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On singular integrals of functions in L^1

by

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Abstract. Let K be a kernel of Calderón-Zygmund type on R^n and p.v. $K*f$ the singular integral of a function f defined by means of it. As a generalization of results of Titchmarsh for conjugate functions sufficient conditions are given for the existence of a family of sets D_σ^* whose measures approach 0 as σ tends to ∞ such that

$$\lim_{\sigma\to\infty}\int_{R^n\sim D_\sigma^*}\text{p.v. } K*f(x)\varphi(x)\,dx = \int f(x)\ \text{p.v. } \check{K}*\varphi(x)\,dx,$$

where $\check{K}(x) = K(-x)$ and f has finite weighted L^1 norm. Analogous results are obtained for convolutions. A generalization of the notion of B-integral is used to prove that if f and p.v. $K*f$ are integrable then the Fourier transform of p.v. $K*f$ equals $\hat{K}\hat{f}$

The conjugate function $\tilde{f}$ of an integrable function f of period 2π is defined a.e. by

$$\tilde{f}(x) = -\frac{1}{\pi}\lim_{\varepsilon\to+0}\int_\varepsilon^\pi [f(x+t)-f(x-t)]\tfrac{1}{2}\cot\tfrac{1}{2}t\,dt.$$

Let now f,φ be periodic and belong to $L^p(0,2\pi)$ and $L^{p'}(0,2\pi)$ respectively, where $1<p<\infty$ and as usual p' is the exponent conjugate to p, i.e., $p^{-1}+(p')^{-1}=1$. A well known equality of Riesz (see, e.g., [1] p. 568) asserts

$$\int_0^{2\pi}\tilde{f}(x)\varphi(x)\,dx = -\int_0^{2\pi} f(x)\tilde{\varphi}(x)\,dx.$$

If p equals 1, $\tilde{f}$ need no longer be integrable. As a substitute for Riesz's equality in the case $p=1$ Titchmarsh proved in [17] that if f is periodic and integrable then there is a sequence $\{E_m\}$ of subsets of the interval $(0,2\pi)$ whose measures tend to 2π such that for any integer k

$$\lim_{m\to\infty}\int_{E_m}\tilde{f}(x)e^{-ikx}\,dx = -i\,\mathrm{sgn}\,k\int_0^{2\pi} f(x)e^{-ikx}\,dx.$$

It was shown more generally by Ulyanov in [18] (see also [1] p. 587) that if $f\in L(0,2\pi)$, $\varphi,\tilde{\varphi}\in L^\infty(0,2\pi)$ then for the same sequence $\{E_m\}$

$$\lim_{m\to\infty}\int_{E_m}\tilde{f}(x)\varphi(x)\,dx = -\int_0^{2\pi} f(x)\tilde{\varphi}(x)\,dx. \tag{1}$$

A corollary of Titchmarsh's result is the theorem that if the conjugate function is integrable then its Fourier series is the conjugate Fourier series of f. This theorem has been extended to singular integrals of periodic functions of several variables by Calderón and Zygmund in [7] using the notion of the so-called B-integral.

The purpose of the present note is to establish a generalization of (1) to several variables and weighted L^p norsam well as analogous results) for convolutions instead of products (Proposition 1 and corollaries). Finally a modification of the definition of the B-integral applicable to the non-periodic case will be used to prove a result analogous to that of Calderón and Zygmund (Proposition 2).

The main concern will be with singular integrals with mixed homogeneity in R^n. Suppose $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ are points in R^n then define, as usual

$$x\cdot y = \sum_{i=1}^{n} x_i y_i, \qquad \|x\|^2 = x\cdot x, \qquad S^{n-1} = \{x\colon \|x\| = 1\}, \qquad |x| = \sum_{i=1}^{n} |x_i|.$$

Let a denote an n-tuple of real numbers $\{a_i\}$ such that $1 = a_1 \leqslant a_2 \leqslant \ldots \ldots \leqslant a_n$. For positive t define the linear transformation t^a by $t^a x = (t^{a_1}x_1, \ldots \ldots, t^{a_n}x_n)$. The assumption that all the a_i are at least 1 implies that

$$[x] = \max_{1\leqslant i\leqslant n} |x|^{1/a_i}$$

defines a metric on R^n. Clearly $[t^a x] = t[x]$ for any $t > 0$.

Let K denote a singular integral kernel with mixed homogeneity which is locally bounded away from the origin. Specifically suppose

$$K(t^a x) = t^{-|a|}K(x) \quad \text{for all} \quad t>0, \tag{2}$$

$$\int\limits_{1<[x]<2} K(x)\,dx = 0, \tag{3}$$

$$\int\limits_{[x]>c_1[y]} |K(x-y)-K(x)|\,dx \leqslant c_2 \quad \text{for all } y\in R^n \quad \text{where} \quad c_1, c_2 > 0, \tag{4}$$

$$|K(x)| \leqslant M < \infty \quad \text{for} \quad [x] = 1 (= \max_{1\leqslant i\leqslant n}|x_i|). \tag{5}$$

For simplicity it will be assumed that $c_1, c_2 \geqslant 1$. The singular integral of a function f which is such that $(1+[x])^{-|a|}f$ is integrable is then defined as the principal value

$$\text{p.v.}\, K*f(x) = \lim_{\varepsilon\to 0} \int\limits_{[y]>\varepsilon} K(x-y)f(y)\,dy.$$

Weighted L^p norms are defined by

$$\|f\|_{p,\alpha} = \left(\int |f(x)|^p [x]^{\alpha p}\,dx\right)^{1/p}$$

and L^p_α will denote the space of functions f for which $\|f\|_{p,\alpha}$ is finite. It is well known that conditions (2)–(5) imply

$$\|\text{p.v.}\, K*f\|_{p,\alpha} \leqslant A_{p,\alpha}\|f\|_{p,\alpha} \quad \text{for } 1<p<\infty,\ -|a|/p < \alpha < |a|/p', \tag{6}$$

where $A_{p,\alpha}$ is some constant depending also on c_1, c_2, M appearing in (4) and (5). In fact if $\alpha = 0$ this becomes a special case of a theorem of Benedek, Calderón and Panzone (Theorem 3 of [3]) if it is noted that the metric $\|\ \|$ used there may be replaced by any metric with mixed homogeneity. The general case is then a consequence of results of Stein [15] and more particularly Sadosky [14].

Remark 1. Condition (3) differs in appearance from the corresponding condition 2) on p. 20 of Fabes and Rivière [10] and was chosen for the sake of simplicity. In [10] a smooth metric with mixed homogeneity ϱ is used to introduce coordinates $\varrho = \varrho(x)$, x' of polar type defined by the equations $x = \varrho(x)^a x'$, $\|x'\| = 1$. It is then shown that Lebesgue measure on R^n can be represented as a product measure on $(0,\infty)\times S^{n-1}$ by the formula $dx = J(x')\,dx'\,\varrho^{|a|-1}d\varrho$ where $J(x')$ is infinitely often differentiable on S^{n-1} and bounded below by a positive number and dx' denotes the element of area on S^{n-1}. In place of condition (3) they have

$$\int\limits_{\|x'\|=1} K(x')J(x')\,dx' = 0.$$

It is not hard to see that under condition (2) this is equivalent to (3).

Remark 2. It can be shown that condition (5) can be replaced by the weaker condition that K be locally in L^q away from the origin, where $1/q = 1-|\alpha|/|a|$.

Define the kernel K by $\check{K}(x) = K(-x)$. If E is a measurable subset of R^n, $|E|$ will denote its measure. For greater precision it will be required to use Lorentz norms. As in [12] for f measurable, set

$$\|f\|^*_{pq} = \left(q/p\int_0^\infty \left(f^*(t)t^{1/p}\right)^q dt/t\right)^{1/q},$$

where f^* denotes the decreasing rearrangement of f on $(0,\infty)$, and let

$$\|f\|_{pq} = \|f^{**}\|^*_{pq},$$

where

$$f^{**}(t) = f^{**}(t, r) = \left(t^{-1}\int_0^t f^*(s)^r\,ds\right)^{1/r}, \qquad r\leqslant 1,\ r\leqslant q,\ r<p.$$

L^{pq} will denote the space of functions f for which $\|f\|_{pq}$ is finite.

The main result is then

Proposition 1. *Suppose K satisfies* (4) *and* (6) *and away from the origin it is locally in L^{q_1}. Suppose also*

$$f\in L^1_{-\alpha}, \qquad \varphi\in L^p_\alpha \cap L^\infty_\alpha, \qquad \text{p.v.}\, \check{K}*\varphi \in L^\infty_\alpha,$$

where $0 \leqslant a < |\alpha|$, $1/q \leqslant 1 - a/|\alpha|$, $p < \infty$. *Then for* $\sigma > 0$ *there exists a set* D_σ^* *such that* $\lim_{\sigma\to\infty} \sigma |D_\sigma^*| = 0$ *and if* χ_σ *denotes the characteristic function of the complement* $\sim D_\sigma^*$ *of* D_σ^* *then*

$$\lim_{\sigma\to\infty} \int \chi_\sigma \,\text{p.v.} K * f(x) \varphi(x)\, dx = \int f(x) \,\text{p.v.}\, \check{K} * \varphi(x)\, dx. \tag{7}$$

Proof. It can be assumed that p is so large that $a < |\alpha|/p'$ and also that $f \notin L_{-a}^\infty$. Otherwise the statement follows from continuity of the singular integral operators $\text{p.v.} K *$, $\text{p.v.} \check{K} *$, in $L_{-a}^{p'}$, L_a^p. Let $F(x) = f(x)[x]^{-a}$ and define $F^{**} = F^{**}(\cdot, 1)$ as above. Then the continuous function F^{**} is strictly decreasing. As in [6] p. 91 let β be its inverse. The integrability of F implies $\lim_{\sigma\to\infty} \sigma\beta(\sigma) = 0$ (write F as the sum of a function in $L^1 \cap L^\infty$ and a function of small L^1 norm).

Let I_0 be a ball in the metric [ ] with center at the origin such that $I_0 = \delta^{|\alpha|}\beta(\sigma)$ (δ is to be determined later). Apply the analogue of the Riesz lemma in [13] p. 51 to $F\chi_{\sim I_0}$, where $\chi_{\sim I_0}$ denotes the characteristic function of $\sim I_0$, to obtain a sequence of non-overlapping n-dimensional intervals $\{I_k\}_{k=1}^\infty$ with the following properties: If $D_\sigma = \bigcup_{k=1}^\infty I_k$ the $|D_\sigma| \leqslant \beta(\sigma)$ (see [6] p. 93),

$$|F(x)| \leqslant \sigma \quad \text{outside } D_\sigma \text{ a.e.} \tag{8}$$

and

$$\sigma < |I_k|^{-1} \int_{I_k \sim I_0} |f(x)|[x]^{-a} dx \leqslant 2^{|\alpha|+n}\sigma \quad \text{for } k = 1, 2, 3, \ldots$$

Furthermore if the sides of I_k raised to the power $1/a_i$ are denoted $d_k^{(i)}$ $(i = 1, \ldots, n)$ then $d_k^{(i)}/d_k^{(j)} \leqslant 2$ for $i, j = 1, \ldots, n$.

If r_0 denotes the radius of I_0 then $\delta^{|\alpha|}\beta(\sigma) = \prod_{i=1}^n 2r_0^{a_i} = 2^n r_0^{|\alpha|}$ or

$$r_0 = 2^{-n/|\alpha|}\delta\beta(\sigma)^{1/|\alpha|}. \tag{9}$$

Let d_k denote the diameter of I_k in the metric [ ] then $d_k = \sup_i d_k^{(i)}$ and

$$d_k^{|\alpha|} \leqslant \prod_{i=1}^n (2d_k^{(i)})^{a_i} = 2^{|\alpha|}|I_k| \leqslant 2^{|\alpha|}\beta(\sigma) \quad \text{or}$$

$$d_k \leqslant 2\beta(\sigma)^{1/|\alpha|}. \tag{10}$$

Let $g(x) = |I_k|^{-1} \int_{I_k \sim I_0} f(x)\, dx$ for $x \in I_k$, and $= f(x)\chi_{\sim I_0}$ otherwise and define h by $f(x)\chi_{\sim I_0}(x) = g(x) + h(x)$. For $x \in I_k$

$$\begin{aligned} |g(x)|[x]^{-a} &\leqslant [x]^{-a}|I_k|^{-1} \int_{I_k \sim I_0} |f(x)|\, dx \\ &\leqslant \sup_{y \in I_k} ([y]/[x])^a |I_k|^{-1} \int_{I_k \sim I_0} |f(y)|[y]^{-a} dy \\ &\leqslant (1 + \sup [y-x][x]^{-1})^a 2^{|\alpha|+n}\sigma. \end{aligned}$$

I_k contains a point x^* outside I_0 hence

$$[x] \geqslant r_0 - [x - x^*] \geqslant r_0 - d_k \geqslant (2^{-n/|\alpha|}\delta - 2)\beta(\sigma)^{1/|\alpha|}$$

hence

$$|g(x)|[x]^{-a} \leqslant (1 + 2(2^{-n/|\alpha|}\delta - 2)^{-1})^a 2^{|\alpha|+n}\sigma \tag{11}$$

so suppose

$$\delta > 2^{1+n/|\alpha|}. \tag{12}$$

Let I_0^* denote a ball with center at the origin and radius $(c_1+1)r_0$ and I_k^* a ball with the same center x_k, say, as I_k and radius $(c_1+1)d_k$ in the metric [ ]. Put $D_\sigma^* = \bigcup_{k=0}^\infty I_k^*$ so that $|D_\sigma^*| \leqslant (c_1+1)^{|\alpha|}(\delta^{|\alpha|} + 2^{n+|\alpha|})\beta(\sigma)$. Now

$$\begin{aligned} &\int_{\sim D_\sigma^*} \text{p.v.} K * f(x)\varphi(x)\, dx - \int f(x)\,\text{p.v.}\,\check{K} * \varphi(x)\, dx \\ &= \int_{\sim D_\sigma^*} \text{p.v.} K * (\chi_{I_0} f)(x)\varphi(x)\, dx + \int_{\sim D_\sigma^*} \text{p.v.} K * h(x)\varphi(x)\, dx + \\ &\quad + \int_{D_\sigma^*} \text{p.v.} K * g(x)\varphi(x)\, dx + \int \big(\text{p.v.} K * g(x)\varphi(x) - f(x)\,\text{p.v.} K * \varphi(x)\big)\, dx \\ &= J_1 + J_2 + J_3 + J_4, \quad \text{say.} \end{aligned}$$

It can be assumed without loss of generality that $\|\varphi\|_{\infty,a} \leqslant 1$ and also that $1/q = 1 - a/|\alpha|$. Let the $L^1 . L^{q_1}$ norms of K restricted to the set $\{x: 1 \leqslant [x] \leqslant 2\}$, say, which by assumption are finite be denoted M_1, M_q respectively. C will denote a constant not necessarily the same at each occurrence whose dependence on a, e.g., may be indicated by the symbol $C(a)$ at the first occurrence and which may also depend on K. Then firstly

$$\begin{aligned} |J_1| &\leqslant \int_{\sim I_\sigma^*} |\text{p.v.} K * (\chi_{I_0} f)(x)|[x]^{-a} dx \\ &= \int_{[x] \geqslant (c_1+1)r_0} [x]^{-a} \int_{[y] \leqslant r_0} |K(x-y)||f(y)|\, dy\, dx \\ &\leqslant \int_{[y] \leqslant r_0} |f(y)| \int_{[x] \geqslant (c_1+1)r_0} [x]^{-a}|K(x-y)|\, dx\, dy \\ &\leqslant C(a)\big((c_1+1)/c_1\big)^{|\alpha|} M_1 \int_{[y] \leqslant r_0} |f(y)| \int_{[x] \geqslant (c_1+1)r_0} [x]^{-|\alpha|-a} dx\, dy \\ &\leqslant CM_q \int_{[y] \leqslant r_0} |f(y)| r_0^{-a} dy \leqslant C \int_{[y] \leqslant r_0} |f(y)|[y]^{-a} dy \end{aligned}$$

which tends to zero as $\sigma\to\infty$ since $r_0\to 0$ as $\sigma\to\infty$.

$$|J_2| \leqslant \sum_{k=1}^{\infty}\int_{\sim(I_0^*\cup I_k^*)}\Big|\int_{I_k} K(x-y)h(y)\,dy\Big|[x]^{-a}\,dx$$
$$= \sum_{k=1}^{\infty}\int_{\sim(I_0^*\cup I_k^*)}\Big|\int_{I_k} \big(K(x-y)-K(x-x_k)\big)h(y)\,dy\Big|[x]^{-a}\,dx$$

since

(13) $$\int_{I_k} h(y)\,dy = 0.$$

So

$$|J_2| \leqslant \sum_{k=1}^{\infty}\int_{I_k}|h(y)|\int_{\sim(I_0^*\cup I_k^*)}|K(x-y)-K(x-x_k)|[x]^{-a}\,dx\,dy.$$

Also

$$\int_{\sim(I_0^*\cup I_k^*)}|K(x-y)-K(x-x_k)|[x]^{-a}\,dx$$
$$\leqslant \int_{[x]\leqslant[x_k]/2}\big(|K(x-x_k)|+|K(x-y)|\big)[x]^{-a}\,dx+$$
$$+2^a[x_k]^{-a}\int_{\sim I_k^*}|K(x-y)-K(x-x_k)|\,dx.$$

Furthermore $[x]\leqslant[x_k]/2$ implies

$$[x-x_k]\geqslant[x_k]/2,\ [x-y]\geqslant[x-x_k]-[x_k-y]\geqslant[x_k]/2-r_k.$$

Consequently $[x-y]\geqslant[x_k]/4$ provided

(14) $$[x_k]\geqslant 4r_k.$$

Also $[x_k]\geqslant r_0-d_k$. Hence it is sufficient to assume $r_0\geqslant 5d_k$. Hence by (9) and (10), (14) is satisfied provided

(15) $$\delta\geqslant 10\cdot 2^{n/|a|}.$$

$[x-x_k]\geqslant(c_1+1)r_k$ implies

$$[x-y]\geqslant[x-x_k]-[x_k-y]\geqslant(c_1+1-1)r_k = c_1r_k.$$

Thus

$$\int_{\sim(I_0^*\cup I_k^*)}|K(x-y)-K(x-x_k)|[x]^{-a}\,dx$$
$$\leqslant 2^{2|a|+1}[x_k]^{-|a|}\int_{[x_k]/4\leqslant[x-y]\leqslant2[x_k]}|K(x-y)|\,dx+2^a c_2[x_k]^{-a}.$$

Now $[\cdot]^{-a}\in L^{q'\infty}$ and the L^{q1} norm of $K(x-y)$ restricted to the set of all x such that $[x_k]/4\leqslant[x-y]\leqslant 2[x_k]$ is at most $C(a)M_q[x_k]^{-|a|/q'} = C(a)M_q[x_k]^{-a}$

hence the preceding expression is at most equal to

$$(CM_q+2^a c_2)[x_k]^{-a}.$$

$[x_k]\geqslant[y]-r_k\geqslant[y]-[x_k]/4$ implies $[y]\leqslant 5/4[x_k]$. Thus

$$|J_2|\leqslant C\sum_{k=1}^{\infty}\int_{I_k}|h(y)|[y]^{-a}dy\,(5/4)^a(M_q+c_2) = C\sum_{k=1}^{\infty}\int_{I_k}|f(y)-g(y)|[y]^{-a}dy$$
$$\leqslant C\int_{D_\sigma}|f(y)|[y]^{-a}dy \quad \text{(see (11))}$$

and this tends to zero as $\sigma\to\infty$ since $|D_\sigma|\to 0$. Next

$$|J_3|\leqslant\Big(\int_{D_\sigma^*}\big(|\mathrm{p.v.}K*g(x)|[x]^{-a}\big)^{p'}dx\Big)^{1/p'}\cdot|D_\sigma^*|^{1/p}\leqslant A_{p,a}\|g\|_{p',-a}|D_\sigma^*|^{1/p}$$
$$\leqslant C\sigma^{1/p}\|g\|_{1,-a}|D_\sigma^*|^{1/p}\leqslant C(\sigma|D_\sigma^*|)^{1/p}\|f\|_{1,-a} \quad \text{(by (11))}$$

hence $J_3\to 0$ as $\sigma\to\infty$. Finally

$$\int \mathrm{p.v.}K*g(x)\varphi(x)\,dx = \int g(x)\,\mathrm{p.v.}\check{K}*\varphi(x)\,dx$$

by continuity since $g\in L^{p'}_{-a}$, thus

$$|J_4| = \Big|\int\big(g(x)-f(x)\big)\mathrm{p.v.}\check{K}*\varphi(x)\,dx\Big|\leqslant\|g-f\|_{1,-a}\|\mathrm{p.v.}\check{K}*\varphi\|_{\infty,a}$$
$$\leqslant\big(\|f\chi_{I_0}\|_{1,-a}+\|h\|_{1,-a}\big)\|\mathrm{p.v.}\check{K}*\varphi\|_{\infty,a}$$
$$\leqslant C\int_{I_0\cup D_\sigma}|f(x)|[x]^{-a}dx\,\|\mathrm{p.v.}\check{K}*\varphi\|_{\infty,a}\to 0 \quad \text{as } \sigma\to\infty.$$

Remark 3. By Remark 2 it follows that Proposition 1 applies to kernels K satisfying conditions (2)–(4) and which are locally in L^{q1}, $1/q = 1-a/|a|$.

Remark 4. The proof extends to weight functions $\omega([x])$ instead of $[x]^{-a}$ where $\omega(\tau)$ is (almost) decreasing (i.e., there is a constant $C\geqslant 1$ such that $\tau_1<\tau_2$ implies $\omega(\tau_2)\leqslant C\omega(\tau_1)$) and for some $a<|a|$ $\omega(\tau)\tau^a$ is (almost) increasing, if use is made of a result of Chen in [9]:

(16) $$\|\mathrm{p.v.}K*f\|_{p,\omega}\leqslant A_{p,\omega}\|f\|_{p,\omega}$$

which generalizes (6) and is valid for $1<p<|a|/a$.

To obtain substitutes for Riesz's formula several authors have made use of generalizations of the Lebesgue integral. One of these is the A-integral defined as follows. Let the distribution function of a measurable function h be denoted λ_h, i.e.,

$$\lambda_h(\sigma) = |\{x:\ |h(x)|>\sigma\}|.$$

Then h is called A-integrable (see [1] p. 585) if

$$\lambda_h(\sigma) = o(\sigma^{-1}) \quad \text{as} \quad \sigma \to \infty \tag{17}$$

and $\lim_{\sigma\to\infty} \int [h(x)]_\sigma dx$ exists where $[h(x)]_\sigma$ denotes the function equal to $h(x)$ for $|h(x)| \leqslant \sigma$ and zero otherwise. The limit is denoted (A) $\int h(x)\,dx$. A result of Ulyanov about conjugate functions (see [17] and [1] p. 587) may then be extended as follows.

COROLLARY 1. *If K, f, φ are as in Proposition 1 then* (p.v. $K*f)\varphi$ *is A-intergrable and*

$$(\mathrm{A}) \ \int \mathrm{p.v.}\, K*f(x)\varphi(x)\,dx = \int f(x)\,\mathrm{p.v.}\, \check{K}*\varphi(x)\,dx.$$

Proof. Since by [14] there exists $A_a < \infty$ such that

$$\|\mathrm{p.v.}\,K*f[\cdot]^{-a}\|_{1\infty}^* \leqslant A_a \|f\|_{1,-a}$$

(17) follows. By Proposition 1 it is sufficient to show that

$$\lim_{\sigma\to\infty} \Big(\int \chi_\sigma\, \mathrm{p.v.}\, K*f(x)\varphi(x)\,dx - \int [\mathrm{p.v.}\, K*f(x)\varphi(x)]_\sigma\,dx \Big) = 0.$$

Now

$$\lim_{\sigma\to\infty} \Big| \int_{D_\sigma^*} [\mathrm{p.v.}\, K*f(x)\varphi(x)]_\sigma\,dx \Big| \leqslant \lim_{\sigma\to\infty} \sigma |D_\sigma^*| = 0.$$

Moreover if $E_\sigma = \{x\colon |\mathrm{p.v.}\,K*f(x)|\,|\varphi(x)| > \sigma\}$ then

$$\begin{aligned}
&\int \chi_\sigma |\mathrm{p.v.}\,K*f(x)\varphi(x) - [\mathrm{p.v.}\,K*f(x)\varphi(x)]_\sigma|\,dx \\
&\quad = \int_{E_\sigma} \chi_\sigma |\mathrm{p.v.}\,K*f(x)\varphi(x)|\,dx \\
&\quad \leqslant \int \chi_\sigma |\mathrm{p.v.}\,K*(\chi_{I_0} f)(x)|\,|\varphi(x)|\,dx + \int \chi_\sigma |\mathrm{p.v.}\,K*h(x)|\,|\varphi(x)|\,dx + \\
&\qquad + \int_{E_\sigma} |\mathrm{p.v.}\,K*g(x)|\,|\varphi(x)|\,dx = J_1' + J_2' + J_3', \quad \text{say}.
\end{aligned}$$

J_1', J_2' have been estimated above. The fact that $J_3' \to 0$ is proved as is $J_3 \to 0$ by use of the fact that $|E_\sigma| = o(\sigma^{-1})$ and $[x]^{-a}|g(x)| \leqslant C\sigma$.

Remark 5. It was tacitly assumed that $a > 0$, if $a = 0$, I_0 is to be deleted from the argument and the estimates simplify. In this case the kernel need no longer be homogeneous, that is, K need only be locally integrable away from the origin and only (4) and (6) with $a = 0$ need be satisfied.

The preceding remark implies, in particular, that the conclusion of Proposition 1 applies to Bessel potentials of purely imaginary order. Let f be a tempered distribution. The Bessel potential of complex order z of f is defined by

$$(J^z f)\hat{\ }(x) = (1 + \|x\|^2)^{-z/2} \hat{f}(x).$$

Here the Fourier transform is defined by $\hat{\varphi}(x) = \int e^{-ix\cdot y}\varphi(y)\,dy$ for any test function φ. It is well known that for $f \in L^p$, $1 \leqslant p \leqslant \infty$, $0 < \operatorname{Re} z < \infty$

$$J^z f = G_z * f,$$

where

$$G_z(x) = 2^{-n}\pi^{-n/2}\big(\Gamma(z/2)\big)^{-1} \int_0^\infty e^{-t-\|x\|^2/4t}\, t^{-n/2+z/2-1}\,dt \tag{18}$$

(see, e.g., [11] p. 392). It is also well known that $\|J^z f\|_p \leqslant C_{p,z}\|f\|_p$ for $1 < p < \infty$ where $C_{p,z}$ is bounded for z varying in any bounded subset of the right half-plane (see Theorems 1, 2 of [4]). Therefore by Theorem 2 of [3] to prove that

$$\|J^z f\|_{1\infty} \leqslant C_z \|f\|_1$$

with C_z bounded on bounded subsets of the right half-plane it is sufficient to prove that there exists a $C > 0$ such that for any $y \in R^n$

$$\int_{\|x\| > 2\|y\|} |G_z(x-y) - G_z(x)|\,dx \leqslant C_z. \tag{19}$$

The proof of Theorem 4 of [4] shows that

$$|\operatorname{grad} G_z(x) \leqslant C_z e^{-\|x\|/2} \|x\|^{-n-1-u} \qquad (u = \operatorname{Re} z),$$

where C_z is locally bounded in the closed right half-plane. This follows similarly from the representation (18):

$$\begin{aligned}
\Big| \frac{\partial}{\partial x_i} G_z(x) \Big| &\leqslant C_z \|x\| \int_0^\infty e^{-t-\|x\|^2/4t}\, t^{-n/2+u/2-2}\,dt \\
&\leqslant C_z |x| \Big(e^{-\|x\|2/8} \int_0^1 e^{-\|x\|^2/8t}\, t^{-n/2+u/2-2}\,dt + \\
&\qquad + \sup_{t>1} t^{-n/2+u/2-2} e^{-t/4} \sup_{t>1} (e^{-\|x\|^2/4t - t/2}) \int_1^\infty e^{-t/4}\,dt \Big) \\
&\leqslant C_z \|x\| (e^{-\|x\|/2} \|x\|^{-n+u-2} + e^{-\|x\|/\sqrt{2}}) \leqslant C_z \|x\|^{-n+u-1} e^{-\|x\|/2}.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{\|x\|>2\|y\|} |G_z(x-y) - G_z(x)|\,dx &\leqslant C_z 2^{n-u+1} \|y\| \int_{\|x\|>2\|y\|} \|x\|^{-n+u-1} e^{-\|x\|/2}\,dx \\
&\leqslant C_z \|y\| \int_{2\|y\|}^\infty e^{-r} r^{u-2}\,dr \leqslant C_z \|y\| \int_{2\|y\|}^\infty r^{-2}\,dr \leqslant C_z
\end{aligned}$$

since $e^{-r} \leqslant C_u r^{-u}$ for $u \geqslant 0$. Since for $f \in L^2$, $J^z f \to J^{iv} f$ in L^2 and hence in measure as $z \to iv$ in the right half plane it follows that

$$\lambda_{J^{iv} f}(\sigma) \leqslant C_z \|f\|_1 \sigma^{-1} \quad \text{for } f \in L^1 \cap L^2$$

and since $L^1 \cap L^2$ is dense in L^1, J^{iv} extends to a continuous operator $\tilde{J}^{iv}$, say, from L^1 to $L^{1\infty}$. Also if $f \in L^1$ then

$$J^{u+iv} f = \tilde{J}^{iv}(J^u f) \to \tilde{J}^{iv} f \quad \text{in } L^{1\infty} \quad \text{as } u \downarrow 0 \tag{20}$$

since G_u is an approximate identity. With these definitions there holds the following version of Proposition 1.

Corollary 2. *If $f \in L^1$, $\varphi \in L^p \cap L^\infty$ for some $p < \infty$, $J^{iv}\varphi \in L^\infty$ then there is a function D^*_σ from the interval $(0, \infty)$ to the set of measurable sets such that $\lim\limits_{\sigma\to\infty} \sigma |D^*_\sigma| = 0$ and if χ_σ is as in Proposition 1 then*

$$\lim_{\sigma\to\infty} \int \chi_\sigma(x) \tilde{J}^{iv} f(x) \varphi(x)\, dx = \int f(x) J^{iv}\varphi(x)\, dx. \tag{21}$$

If $\tilde{J}^{iv} f$ is locally integrable in an open set then it agrees there with the distribution $J^{iv} f$.

Proof. After the above discussion to prove the first part it only remains to note that if g, h are as in the proof of Proposition 1 then (19) holds for purely imaginary $z = iv$ and in the complement of $D^*_\sigma = \bigcup\limits_{K=1}^{\infty} I^*_k$, $\tilde{J}^{iv} h = G_{iv} * h$. To prove the latter assertion let $\psi \in L^1$. Then for $u > 0$ $J^{u+iv}\psi = G_{u+iv} * \psi$ tends to $G_{iv} * \psi$ locally in L^1 away from the support of ψ, because G_{u+iv} tends to G_{iv} in $L^1(\{x\colon |x| \geqslant \delta\})$ for any $\delta > 0$. It follows that $\tilde{J}^{iv}\psi = G_{iv} * \psi$ away from the support of ψ. Hence if h_k denotes the restriction of the function h to I_k $\tilde{J}^{iv} h_k = G_{iv} * h_k$ outside I_k. (13) and (18) imply

$$\int_{\sim I^*_k} |G_{iv} * h_k|\, dx \leqslant C \|h_k\|_1$$

and hence

$$G_{iv} * h = \sum_{i=1}^{\infty} G_{iv} * h_k = \sum_{i=1}^{\infty} \tilde{J}^{iv} h_k = \tilde{J}^{iv} h \qquad \text{a.e.}$$

outside $\bigcup\limits_{k=1}^{\infty} I^*_k$ by continuity of $\tilde{J}^{iv}\colon L^1 \to L^{1\infty}$.

The last part of the corollary follows by application of (21) to any test function in the given open set.

Recall that any of the various singular integral kernels K gives rise to a tempered distribution, to be denoted p.v.K. It is defined by

$$(\text{p.v.} K, \varphi) = \text{p.v.} K * \check{\varphi}(0)$$

where φ is any test function and $\check{\varphi}(x) = \varphi(-x)$. The Fourier transform of p.v.K will be denoted $\hat{K}$. The next corollary gives information about the relation between p.v.$K*f$ and $\hat{K}\hat{f}$. As usual $L^p + L^\infty$ will denote the space of functions f which are sums of functions f_1, f_2 in L^p and L^∞ respectively. $L^p + L^\infty$ is normed by

$$\|f\|[L^p + L^\infty] = \inf\{\|f_1\|_p + \|f_2\|_\infty\colon f = f_1 + f_2\}.$$

Corollary 3. *Suppose K is as in Proposition 1 or more generally is locally integrable in $R^n \sim \{0\}$ and $\hat{K}$ is bounded and (4) and therefore (by [3]) (6) with $a = 0$ hold. Then for any $p < \infty$*

$$\lim_{\sigma\to\infty} \Big[\sup_x \Big| \lim_{r\to\infty} (\text{in } L^p + L^\infty) \int_{|y|\leqslant r} \chi_\sigma(y)\, \text{p.v.} K * f(y) e^{-ix\cdot y} dy - \hat{K}(x)\hat{f}(x)\Big|\Big] = 0. \tag{22}$$

Proof.

$$\int_{|y|\leqslant r} \chi_\sigma(y)\, \text{p.v.} K * f(y) e^{-ix\cdot y} dy - \hat{K}(x)\hat{f}(x)$$
$$= \int \chi_\sigma(y)\, \text{p.v.} K * h(y) e^{-ix\cdot y} dy - \int_{D^*_\sigma} \text{p.v.} K * g(y) e^{-ix\cdot y} dy +$$
$$+ \Big(\int_{|y|\leqslant r} \text{p.v.} K * g(y) e^{-ix\cdot y} dy - \hat{K}(x)\hat{g}(x)\Big) + \hat{K}(x)\big(\hat{g}(x) - \hat{f}(x)\big)$$
$$= \sum_{l=1}^{3} J_l(x, r) + J_4(x), \quad \text{say.}$$

$J_1(\cdot, r) \to J_1(\cdot, \infty)$ in L^∞ and for all x, r

$$|J_1(x, r)| \leqslant \|\text{p.v.} K * h\chi_\sigma\|_1 \leqslant c_2 \|h\|_1 \to 0 \quad \text{as} \quad \sigma \to \infty.$$

$J_2(\cdot, r) \to J_2(\cdot, \infty)$ in L^∞ and $|J_2(x, r)| \leqslant \int_{D^*_\sigma} |\text{p.v.} K * g(y)|\, dy \to 0$ as $\sigma \to \infty$ (see the estimates for the previous J_2, J_3). By (6) and the Hausdorff–Young theorem, if $2 \leqslant p < \infty$, then $\|J_3(\cdot, r)\|_p \to 0$ since p.v.$K * g \in L^{p'}$ for $1 < p' \leqslant 2$ as $g \in L^1 \cap L^\infty$. Also

$$|J_4(x)| \leqslant \|\hat{K}\|_\infty \|\hat{f} - \hat{g}\|_\infty \leqslant \|\hat{K}\|_\infty \|h\|_1 \to 0 \quad \text{as} \quad \sigma \to \infty.$$

Remark 6. With the notation $h(x)^\varepsilon = h(x)$ if $|h(x)| > \varepsilon$, 0 otherwise and $|||g||| = \max(\|g\|_1, \|g\|_\infty)$ the conclusion can be strengthened to: there exists a $C > 0$ such that for any $\varepsilon > 0$

$$\lim_{r\to\infty} \int \Big[\exp\Big\{C^{-1} |||g|||^{-1} \times$$
$$\times \Big|\Big[\int_{|y|\leqslant r} \chi\sigma(y)\, \text{p.v.} K * f(y) e^{-ix\cdot y} dy - \hat{K}(x)\hat{g}(x)\Big]^\varepsilon\Big|\Big\} - 1\Big] dx = 0.$$

It is sufficient to show that

$$\lim_{r\to\infty} \int \{\exp(C^{-1} |||g|||^{-1} |[J_l(x, r)]^{\varepsilon/3}|) - 1\}\, dx = 0$$

for $l = 1, 2, 3$. For J_1, J_2 this is obvious. As regards J_3 note that Theorem 2 of [3] gives $A_p = A_{p,0} = O(p')$ in (6) as $p \downarrow 1$. Hence by the Hausdorff–Young inequality

$$\|J_3(\cdot, r)\|_p \leqslant C \|\text{p.v.} K * g\|_{p'} \leqslant Cp \|g\|_{p'} \leqslant Cp\, |||g|||.$$

The assertion then follows from the following complementary remark to [19], II Theorem 4.41 applied to $J_3(\cdot, r)$.

LEMMA 1. *Let* $\{\Phi_r\}_{r>0}$ *be a family of functions such that* $\|\Phi_r\|_p \leqslant Cp$ *for all* p *satisfying* $p_0 \leqslant p < \infty$ *and* $\|\Phi_r\|_{p_0} \to 0$ *as* $r \to \infty$. *Then for any* $\varepsilon > 0$

$$\lim_{r\to\infty} \int \left[\exp\left((2eC)^{-1}|\Phi_r^\varepsilon(x)|\right) - 1\right] dx = 0. \tag{23}$$

Proof. Let $E_{r,\varepsilon} = \{x\colon |\Phi_r(x)| > \varepsilon\}$ then

$$\lambda_{\Phi_r}(\varepsilon) = |E_{r,\varepsilon}| \leqslant \varepsilon^{-p_0}\|\Phi_r\|_{p_0}^{p_0},$$

$$\int \left[\exp\left((2eC)^{-1}|\Phi_r^\varepsilon(x)|\right) - 1\right] dx$$

$$\leqslant \sum_{p=1}^{N-1} (2eC)^{-p}(p!)^{-1}\|\Phi_r^\varepsilon\|_N^p(\|\Phi_r\|_{p_0}^{p_0}\varepsilon^{-p_0})^{1-p/N} + \sum_{p=N}^{\infty}(p!)^{-1}(2e)^{-p}p^p$$

hence

$$\limsup_{r\to\infty} \int \left[\exp\left((2eC)^{-1}|\Phi_r^\varepsilon(x)|\right) - 1\right] dx \leqslant \sum_{N}^{\infty}(p!)^{-1}(2e)^{-p}p^p \quad (N \geqslant p_0).$$

(23) follows if N is made to approach infinity.

Corollary 3 implies

COROLLARY 4. *If* p.v. $K*f$ *is integrable then* $(\text{p.v.}\,K*f)^\wedge = \hat{K}\hat{f}$ *a.e. Hence if* $\hat{K}$ *is continuous except at the origin* $(\text{p.v.}\,K*f)^\wedge = \hat{K}\hat{f}$ *and* $\hat{f}(0) = \int f(x)\,dx = 0$.

The remaining corollaries deal with convolutions involving singular integrals.

COROLLARY 5. *Let* $f, \varphi \in L^1$, p.v. $K*f \in L^1$ *then*

$$(\text{p.v.}\,K*f)*\varphi = \text{p.v.}\,K*(f*\varphi) \quad \textit{a.e.}$$

Proof. With $f = g + h$ as above

$$\lim_{\sigma\to\infty} \|(\chi_\sigma \text{p.v.}\,K*f)*\varphi - (\text{p.v.}\,K*f)*\varphi\| = 0,$$

$$\lim_{\sigma\to\infty} \|\chi_\sigma \text{p.v.}\,K*f - \text{p.v.}\,K*g\| = 0,$$

hence $\lim\limits_{\sigma\to\infty} \|(\text{p.v.}\,K*g)*\varphi - (\text{p.v.}\,K*f)*\varphi\| = 0$. Also

$$(\text{p.v.}\,K*g)*\varphi = \text{p.v.}\,K*(g*\varphi)$$

and $\lim\limits_{\sigma\to\infty} g*\varphi = f*\varphi$ in L^1. Thus $\lim\limits_{\sigma\to\infty} \text{p.v.}\,K*(g*\varphi) = \text{p.v.}\,K*(f*\varphi)$, hence

$$(\text{p.v.}\,K*f)*\varphi = \text{p.v.}\,K*(f*\varphi)$$

in $L^{1\infty}$, that is, a.e.

In what follows attention will be restricted to homogeneous kernels with $a_1 = \ldots = a_n = 1$, *for simplicity*. In extending Corollary 5 to convolutions in weighted norms it appears necessary to obtain a norm inequality for convolutions in weighted norms by interpolation between the fractional integration theorem in weighted norms and Young's inequality for convolutions. This may be of some independent interest. The full generality of the following discussion is in fact not necessary to prove the remaining corollaries of Proposition 1.

Define the weighted Lorentz norms $\|\cdot\|_{pq,a}$ by

$$\|f\|_{pq,a} = \|f[\cdot]^a\|_{pq}$$

and let $L_a^{pq} = \{f\colon \|f\|_{pq,a} < \infty\}$. Also let a^+ denote the positive part $\max(a, 0)$ of a. A slightly more general version of the fractional integration theorem in weighted norms given in [16] can then be stated as follows:

The bilinear operator: $(f, g) \to f*g$ is bounded from $L_a^{pu} \times L_\gamma^\infty$ to $L_{-\beta}^{qv}$ if

$$1/q + 1/p' = (a+\beta+\gamma)/n, \quad a+\beta \geqslant 0,\ 0 \leqslant \gamma < n,\ a^+/n \leqslant 1/p',$$

$$\beta^+/n \leqslant 1/q, \quad 0 < u \leqslant v$$

except if $1/p' = a^+/n$ or $1/q = \beta^+/n$ in which case it is required that $1 = u = v'$.

By interpolation between this result and the statement obtained from it by interchanging $p, \infty (= r)$ and a, γ respectively or by passing to the adjoint or both and the convolution theorem for L^{pq} spaces (Theorem 4.10 of [12]) further (quasi-) norm inequalities can be obtained. That is, the inequality

$$\|f*g\|_{qw,-\beta} \leqslant C\|f\|_{pu,a}\|g\|_{rv,a} \quad (C = C(p, r, q, a, \ldots, w)) \tag{24}$$

holds for some u, v, w provided $(1/p, 1/r, 1/q, a/n, \beta/n, \gamma/n)$ is in the convex hull in the hyperplane

$$1/p + 1/r - 1/q + a/n + \beta/n + \gamma/n - 1 = 0 \tag{25}$$

of R^6, of the convex sets determined by

$$0 \leqslant 1/p \leqslant 1,\ 1/r = 0,\ a/n + 1/p \leqslant 1,\ \beta/n - 1/q \leqslant 0,$$

$$a/n + \beta/n \geqslant 0,\ 0 \leqslant \gamma/n < 1$$

and the three sets obtained from it by interchanging p and r, a and γ in the preceding inequalities or interchanging $1/p$ with $1/q'$ provided L^{qw} is a normed space, and a with β, or both as well as the set defined by

$$0 \leqslant 1/p \leqslant 1,\ 0 \leqslant 1/r \leqslant 1,\ 1/p + 1/r \geqslant 1,\ a = \beta = \gamma = 0.$$

It is well known that such a convex hull is again determined by a finite number of linear inequalities in $1/p$, $1/r$, $1/q$, a/n, β/n, γ/n.

Interpolation between $L_{a^*}^{p^*u^*} \times L_{\gamma^*}^{\infty} \to L_{-\beta^*}^{q^*v^*}$ and $L^{1/(1-t),w} \times L^1 \to L^{1/(1-t),w}$ where $t = 0$, $w = 1$ or $0 < t < 1$, $1 \leqslant w \leqslant \infty$ or $t = 1$, $w = \infty$ (a consequence of Minkowski's inequality for integrals) gives

$$(26) \qquad L_a^{pu} \times L_\gamma^r \xrightarrow{*} L_{-\beta}^{qv} \quad \text{where } 1/p = (1-t)/r + 1/(r'p^*), \quad 1/q = (1-t)/r + 1/(r'q^*)$$

and

$$(27) \qquad \begin{cases} 1/u = 1/(rw) + 1/(r'u^*), & 1/v = 1/(rw) + 1/(r'v^*), \\ a = a^*/r', \ \beta = \beta^*/r', \ \gamma = \gamma^*/r'. \end{cases}$$

For $1 < r < \infty$ the conditions on p^*, q^*, u^*, v^*, a^*, β^*, γ^* become

$$(28) \qquad \begin{cases} 1/q = (a+\beta+\gamma)/n + 1/p + 1/r - 1, \ a+\beta \geqslant 0, \ 0 \leqslant \gamma < n/r', \\ a^+/n = a^{*+}/(r'n) \leqslant 1/(r'p^{*\prime}) = (1-t)/r + 1/r' - 1/p, \\ \beta^+/n = \beta^{*+}/(r'n) \leqslant 1/q - (1-t)/r. \end{cases}$$

The latter two relations can be satisfied provided there exists t such that

$$1 - r(1/p' - a^+/n) \leqslant 1 - t \leqslant r(1/q - \beta^+/n)$$

hence if

$$a^+/n \leqslant 1/p', \quad \beta^+/n \leqslant 1/q, \quad a^+/n + \beta^+/n \leqslant 1/p' + 1/q - 1/r.$$

If all three inequalities are strict $0 \leqslant 1/v^* \leqslant 1/u^* < \infty$, $0 \leqslant 1/w \leqslant 1$ and hence from (27) $0 < u \leqslant v \leqslant \infty$ otherwise $u^* = 1$, $v^* = 0$. Thus if $a^+/n = 1/p'$ t must be zero and $w = 1$, $1/u = 1/r + 1/r' = 1$, $v = r$. In particular

$$(29) \qquad L_a^1 \times L_\gamma^r \xrightarrow{*} L_{-\beta}^{qr}$$

if $0 < 1/q$, $\beta \geqslant -a \geqslant 0$, $\beta/n \leqslant 1/q - 1/r$, $0 \leqslant \gamma < n/r'$. If $\beta^+/n = 1/q$ t must equal 1 and $w = \infty$, $u = r'$, $v = \infty$. If thirdly

$$a^+/n < 1/p', \ \beta^+/n < 1/q, \ (a^+ + \beta^+)/n = 1/p' + 1/q - 1/r$$

then $1 \leqslant u \leqslant r'$, $1/v \leqslant 1/u - 1/r'$.

The following lemma is obtained by further interpolation (cf. the proof of Theorem 4.10 of [12]). It clearly does not contain the endpoint results just stated. But even if these are included (30) below apparently only gives part of the convex hull referred to above.

LEMMA 2. $L_a^{pu} \times L_\gamma^{rv} \xrightarrow{*} L_{-\beta}^{qw}$ provided (25) and $u, v > 0$, $1/w \leqslant 1/u + 1/v$

$$(30) \qquad 1 < p, q, r < \infty, \ a/n < 1/p', \ \beta/n < 1/q, \ 0 \leqslant \gamma/n < 1/r', \quad a/n + \beta/n < 1/p' + 1/q - 1/r.$$

Proof. If (30) holds then there exist p_0, p_1, $p_0 < p < p_1$, q_i corresponding to p_i such that (25) and (30) are satisfied for $p_i, r, q_i, a, \beta, \gamma$, $i = 0, 1$. Hence by the Marcinkiewicz interpolation theorem for Lorentz spaces (see, e.g., [12]) and the discussion preceding the lemma

$$L_a^{p\infty} \times L_\gamma^{ru} \xrightarrow{*} L_{-\beta}^{qu}, \quad u > 0.$$

Also since $1 < r < \infty$ by the same argument

$$L_a^{pu} \times L_\gamma^{r\infty} \xrightarrow{*} L_{-\beta}^{qu}, \quad u > 0.$$

Now the complex method of interpolation yields Lemma 2 (the spaces involved are Banach spaces only if $u, v, w \geqslant 1$, therefore [8], [12] instead of [6] may have to be used).

Remark 7. Let φ be non-negative and decreasing and $f(x) = \varphi(|x|)$ then

$$\|f\|_{rv,\gamma} = C\Big(\int_0^\infty [\varphi(t^{1/n}) t^{\gamma/n+1/r}]^v \, dt/t\Big)^{1/v} = C\|f\|_{s,v}$$

provided $1/s = 1/r + \gamma/n$. It follows that Lemma 2 with $\beta = 0$ and the rearrangement theorem of F. Riesz imply Young's inequality for convolutions in L^{pq} spaces (i.e. Theorem 4.10 of [12]). (The theorem of F. Riesz says, in particular, that the radial rearrangement of the convolution of two non-negative functions is at most equal to the convolution of the radial rearrangements of the functions.)

The desired extension of Corollary 5 to weighted L^p spaces then is

COROLLARY 6. *With the assumptions and notation of Proposition* 1 (*restricted to the case* $a_1 = \ldots = a_n = 1$) *suppose* $f \in L^1 a$, $\varphi \in L_\gamma^r$ *where*

$$1 < r < \infty, \quad a \leqslant 0, \quad 1/q = (a+\beta+\gamma)/n + 1/r, \quad 0 \leqslant \gamma < n/r', \quad \beta < n/q - n/r, \quad a+\beta \geqslant 0.$$

Then

$$\lim_{\sigma\to\infty} (\chi_\sigma \text{p.v.} K * f) * \varphi = f * (\text{p.v.} K * \varphi) = \text{p.v.} K * (f * \varphi) \quad \text{a.e.}$$

This is again proved similarly as Proposition 1 by the observation that $a^+ = 1/p' = 0$ and by choosing $p_1 > 1$ sufficiently close to 1 so that there exists a $\beta_1 > \beta$ such that .

$$1/q = (a+\beta_1+\gamma)/n + 1/p_1 + 1/r - 1, \quad \beta_1 < n/q - n/r.$$

The equality

$$(\text{p.v.} K * g) * \varphi = g * (\text{p.v.} K * \varphi)$$

certainly holds if g, φ are test functions hence by continuity of both terms of the equality as functions from $L_a^{p_1} \times L_\gamma^r$ to $L_{-\beta_1}^{qr}$ (see (6) and Lemma 2) it must be true for $g \in L_a^{p_1}$, $\varphi \in L_\gamma^r$. Hence

$$(\chi_\sigma \text{p.v.} K * f) * \varphi = g * (\text{p.v.} K * \varphi) - \big((1-\chi_\sigma)\text{p.v.} K * g\big) * \varphi + (\chi_\sigma \text{p.v.} K * h) * \varphi.$$

By means of (29) and Lemma 2 it follows similarly as before that in $L^{qr}_{-\beta}$

$$g*(\text{p.v.}K*\varphi)\to f*(\text{p.v.}K*\varphi),$$

$$\big((1-\chi_\sigma)\text{p.v.}K*g\big)*\varphi\to 0, \qquad (\chi_\sigma\text{p.v.}K*h)*\varphi\to 0$$

as $\sigma\to\infty$.

Remark 8. The analogous result in case $r=\infty$ (fractional integration) can be proved similarly. Hence if $\text{p.v.}K*f\in L^1$, $0<\gamma<n$ then

$$\text{p.v.}K*(|\cdot|^{-\gamma}*f) = |\cdot|^{-\gamma}*(\text{p.v.}K*f)$$

(the left hand side is well defined since the linear operator $\text{p.v.}K*$ is bounded on $L^{n/\gamma\infty}$). Corollary 5, however, already implies a more general result.

Let f and $\text{p.v.}K*f$ be integrable and $\varphi\in L^{ru}$ for $1<r<\infty$, $u>0$. Then, since $L^1\cap L^{ru}$ is dense in L^{ru} and since both $\varphi\to\varphi*(\text{p.v.}K*f)$ and $\varphi\to\text{p.v.}K*(\varphi*f)$ are continuous in L^{ru} Corollary 5 implies $\varphi*(\text{p.v.}K*f)$ $=\text{p.v.}K*(\varphi*f)$.

Corollary 7. *If in the situation of Corollary 6 one defines*

$$[\text{p.v.}K*f(y)]_{\sigma,\alpha} = [\text{p.v.}K*f(y)|y|^{\alpha}]_\sigma |y|^{-\alpha}$$

then

$$\lim_{\sigma\to\infty}[\text{p.v.}K*f]_{\sigma,\alpha}*\varphi = f*(\text{p.v.}K*\varphi) \quad \text{in } L^{qr}_{-\beta}.$$

By corollary 6 it suffices to show

$$\int\chi_\sigma(y)\,\text{p.v.}K*f(y)\varphi(\cdot-y)\,dy-\int[\text{p.v.}Kf*(y)]_{\sigma,\alpha}\varphi(\cdot-y)\,dy\to 0$$

in $L^{qr}_{-\beta}$. But $(\text{p.v.}K*f)\chi_\sigma-[\text{p.v.}K*f]_{\sigma,\alpha}\to 0$ in L^1_α as follows by replacing $\varphi(y)$ by $|y|^\alpha$ in the proof of Corollary 1.

The last corollary is analogous to the "multiplication formula" for Fourier transforms.

Corollary 8. *Let $f\in L^1$, $\varphi\in L^p\cap L^1$ for some $p>1$, then*

$$\lim_{\sigma\to\infty}\int\chi_\sigma\big(\text{p.v.}K*f(x)\big)\hat\varphi(x)\,dx = \int\hat K(x)\hat f(x)\varphi(x)\,dx.$$

The proof is similar if use is made of the fact that

$$\int\text{p.v.}K*g(x)\hat\varphi(x)\,dx = \int(\text{p.v.}K*g)\hat{\ }(x)\varphi(x)\,dx$$

since $\text{p.v.}K*g\in L^p$ $(p\leqslant 2)$ and also $(\text{p.v.}K*g)\hat{\ } = \hat K\hat g$.

A different proof of Corollary 4 can be based on the following proposition which may be considered as an extension to non-periodic functions of the theorem that the Fourier B-integral of the singular integral $\text{p.v.}K*f$ equals $\hat K\hat f$ (see [7]). As usual denote the space of continuous functions of compact support by $\mathscr{K}(R^n)$. For any set E, $\mathscr{K}(E)$ will denote the subspace consisting of those elements of $\mathscr{K}(R^n)$ whose support is contained in E. $\mathscr{M}(R^n)$ denotes the dual of $\mathscr{K}(R^n)$ consisting of all (Radon-) measures. A measure is called discrete if its support is discrete, i.e., intersects any compact set in a finite set of points.

Proposition 2. *Suppose K is locally integrable away from the origin and satisfies (2), (3), (4) and suppose $f\in L^1$. Let β be a continuously differentiable function on R which equals 1 in $(0,\frac12)$ and vanishes in $(1,\infty)$. Then*

$$\lim_{\delta\to 0}\Big(\lim_{\mu\to\beta(\delta[\cdot])}\int\text{p.v.}K*f(t-\cdot)e^{-ix\cdot(t-\cdot)}\,d\mu(t)\Big) = \hat K(x)\hat f(x) \tag{31}$$

for all x. Here $\lim\limits_{\mu\to\beta(\delta[\cdot])}$ means limit in L^∞_{loc} as the discrete measure μ tends to $\beta(\delta[\cdot])$ weakly with respect to the pairing of $\mathscr{M}(R^n)$ with $\mathscr{K}(R^n)$ while the support of μ remains in an arbitrary bounded set. The inner limit is a bounded function on R^n with a bound independent of x,δ which tends to $\hat K(x)\hat f(x)$ as δ tends to zero.

Proof. Let the net of measures $\{\mu_i\}$ converge weakly to 0 in such a manner that the supports of the μ_i are all contained in a compact set A. It follows by the uniform boundedness theorem that there is $C<\infty$ such that $\|\mu_i\|\,(=|\mu_i|(R^n))\leqslant C$ for all i. Let B be another compact set and set $A-B=\{x-y\colon x\in A,\ y\in B\}$. Then if $F\in L^1(A-B)$

$$\int_A F(t-\cdot)\,d\mu_i(t)\to 0 \quad \text{in } L^1(A-B). \tag{32}$$

For

$$\int_A\Big|\int_B F(t-y)\,d\mu_i(t)\Big|\,dy\leqslant\int_{A-B}|F(t)|\,dt\,\|\mu_i\|\leqslant C\int_{A-B}|F(t)|\,dt$$

and for $F\in\mathscr{K}(R^n)$ (32) holds pointwise in A. Hence by dominated convergence it holds for $F\in\mathscr{K}(R^n)$, Since $\mathscr{K}(R^n)$ (or rather the set of restrictions of the elements of $\mathscr{K}(R^n)$ to $A-B$) is dense in $L^1(A-B)$ (32) is valid for $F\in L^1(A-B)$.

Let $\sigma>0$, $f=f^\sigma+f_\sigma$, where f^σ is as defined in Remark 6. Define

$$\hat K(x,t,\delta) = \text{p.v.}\int K(u)\beta(\delta[u-t])e^{-ix\cdot u}\,du,$$

$f_\sigma\in L^1\cap L^\infty$ and therefore

$$\int\text{p.v.}K*f_\sigma(t-y)e^{-ix\cdot(t-y)}\beta(\delta[t])\,dt = \int f_\sigma(t-y)\,\text{p.v.}\check K*\big(e^{-ix\cdot(\cdot-y)}\beta(\delta[\cdot])\big)(t)\,dt$$

$$=\int f_\sigma(t-y)e^{-ix\cdot(t-y)}\hat K(x,-t,\delta)\,dt.$$

Hence

$$\int \text{p.v.}K*f(t-y)e^{-ix\cdot(t-y)}d\mu(t)-\int f(t-y)e^{-ix\cdot(t-y)}\hat{K}(x,-t,\delta)\,dt$$
$$=\int \text{p.v.}K*f_\sigma(t-y)e^{-ix\cdot(t-y)}\big(d\mu(t)-\beta(\delta[t])\,dt\big)+$$
$$+e^{ix\cdot y}(\text{p.v.}K*f^\sigma)*(e^{ix\cdot(\cdot)}\check{\mu})(-y)-\int f^\sigma(t-y)e^{-ix\cdot(t-y)}\hat{K}(x,-t,\delta)\,dt$$
$$=\sum_{l=1}^{3} I_l(x,y,\delta),\quad \text{say.}$$

Since $\text{p.v.}K*f_\sigma\in L^p$ for $1<p<\infty$ it is locally integrable and so by (32) for any compact set B $\|I_1(x,\cdot,\delta)\chi_B\|_1\to 0$ as $\mu\to\beta(\delta[\cdot])$ in the described manner. Next since convolution with $e^{ix\cdot(\cdot)}\check{\mu}$ is a finite linear combination of translations

$$\text{p.v.}K*f^\sigma*(e^{ix\cdot(\cdot)}\check{\mu}) = \text{p.v.}K*\big(f^\sigma*(e^{ix\cdot(\cdot)}\check{\mu})\big).$$

Hence as the hypotheses imply that $\text{p.v.}K*$ is bounded from L^1 to $L^{1\infty}$ (see [3])

$$\|I_2(x,\cdot,r)\|^*_{1\infty}\leqslant C\|f^\sigma*(e^{ix\cdot(\cdot)}\check{\mu})\|_1\leqslant C\|f^\sigma\|\,\|\mu\|\leqslant C\|f^\sigma\|\to 0$$

as $\sigma\to\infty$.

It will next be proved that

$$\lim_{\delta\to 0}\hat{K}(x,t,\delta) = \hat{K}(x) \tag{33}$$

the convergence being uniformly bounded in x and t.

Let $x^* = \varrho(\pi^{-1/2}x)^{-2a}x$, where ϱ is the metric with mixed homogeneity a referred to in Remark 1, i.e., $\varrho=1$ on the euclidean unit sphere S^{n-1}. Since for $x\in R^n$, $\lambda>0$

$$\min(\lambda,\lambda^{1/a_n})\,\varrho(x)\leqslant\varrho(\lambda x)\leqslant\max(\lambda,\lambda^{1/a_n})$$

it follows that

$$\pi^{1/a_n}\varrho(x)^{-1}\leqslant\varrho(x^*)\leqslant\pi\varrho(x)^{-1}.$$

ϱ and $[\cdot]$ having the same mixed homogeneity are equivalent, i.e., $\varrho/[\cdot]$ and $[\cdot]/\varrho$ are bounded on R^n.

$$x\cdot x^* = \|\varrho(\pi^{-1/2}x)^{-a}x\|^2 = \pi\|\varrho(\pi^{-1/2}x)^{-a}\pi^{-1/2}x\|^2 = \pi.$$

Thus

$$\hat{K}(x,t,\delta) = \text{p.v.}\int_{[u]\leqslant c_1[x^*]} K(u)\beta(\delta[u-t])e^{-ix\cdot u}du+$$
$$+1/2\int_{[u]\geqslant c_1[x^*]} K(u)\beta(\delta[u-t])\,(e^{-ix\cdot u}-e^{-ix\cdot(u+x^*)})\,du$$
$$=J_1(x,t,\delta)+J_2(x,t,\delta),\quad\text{say.}$$

$$J_1 = \text{p.v.}\int_{[u]\leqslant\min(4/\delta,c_1[x^*])} K(u)\big[\beta(\delta[u-t]e^{-ix\cdot u}-\beta(\delta[t])\big]du+$$
$$+\int_{4/\delta\leqslant[u]\leqslant c_1[x^*]} K(u)\beta(\delta[u-t])e^{-ix\cdot u}du = J_{11}+J_{12},\quad\text{sa .}$$

$$|J_{11}|\leqslant\int_{[u]\leqslant c_1[x^*]}|K(u)|\,|\beta(\delta[u-t])|\,|e^{-ix\cdot u}-1|\,du+$$
$$+\int_{[u]\leqslant 4/\delta}|K(u)|\,|\beta(\delta[u-t])-\beta(\delta[t])|\,du$$
$$\leqslant\|\beta\|_\infty\int_{[u]\leqslant c_1[x^*]}|K(u)|\,|x\cdot u|\,du+\|\beta'\|_\infty\,\delta\int_{[u]\leqslant 4/\varepsilon}|K(u)|\,[u]\,du.$$

Furthermore for $[u]\leqslant c_1[x^*]$

$$|x\cdot u| = |[x^*]^a x\cdot[x^*]^{-a}u|\leqslant\sqrt{n}\,c_1^{a_n}\|[x^*]^a x\|\,[x^*]^{-1}[u]\leqslant\sqrt{n}(c_1\pi)^{a_n}[x^*]^{-1}[u]$$

since for $[y]\leqslant 1$, $\|y\|\leqslant\sqrt{n}\sup_i|y_i|\leqslant\sqrt{n}[y]$. Hence

$$|J_{11}|\leqslant C[x^*]^{-1}\int_{[u]\leqslant c_1[x^*]}|K(u)|\,[u]\,du+C\delta\int_{[u]\leqslant 4/\delta}|K(u)|\,[u]\,du\leqslant CM_1,$$

where M_1 is as in the proof of Proposition 1.

Consider next J_{12}. If $\delta[t]<3$ then $J_{12}=0$. If $\delta[t]>3$

$$|J_{22}|\leqslant\int_{4/\delta\leqslant[u]\leqslant c_1[x^*}|K(u)|\,|\beta(\delta[u-t])|\,du\leqslant C\int_{2/3[t]\leqslant[u]\leqslant 4/3[t]}|K(u)|\,du = CM_1$$

hence altogether

$$|J_1(x,t,\delta)|\leqslant C.$$

On the other hand

$$|J_2(x,t,\delta)|\leqslant 1/2\int_{[u]\geqslant c_1[x^*]}|K(u)\beta(\delta[u-t])-K(u-x^*)\beta(\delta[u-t-x^*])|\,du+$$
$$+1/2\,\|\beta\|_\infty\int_{(c_1-1)[x^*]\leqslant[u]\leqslant(c_1-1)[x^*]}|K(u)|\,du$$
$$\leqslant\int_{[u]\geqslant c_1[x^*]}|\beta(\delta[u-t-x^*])|\,|K(u)-K(u-x^*)|\,du+$$
$$+\int_{[u]\geqslant c_1[x^*]}|\beta(\delta[u-t])-\beta(\delta[u-t-x^*])|\,|K(u)|\,du+$$
$$+CM_1\log[(c_1+1)(c_1-1)^{-1}]$$
$$\leqslant Cc_2+J'(x,t,\delta)+C,$$

where

$$J' = \int_{[u]\geqslant c_1[x^*]}|\beta(\delta[u-t])-\beta(\delta[u-t-x^*])|\,|K(u)|\,du.$$

Let now $\delta[t] < 2$, then

$$J' \leqslant \|\beta'\|_\infty \delta[x^*] \int\limits_{c_1[x^*] \leqslant [u] \leqslant 3/\delta(1-1/c_1)^{-1}} |K(u)|\, du$$
$$\leqslant c_1^{-1} \|\beta'\|_\infty \delta \int\limits_{[u] \leqslant 3/\delta(1-1/c_1)^{-1}} |K(u)|\, [u]\, du = C.$$

If on the other hand $\delta[t] > 2$, then

$$J' \leqslant \int\limits_{(c_1-1)(2c_1)^{-1}[t] \leqslant [u] \leqslant 3(c_1+1)(2c_1)^{-1}[t]} |K(u)|\, du \leqslant C.$$

This finishes the proof that $\hat{K}(x, t, \delta)$ is bounded independently of x, t, δ.

To prove (33) suppose $\delta[t] \leqslant \frac{1}{4}$, then

$$|\hat{K}(x, t, \delta) - \hat{K}(x, 0, \delta)| = \Big| \int K(u) \big[\beta(\delta[u-t]) - \beta(\delta[u])\big] e^{-ix\cdot u} du \Big|$$
$$\leqslant \int |K(u)|\, |\beta(\delta[u-t]) - \beta(\delta[u])|\, du$$
$$\leqslant \|\beta'\|_\infty \delta[t] \int\limits_{1/(2\delta) - [t] < [u] < 1/\delta + [t]} |K(u)|\, du$$
$$\leqslant C\delta[t] \int\limits_{1/(4\delta) \leqslant [u] \leqslant 5/(4\delta)} |K(u)|\, du.$$

Hence

(34) $$|\hat{K}(x, t, \delta) - \hat{K}(x, 0, \delta)| \leqslant C\delta[t] \quad (\delta[t] \leqslant 1/4).$$

Let $\hat{K}_t(x) = \text{p.v.} \int\limits_{[u] \leqslant t} K(u) e^{-ix\cdot u} du$. It is well known that $\hat{K}_t$ tends boundedly to $\hat{K}$ as $t \to \infty$ (see [3], [10]). Integration by parts shows

$$\hat{K}(x, 0, \delta) = \int\limits_{1/2}^{1} \big(-\beta'(t)\big) \hat{K}_{t/\delta}(x)\, dt.$$

Hence it follows that

(35) $$\lim_{\delta \to 0} \hat{K}(x, 0, \delta) = \hat{K}(x)$$

and so (33) is a consequence of (34) and (35).

Hence finally

$$\|I_3(x, \cdot, \delta)\chi_B\|_1 \leqslant C|B|\, \|f^\sigma\|_1$$

for any measurable set B. If σ is first chosen sufficiently large it follows now that

$$\lim_{n \to \beta(\delta[\cdot])} \int \text{p.v.} K * f(t - \cdot) e^{-ix\cdot(t-\cdot)} d\mu(t) = \int f(t - \cdot) e^{-ix\cdot(t-\cdot)} \hat{K}(x, t, \delta)\, dt$$

in $L^{1\text{oo}}_{\text{loc}}$. (31) follows from (33).

If p.v.$K * f \in L^1$ (32) implies that the iterated limit on the left-hand side of (31) is $(\text{p.v.} K * f)^\wedge$. It follows that if p.v.$K * f \in L^1$ then $(\text{p.v.} K * f)^\wedge(x) = \hat{K}(x)\hat{f}(x)$ for all x.

Remark 9. It is well known and easy to see that a net of positive measures $\{\mu_i\}$ tends weakly to the measure $\beta(\delta[x])dx$ if and only if for any compact set A,

$$\lim_i \int\limits_A d\mu_i = \int\limits_A \beta(\delta[x])\, dx.$$

Remark 10. Corollaries 4 and 5 have been proved for the one-dimensional Hilbert transform by a different method in [2] (Theorems 2.7 and 2.11).

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