mathfrak{h}^{1'}|_{K_1} = \mathfrak{h}^1|_{K_1}$ and $\mathfrak{h}^{1'}|_{X-\mathcal{O}_1} = \Omega|_{X-\mathcal{O}_1}$.

For the next step we take $\mathfrak{h}_0 = \mathfrak{h}^{1'}$, $\mathfrak{h}_1 = \mathfrak{h}^2$, $K = K_2$, $\mathcal{O} = \mathcal{O}_2$ and obtain a function $\mathfrak{h}^{2'}$ satisfying the conditions

$$\mathfrak{h}^{2'}|_{K_1} = \mathfrak{h}^1|_{K_1}, \quad \mathfrak{h}^{2'}|_{K_2} = \mathfrak{h}^2|_{K_2}, \quad \mathfrak{h}^{2'}|_{X-\mathcal{O}_1\cup\mathcal{O}_2} = \Omega|_{X-\mathcal{O}_1\cup\mathcal{O}_2}.$$

After n steps we get the desired function $\mathfrak{h}_\varepsilon$.

From this construction and Lemma 2.1 it follows that, for any $\mathfrak{h} \in \Gamma$, $|\ln(\mathfrak{h}_\varepsilon(\cdot)|\mathfrak{h}(\cdot))|$ can be bounded by a continuous function which has a compact support and does not depend on ε. A calculation analogous to that applied in the proof of Lemma 2.2 shows that if $\varepsilon \to 0$, then

$$\mathfrak{h}_\varepsilon \to (\overset{n}{\underset{i=1}{\otimes}} \mathfrak{h}^i_{X_i}) \otimes (\overset{\infty}{\underset{i=n+1}{\otimes}} \Omega_{X_i}).$$

LEMMA 2.5. *The mapping $\mathfrak{h} \mapsto \underset{i}{\otimes}\mathfrak{h}_{X_i}$ can be (uniquely) extended to a unitary operator $U_{(X_i)}\colon \overset{\Gamma,\Omega}{\underset{X}{\otimes}} \mathfrak{H}_X \to \overset{(\Omega_{X_i})}{\underset{i}{\otimes}} \mathfrak{H}_{X_i}$. The operators $U_{(X_i)}$ define a tensor structure in $\overset{\Gamma,\Omega}{\underset{X}{\otimes}} \mathfrak{H}_X$.*

Proof. From the proof of Lemma 2.3 it follows that the mapping $\overset{\Gamma,\Omega}{\underset{X}{\otimes}} \mathfrak{H}_X \supset \Gamma \ni \mathfrak{h} \mapsto \underset{i}{\otimes} \mathfrak{h}_{X_i} \in \overset{(\Omega X_i)}{\underset{i}{\otimes}} \mathfrak{H}_{X_i}$ preserves the scalar product, and from Lemma 2.4 we know that it maps a total set onto a total set, and so it can be extended to a unitary operator. The fulfilment of the associativity condition is obvious.

LEMMA 2.6. *Vectors $\mathfrak{h} \in \Gamma$ are product vectors of the tensor structure in $\overset{\Gamma,\Omega}{\underset{X}{\otimes}} \mathfrak{H}_x$ defined in Lemma 2.5.*

Proof. The lemma follows immediately from Lemma 2.5.

Now we can make use of the Araki–Woods theorem. As a result, we get part 1° of Theorem 1 and, moreover, we know that the tensor structure constructed in Lemmas 2.3–2.5 under the isomorphism U of the Araki–Woods theorem becomes the canonical tensor structure in $e^{\int_X H_x d\mu}$ (i. e. such as the one defined in the Example).

Now we have to prove part 2° of Theorem 1.

Let us take a countable subset $\Gamma_0 \subset \Gamma$ such that vectors $\mathfrak{h} \in \Gamma_0$ span the space $\overset{\Gamma,\Omega}{\underset{X}{\otimes}} \mathfrak{H}_x$ and for μ-almost every x vectors, $\mathfrak{h}(x)$, $\mathfrak{h} \in \Gamma_0$ span $\mathfrak{H}_x$. The existence of such a subset follows from the assumption that X is countable at infinity and metric.

From part 1° of Theorem 1 we know that to the vectors $\mathfrak{h}^i \in \Gamma_0$ correspond $e^{h_i} \in e^{\int_X H_x d\mu}$, and from the correspondence of the tensor structures in $\overset{\Gamma,\Omega}{\underset{X}{\otimes}} \mathfrak{H}_X$ and in $e^{\int_X H_x d\mu}$ we obtain for each measurable subset $Z \subset X$:

$$(\mathfrak{h}_Z^i|\mathfrak{h}_Z^j) = \exp\Big(\int_Z \ln\big(\mathfrak{h}^i(x)|\mathfrak{h}^j(x)\big)\,d\mu\Big) = (e^{P_Z h^i}|e^{P_Z h^j}) = \exp\Big(\int_Z (h_x^i|h_x^j)\,d\mu\Big).$$

This implies that

$$\int_Z \ln\big(\mathfrak{h}^i(x)|\mathfrak{h}^j(x)\big)\,d\mu = \int_Z (\mathfrak{h}_x^i|h_x^j)\,d\mu + 2\pi i\cdot n(Z)$$

and

$$\frac{1}{2\pi i}\int_Z [\ln\big(\mathfrak{h}^i(x)|\mathfrak{h}^j(x)\big) - (h_x^i|h_x^j)]\,d\mu = n(Z).$$

We obtain in this way an integrable function, the integral of which, taken over any measurable set Z, is an integer $n(Z)$.

LEMMA 2.7. *Let f be such an integrable function that, for any measurable set Z, $\int_Z f d\mu = n(Z)$ is an integer and let μ be a non-atomic measure; then $f = 0$ μ-almost everywhere.*

Proof. Let us assume that there exists an interval $[a, b]$, $b > a > 0$, such that $\mu\big(f^{-1}([a, b])\big) > 0$. Since the measure μ is non-atomic, one can divide the set $f^{-1}([a, b])$ into a finite number of measurable parts Z_i such that $\mu(Z_i) < 1/b$. Then

$$0 \leqslant \int_{Z_i} f d\mu \leqslant \sup_{Z_i} f\cdot\mu(Z_i) < b\cdot\frac{1}{b} = 1,$$

but $\int_{Z_i} f d\mu$ should be an integer, and so $\int_{Z_i} f d\mu = 0$ and $\int_{f^{-1}([a,b])} f d\mu = 0$.

On the other hand,

$$0 = \int_{f^{-1}([a,b])} f d\mu \geqslant \inf_{f^{-1}([a,b])} f\cdot\mu\big(f^{-1}([a, b])\big) \geqslant a\cdot\mu\big(f^{-1}([a, b])\big) > 0.$$

We have a contradiction: $0 > 0$.

From Lemma 2.7 we get

$$\big(\mathfrak{h}^i(x)|\mathfrak{h}^j(x)\big) = \exp(h_x^i|h_x^j) \quad \text{for } \mu\text{-almost all } x \in X.$$

Since Γ_0 is countable, one can take a set X' with full measure such that

$$\big(\mathfrak{h}^i(x)|\mathfrak{h}^j(x)\big) = \exp(h_x^i|h_x^j) \quad \text{for arbitrary } i, j \text{ and } x \in X'.$$

The set $\big(\mathfrak{h}^i(x)\big)_{i=1}^{\infty}$ spans $\mathfrak{H}_x$ for μ-almost all $x \in X$ and the set $(h_x^i)_{i=1}^{\infty}$ spans H_x for μ-almost all $x \in X$ (this follows from the observation that if a set $(e^{h^i})_{i=1}^{\infty}$ spans e^H, then the set $(h^i)_{i=1}^{\infty}$ spans H, whence the sets $(h_x^i)_{i=1}^{\infty}$ span H_x for μ-almost all $x \in X$). Taking the intersection of sets with full measure, we obtain the isomorphisms

$$U_x\colon \mathfrak{H}_x \to e^{H_x} \quad \text{for } \mu\text{-almost all } x \in X$$

with property $U_x h(x) = e^{h_x}$, where $\mathfrak{h} \in \Gamma$ and $U\mathfrak{h} = e^h$ (U being the isomorphism from part 1° of Theorem 1). The proof is complete.

3. AN APPLICATION OF THEOREM 1

Theorem 1 can be applied to the theory of continuous tensor products of Banach $*$-algebras. That theory has been developed by Guichardet in [3], [4], [5]. We adopt some of the notation used there but to avoid confusion with the notation used in present paper we must change some of the symbols. A continuous family of Banach algebras will be denoted by $\big((\mathfrak{A}_x)_{x\in X}, \overline{\Gamma}\big)$ (instead of $\big((\mathfrak{A}_i)_{i\in I}, \Gamma\big)$). Elements of algebras $\mathfrak{A}_x$ will be denoted by $f_x, g_x, \mathfrak{h}_x$ instead of x_i, y_i, z_i.

Let $\big((\mathfrak{A}_x)_{x\in X}, \overline{\Gamma}\big)$ be a continuous family of Banach $*$-algebras and $(\varphi_x)_{x\in X}$ a family of states on $\mathfrak{A}_x$ such that the function $x \mapsto \big(\varphi_x(f_x) - 1\big)$

belongs to $C_0(X)$ for each family $(f_x)_{x\in X}\in\bar{\Gamma}$. It is possible to define a functional $\hat{\otimes}\varphi_x$ on $\hat{\otimes}\mathfrak{A}_x$.

The problem is to find the conditions under which, for the family $(\varphi_x)_{x\in X}$, the functional $\hat{\otimes}\varphi_x$ is a state on $\hat{\otimes}\mathfrak{A}_x$.

Guichardet has shown [4] that if $\hat{\otimes}\varphi_x$ is a state, then one can construct a positive tensor product of Hilbert spaces as follows.

Let N_x denote the left ideal of $\mathfrak{A}_x$ formed of elements f_x such that $\varphi_x(f_x^*f_x)=0$, and let $\mathfrak{H}_x$ be the Hilbert space formed from $\mathfrak{A}_x/N_x$ by completion. Let L_x be the canonical mapping from $\mathfrak{A}_x$ into $\mathfrak{H}_x$, let $\Omega_x=L_xe_x$ (the unity of $\mathfrak{A}_x$) and let Γ be the family of functions $(L_xf_x)_{x\in X}$, where $(f_x)_{x\in X}\in\bar{\Gamma}$. Γ in general does not define a continuous family of Hilbert spaces because conditions 2° and 5° are not always fulfilled. These conditions are not necessary for the construction of the continuous tensor product of Hilbert spaces; it suffices to extend the definition of scalar product in $\Gamma\times\Gamma$ by putting $(\mathfrak{h}_1|\mathfrak{h}_2)=0$ if, for some x, $(\mathfrak{h}_1(x)|\mathfrak{h}_2(x))=0$. If $\hat{\otimes}\varphi_x$ is a state on $\hat{\otimes}\mathfrak{A}_x$, then the continuous tensor product $\overset{\Gamma,\Omega}{\underset{X}{\otimes}}\mathfrak{A}_x$ is positive.

From Theorem 1 we get the following criterion for the functiona- $\hat{\otimes}\varphi_x$ to be a state.

PROPOSITION. *Let $(\varphi_x)_{x\in X}$ be a family of states on $\mathfrak{A}_x$ such that condil tions 2° and 5° are fulfilled. Let Γ be equi-contractible.*

Then $\hat{\otimes}\varphi_x$ is a state if and only if for almost all $x\in X$ there exist isomorphisms $U_x:\mathfrak{H}_x\to e^{H_x}$ such that

$$U_x\big(\{L_xf_x:(f_x)_{x\in X}\in\bar{\Gamma}\}\big)\subset\{ce^{h_x}:h_x\in H_x,\ c\in C^1\}.$$

II. Product operators

1. THE OPERATION ⊙

Let us have an exponential Hilbert space $(H,e^H,e^{(\cdot)})$. From Lemma I. 1.3.2 it follows that we can consider e^H as a direct sum

$$e^H=C^1\oplus H\oplus(\otimes H)^2_s\oplus\ldots,$$

and the mapping $e^{(\cdot)}$ has the form

$$e^h=1+h+\frac{h\otimes h}{\sqrt{2!}}+\ldots+\frac{(\otimes h)^n}{\sqrt{n!}}+\ldots$$

Let $\mathfrak{h}\in e^H$; then $\mathfrak{h}=\sum_{i=1}^{\infty}\mathfrak{h}_i$, $\mathfrak{h}_i\in H_i=(\otimes H)^i_s$; and let

$$D:=\Big\{\mathfrak{h}\in e^H:\bigvee_C\bigwedge_j\|\mathfrak{h}_j\|<\frac{C}{2^j}\Big\}.$$

We define

$$\mathfrak{h}\odot\mathfrak{k}:=\sum_{k=0}^{\infty}\sum_{i+j=K}\sqrt{\binom{k}{i}}(\mathfrak{h}_i\otimes\mathfrak{k}_j)_s,\qquad \mathfrak{h},\mathfrak{k}\in D,$$

where $(\mathfrak{h}_i\otimes\mathfrak{k}_j)_s$ denotes the projection of $\mathfrak{h}_i\otimes\mathfrak{k}_j\in(\otimes H)^i_s\otimes(\otimes H)^j_s$ on the space $(\otimes H)^{i+j}_s$ (i. e. symmetrization).

LEMMA 1.1 *If $\mathfrak{h},\mathfrak{k}\in D$, then the sum in the definition of $\mathfrak{h}\odot\mathfrak{k}$ is convergent.*

Proof.

$$\|\mathfrak{h}\odot\mathfrak{k}\|^2=\Big\|\sum_{k=0}^{\infty}\sum_{i+j=k}\sqrt{\binom{k}{i}}(\mathfrak{h}_i\otimes\mathfrak{k}_j)_s\Big\|^2=\sum_{k=0}^{\infty}\Big\|\sum_{i+j=k}\sqrt{\binom{k}{i}}(\mathfrak{h}_i\otimes\mathfrak{k}_j)\Big\|^2$$

$$=\sum_{k=0}^{\infty}\sum_{i,j=0}^{k}\sqrt{\binom{k}{i}\binom{k}{j}}\big((\mathfrak{h}_i\otimes\mathfrak{k}_{k-i})_s\,|\,(\mathfrak{h}_j\otimes\mathfrak{k}_{k-j})_s\big)$$

$$\leqslant\sum_{k=0}^{\infty}\sum_{i,j=0}^{k}\sqrt{\binom{k}{i}\binom{k}{j}}\big|\big((\mathfrak{h}_i\otimes\mathfrak{k}_{k-i})_s\,|\,(\mathfrak{h}_j\otimes\mathfrak{k}_{k-j})_s\big)\big|.$$

We use the inequalities

$$\sqrt{\binom{k}{i}\binom{k}{j}}\leqslant\frac{1}{2}\left(\binom{k}{i}+\binom{k}{j}\right).$$

and

$$\big|\big((\mathfrak{h}_i\otimes\mathfrak{k}_{k-i})_s\,|\,(\mathfrak{h}_j\otimes\mathfrak{k}_{k-j})_s\big)\big|\leqslant\|(\mathfrak{h}_i\otimes\mathfrak{k}_{k-i})_s\|\cdot\|(\mathfrak{h}_j\otimes\mathfrak{k}_{k-j})_s\|\leqslant\|(\mathfrak{h}_i\otimes\mathfrak{k}_{k-i})\|\cdot\|(\mathfrak{h}_j\otimes\mathfrak{k}_{k-j})\|$$

$$\leqslant\frac{c_{\mathfrak{h}}}{2^i}\cdot\frac{C_{\mathfrak{k}}}{2^{k-i}}\cdot\frac{C_{\mathfrak{h}}}{2^j}\cdot\frac{C_{\mathfrak{k}}}{2^{k-j}}=\frac{C_{\mathfrak{h}}^2\cdot C_{\mathfrak{k}}^2}{2^{2k}}.$$

We get

$$\|\mathfrak{h}\odot\mathfrak{k}\|^2\leqslant\sum_{k=0}^{\infty}\sum_{i,j=0}^{k}\frac{1}{2}\left(\binom{k}{i}+\binom{k}{j}\right)\frac{C_{\mathfrak{h}}^2\cdot C_{\mathfrak{k}}^2}{2^{2k}}$$

$$=\sum_{k=0}^{\infty}\frac{C_{\mathfrak{h}}^2\cdot C_{\mathfrak{k}}^2}{2^{2k}}k\cdot 2^k=C_{\mathfrak{h}}^2\cdot C_{\mathfrak{k}}^2\sum_{k=0}^{\infty}\frac{k}{2^k}<\infty.$$

LEMMA 1.2. *The operation ⊙ has the following properties:*

1° $(\mathfrak{h}\odot\mathfrak{k}|e^h)=(\mathfrak{h}|e^h)\cdot(\mathfrak{k}|e^h)$,

2° $e^{h_1}\odot e^{h_2}=e^{h_1+h_2}$,

3° ⊙ *is bilinear (but not continuous).*

Proof. The bilinearity of the operation $\odot$ follows from its definition. Let us calculate

$$(\mathfrak{h}\odot\mathfrak{k}|e^h) = \sum_{k=0}^{\infty}\sum_{i+j=k}\frac{\sqrt{\binom{k}{i}}}{\sqrt{k!}}\big((\mathfrak{h}_i\otimes\mathfrak{k}_j)_s|(\otimes h)^k\big)$$

(we use the equality $\frac{1}{\sqrt{k!}} = \frac{1}{\sqrt{i!}\cdot\sqrt{j!}\cdot\sqrt{\binom{k}{i}}}$; $j = k-i$)

$$= \sum_{k=0}^{\infty}\sum_{i+j=k}\frac{1}{\sqrt{i!\cdot j!}}\big((\mathfrak{h}_i\otimes\mathfrak{k}_j)_s|(\otimes h)^k\big)$$

We may omit the sign of symmetrisation because $(\otimes h)^k$ is symmetric. We use the equality $(\otimes h)^k = (\otimes h)^i\otimes(\otimes h)^j$.

$$= \sum_{i=0}^{\infty}\frac{(\mathfrak{h}_i|(\otimes h)^i)}{\sqrt{i!}}\cdot\sum_{j=0}^{\infty}\frac{(\mathfrak{k}_j|(\otimes h)^j)}{\sqrt{j!}} = (\mathfrak{h}|e^h)\cdot(\mathfrak{k}|e^h).$$

To prove 2° we compute

$$(e^{h_1}\odot e^{h_2}|e^h) = (e^{h_1}|e^h)\cdot(e^{h_2}|e^h)$$
$$= \exp\big((h_1|h)+(h_2|h)\big) = (e^{h_1+h_2}|e^h).$$

Vectors of the form e^h, $h\in H$, form a total set; hence

$$e^{h_1}\odot e^{h_2} = e^{h_1+h_2}.$$

2. EXPONENTIAL GELFAND TRIPLETS

2.1. Exponential nuclear space. Let $\Phi\subset H$ be a nuclear space embedded in a Hilbert space H. We shall construct a nuclear space $e^\Phi\subset e^H$. Let us denote by Φ_a the pre-Hilbert space formed from Φ by a continuous Hilbertian seminorm p_a. The scalar product corresponding to p_a will be denoted by $(\cdot|\cdot)_a$. Let us take in e^H the subset consisting of vectors e^φ, $\varphi\in\Phi$, and denote its linear hull by $\widetilde{e^\Phi}$. In $\widetilde{e^\Phi}$ we take the weakest topology satisfying the condition that the mappings $\widetilde{e^\Phi}\to e^{\Phi_a}$ (defined by the mappings $\Phi\to\Phi_a$) are continuous. By e^Φ we denote the completion of $\widetilde{e^\Phi}$.

The space e^Φ can be considered as a subset of e^H. From the decomposition of e^H (Lemma I. 1.3.2) we get the decomposition

$$f = \sum_{n=0}^{\infty}f_n;\quad f\in e^\Phi,\quad f_n\in(\hat{\otimes}\Phi)_s^n\subset(\otimes H)_s^n,$$

and the components f_n satisfy the condition

$$\sum_{n=0}^{\infty}\big(p_a^n(f_n)\big)^2<\infty,$$

where p_a^n denotes the Hilbertian seminorm in $(\otimes\Phi_a)_s^n$ generated by p_a.

The topology in e^Φ is generated by seminorms e^{p_a} defined as follows:

$$e^{p_a}(f) := \Big(\sum_{n=0}^{\infty}\big(p_a^n(f_n)\big)^2\Big)^{1/2},$$

where p_a is a continuous Hilbertian seminorm in Φ.

Remark. To construct the topology in e^Φ we take all continuous Hilbertian seminorms in Φ, not only a fundamental family. This is necessary, because for equivalent seminorms p_a and p_β the seminorms e^{p_a} and e^{p_β} are, in general, not equivalent. Similarly, $p_a\prec p_\beta$ does not imply $e^{p_a}\prec e^{p_\beta}$ (but from Lemma I. 1.3.3 we have $p_a\leqslant p_\beta\Rightarrow e^{p_a}\leqslant e^{p_\beta}$).

LEMMA 2.1.1. *If Φ is a nuclear space, then e^Φ is also nuclear.*

Proof. Let $p_a\leqslant p_\beta$. Then

$$(\varphi|\psi)_a = (\varphi|A_{a\beta}\psi)_\beta;\qquad \varphi,\psi\in\Phi,\ A_{a\beta}\in L(\Phi_\beta).$$

The nuclearity of the space Φ means that for each p_a there exists a p_β such that $\mathrm{Tr}(A_{a\beta})<\infty$. Multiplying p_β by a constant, we can get $\mathrm{Tr}(A_{a\beta})<1$. Since $e^{p_a}\leqslant e^{p_\beta}$, we have

$$(f|g)e^{p_a} = (f|B_{a\beta}g)e^{p_\beta};\quad f,g\in e^\Phi,\ B_{a\beta}\in L(e^{\Phi_\beta}).$$

From the decomposition of e^Φ and definition of e^{p_a} we have

$$\mathrm{Tr}(B_{a\beta})\leqslant 1+\mathrm{Tr}(A_{a\beta})+\big(\mathrm{Tr}(A_{a\beta})\big)^2+\ldots<\infty.$$

This means that e^Φ is nuclear.

Remark. The embedding $\Phi\subset H$ is not necessary for the construction of e^Φ, but it simplifies some notation and gives at the same time the construction of a Gelfand triplet $e^\Phi\subset e^H\subset(e^\Phi)'$.

LEMMA 2.1.2. *Let $\xi\in\Phi'$; then the functional e^ξ defined on e^Φ by*

$$\langle f, e^\xi\rangle := \sum_{n=0}^{\infty}\frac{\langle f_n,\xi^n\rangle}{\sqrt{n!}},\qquad f=\sum_{n=0}^{\infty}f_n,$$

where $f\in e^\Phi$, $f_n\in(\hat{\otimes}\Phi)_s^n$, $\xi^n = \xi\otimes\xi\otimes\ldots\otimes\xi\in(\hat{\otimes}\Phi)_s^{n'}$, is linear and continuous.

Proof. If ξ is continuous with respect to the seminorm p_a, then e^ξ is continuous with respect to the seminorm e^{p_a}. The linearity of e^ξ is obvious.

If the strong topology of the space Φ' is also generated by Hilbertian seminorms, then one can construct a space $e^{\Phi'}$ in the same way as e^{Φ} was constructed from Φ. The spaces $e^{\Phi'}$ and $(e^{\Phi})'$ are, in general, not isomorphic.

EXAMPLE. Let $\Phi = C^1$. Then the continuous Hilbertian seminorms are $p_a(\varphi) = a\cdot|\varphi|$; $\varphi \in C^1$, $a \geqslant 0$. The elements of the space e^{Φ} can be represented by sequences $(x_n)_1^{\infty}$; $x_n \in (\hat{\otimes}\, C^1)_S^n = C^1$ such that

$$\sum_{n=0}^{\infty} (|x_n|\cdot a^n)^2 < \infty \quad \text{for each } a \geqslant 0$$

and

$$e^{p_a}\big((x_n)\big) = \Big(\sum_{n=0}^{\infty} (|x_n|\cdot a^n)^2\Big)^{1/2}.$$

This means that the elements of $(e^{\Phi})'$ can be represented by sequences $(y_n)_1^{\infty}$ such that

$$|y_n| \leqslant c a^n \quad \text{for some } a > 0$$

and

$$\langle (x_n), (y_n)\rangle = \sum_{n=0}^{\infty} x_n y_n.$$

We see that the elements of e^{Φ} are fast decreasing sequences whereas the elements of $(e^{\Phi})'$ are slowly increasing sequences.

The spaces Φ and Φ' are in this case isomorphic; hence, e^{Φ} and $e^{\Phi'}$ are isomorphic. Combining these facts, we see that even if $e^{\Phi'}$ and $(e^{\Phi})'$ were isomorphic, the isomorphism would not be generated by the formula of Lemma 2.1.2.

Let Φ'_{-p_a} denote the Hilbert space conjugate to Φ_a, considered as a subset of Φ', and let $e^{\Phi'-p_a}$ denote an analogous subset of $(e^{\Phi})'$.

LEMMA 2.1.3. *Let $p_a \leqslant p_\beta$. Then $e^{\Phi'-p_a} \subset e^{\Phi'-p_\beta}$, and for arbitrary $\varrho \in e^{\Phi'-p_a}$ the following inequalities are fulfilled:*

$$\|\varrho_n\|_{(-2p_\beta)^n} < \frac{C}{2^n}, \quad \text{where } \varrho = \sum_{n=0}^{\infty} \varrho_n,\ \varrho_n \in (\otimes \Phi'_{-p_a})_S^n.$$

Proof. The sequence $\|\varrho_n\|_{(-p_a)^n}$ is square summable, whence it is bounded. Let $\|\varrho_n\|_{(-p_a)^n} < C$. But

$$\|\varrho_n\|_{(-p_a)^n} \geqslant 2^n \|\varrho_n\|_{(-2p_\beta)^n}, \quad \text{whence } \|\varrho_n\|_{(-2p_\beta)^n} < \frac{C}{2^n}.$$

From Lemmas 1.1 and 2.1.3 it follows that in the space $(e^{\Phi})'$ one can define the operation $\odot$ by the same formula as in Lemma 1.1. If $\varrho \in e^{\Phi'-p_a}$, $\sigma \in e^{\Phi'-p_\beta}$, then $\varrho \odot \sigma \in e^{\Phi'-p_\gamma}$, where $p_\gamma \geqslant p_a$ and $p_\gamma \geqslant p_\beta$. The operation $\odot$ is defined for every pair ϱ, σ of elements of $(e^{\Phi})'$.

LEMMA 2.1.4. *The operation $\odot$ has the following properties:*
1° $\odot$ is bilinear and continuous on $(e^{\Phi})' \times (e^{\Phi})'$;
2° if $\varphi \in \Phi$, then

$$\langle e^{\varphi}, \varrho \odot \sigma\rangle = \langle e^{\varphi}, \varrho\rangle \langle e^{\varphi}, \sigma\rangle.$$

Proof. The lemma follows immediately from Lemmas 1.1, 1.2 and 2.1.3.

2.2. Continuous family of Gelfand triplets. Let $\mathfrak{H} = e^H$ be an exponential Hilbert space and $H = \int_X H_x d\mu$, X, μ being a space with measure. We denote e^{H_x} by $\mathfrak{H}_x$. Let $\Phi \subset H$ be a nuclear space whose elements are functions on X with values in H_x. We assume that for each $h_x \in H_x$ the functional

$$\xi_{h_x}: \Phi \ni \varphi \mapsto \langle \varphi, \xi_{h_x}\rangle := \big(\varphi(x)\,|\,h_x\big)_{H_x}$$

is continuous (i. e., belongs to Φ').

For each $x \in H$ the mapping $\delta_x: \Phi \ni \varphi \mapsto \varphi(x) \in H_x$ is weakly continuous, and by the closed graph theorem it is strongly continuous. Let $\mathfrak{Z} \subset e^{\Phi}$ be the set of vectors e^{φ}, $\varphi \in \Phi$. We define a mapping:

$$\mathfrak{Z} \ni e^{\varphi} \mapsto e^{\varphi(x)} \in \mathfrak{H}_x.$$

This mapping can be (uniquely) extended to a linear continuous mapping $e^{\Phi} \to \mathfrak{H}_x$. We shall denote the extension by e^{δ_x}. The possibility of constructing that extension follows from the fact that each $\xi \in \Phi'$ defines a functional $e^{\xi} \in (e^{\Phi})'$ (see Lemma 2.1.2).

Let Φ_x be the image (in H_x) of Φ under the mapping δ_x. We take in Φ_x the topology induced by that mapping. Then Φ_x is a nuclear space as a quotient space of Φ.

By using the mapping e^{δ_x}, we obtain in the same way the space $(e^{\Phi})_x$. It is easy to see that $(e^{\Phi})_x = e^{\Phi_x}$. We have got two families of Gelfand triplets:

$$\Phi_x \subset H_x \subset \Phi'_x, \qquad e^{\Phi_x} \subset \mathfrak{H}_x \subset (e^{\Phi_x})',$$

and the triplets:

$$\Phi \subset H \subset \Phi', \qquad e^{\Phi} \subset \mathfrak{H} \subset (e^{\Phi})'.$$

3. PRODUCT OPERATORS

Definition. An operator $A \in L(\mathfrak{H})$ is called a *product operator* of a given tensor structure in $\mathfrak{H}$ if for each isomorphism of that structure $U_{(X_i)}: \mathfrak{H} \to \bigotimes_i \mathfrak{H}_{X_i}$ the operator $U_{(X_i)} A U_{(X_i)}^{-1}$ has the form $\bigotimes_i A_{X_i}$.

In each space $\mathfrak{H}_x$ let $A_x \in L(\mathfrak{H}_x)$ be an operator such that $A_x(\{e^{h_x}: h_x \in H_x\}) \subset \{e^{h_x}: h_x \in H_x\}$. If for each $h \in H$ the vector field $(h_{1_x})_{x\in X}$ defined by $A_x e^{h_{1_x}} = e^{h_{1_x}}$ belongs to H and the mapping $e^h \mapsto e^{h_1}$ can be extended to a linear operator $A \in L(\mathfrak{H})$, then we say that A is a continuous product of the family (A_x) and write $A = \prod_x A_x$.

It is easy to see that A is a product operator of the canonical tensor structure in $\mathfrak{H} = e^{\int_X H_x d\mu}$. From the results of Araki and Woods it follows that for each product operator $A \in L(e^{\int_X H_x d\mu})$ one can find a family (A_x) such that $A = \prod_x A_x$, $A_x \in L(e^{H_x})$.

Let us take a product operator A such that $Ae^{\Phi} \subset e^{\Phi}$, A is continuous on e^{Φ} and there exists a family (A_x) such that $A = \prod_x A_x$ and $A_x e^{\Phi_x} \subset e^{\Phi_x}$, and A_x are continuous on e^{Φ_x}. Since A transforms product vectors into product vectors, it preserves the set $\{e^{\varphi}: \varphi \in \Phi\}$ and A_x preserve the sets $\{e^{\varphi(x)}: \varphi(x) \in \Phi_x\}$.

Problem. How to construct the eigenfunctionals of the operator A from the eigenfunctionals of the operators A_x?

LEMMA 3.1. *Let $e_{x_0} \in (e^{\Phi_{x_0}})'$ be an eigenfunctional of A_{x_0}. Then the functional ε_{x_0} defined as*

$$e^{\Phi} \ni f \mapsto \langle f, \varepsilon_{x_0}\rangle := \langle e^{\delta_{x_0}} f, e_{x_0}\rangle$$

is an eigenfunctional of A.

Proof. Let $f = e^{\varphi}$, $\varphi \in \Phi$. Then

$$\langle Af, \varepsilon_{x_0}\rangle = \langle e^{\delta_{x_0}}(Af), e_{x_0}\rangle = \langle (Af)(x_0), e_{x_0}\rangle$$
$$= \langle A_{x_0} f(x_0), e_{x_0}\rangle = \lambda_{e_{x_0}} \langle f(x_0), e_{x_0}\rangle = \langle f, \varepsilon_{x_0}\rangle \cdot \lambda e_{x_0}.$$

The vectors $f = e^{\varphi}$ span the space e^{Φ}, whence

$$\langle Af, \varepsilon_{x_0}\rangle = \lambda e_{x_0} \langle f, \varepsilon_{x_0}\rangle \quad \text{for each } f \in e^{\Phi}.$$

THEOREM 2. *Let A be a product operator such that $Ae^{\Phi} \subset e^{\Phi}$, its restriction $A: e^{\Phi} \to e^{\Phi}$ is continuous and there exists a family (A_x) such that $A = \prod_x A_x$, $A_x e^{\Phi_x} \subset e^{\Phi_x}$ and $A_x: e^{\Phi_x} \to e^{\Phi_x}$ are continuous mappings. Let $(e_{x_i}^{a_i})_{i=1,\ldots,n}$ be the eigenfunctionals of A_{x_i}.*

Then the functional

$$\varepsilon_{x_1}^{a_1} \odot \varepsilon_{x_2}^{a_2} \odot \ldots \odot \varepsilon_{x_n}^{a_n} \in e^{\Phi'}, \quad \textit{where } \varepsilon_{x_i}^{a_i} = e_{x_i}^{a_i} \circ e^{\delta_{x_i}}$$

is an eigenfunctional of the operator A.

Proof. The theorem follows immediately from Lemma 2.1.4 and Lemma 3.1. The eigenvalue is in this case the product of the eigenvalues of A_{x_i} corresponding to the eigenfunctionals $e_{x_i}^{a_i}$.

Remark. The set of eigenvalues of A has the property that if λ_1, λ_2 are eigenvalues of A, then $\lambda_1 \cdot \lambda_2$ is also an eigenvalue of A. Because the points x_i are not necessarily different, each A_x has the same property. Nevertheless, since the whole construction depends on the choice of the space Φ, we cannot conclude that the spectra of A and A_x have the same property.

THEOREM 3. *Let the assumptions of Theorem 2 be fulfilled. Then the functionals $\varepsilon_{x_1}^{a_1} \odot \ldots \odot \varepsilon_{x_n}^{a_n}$ separate the vectors of e^{Φ}.*

Proof. Let $f \in e^{\Phi}$. Then $f = \sum_{n=0}^{\infty} f_n$, $f_n \in (\hat{\otimes}\Phi)_s^n$.

Since the elements of Φ are functions on X with values in H_x, the elements of $(\hat{\otimes}\Phi)_s^n$ can be considered as functions on $(X \times X \times \ldots \times X)$ with values in $H_{x_1} \otimes H_{x_2} \otimes \ldots \otimes H_{x_n}$. The condition $\langle f, \varepsilon_{x_1}^{a_1}\rangle = 0$ for an arbitrary functional $\varepsilon_{x_1}^{a_1}$ gives $f_0 = 0$, $f_1 = 0$, $f_n(x, x, \ldots, x) = 0$ for every $x \in X$. From the condition $\langle f, \varepsilon_{x_1}^{a_1} \odot \varepsilon_{x_2}^{a_2}\rangle = 0$ we get more complicated relations, which imply $f_2 = 0$. Taking the conditions $\langle f, \varepsilon_{x_1}^{a_1} \odot \varepsilon_{x_2}^{a_2} \odot \ldots \odot \varepsilon_{x_k}^{a_k}\rangle = 0$ for $k = 1, \ldots, n$, we get $f_k = 0$ for $k \leqslant n$; hence, if $\langle f, \varepsilon_{x_1}^{a_1} \odot \varepsilon_{x_2}^{a_2} \odot \ldots \odot \varepsilon_{x_n}^{a_n}\rangle = 0$ for every collection of functionals $\varepsilon_{x_i}^{a_i}$, then $f = 0$.

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DEPARTAMENT OF MATHEMATICAL METHODS IN PHYSICS
THE UNIVERSITY OF WARSAW

Reçu par la Rédaction le 2. 7. 1970