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Extremal problems in the class of close-to-convex functions

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Introduction. Let S^0 denote the class of functions $f(z) = z+a_2z^2+\ldots$ regular for $|z| < 1$ and mapping the unit circle on a convex region.

In 1952 Kaplan [6] has introduced the class L of close-to-convex functions. This class has been defined as follows: a function $f(z) = z+A_2z^2+\ldots$, $|z| < 1$ belongs to L if and only if

$$(1) \qquad \operatorname{re}\frac{f'(z)}{\tilde{\varphi}'(z)} > 0$$

for some function $\tilde{\varphi}(z) = \lambda e^{i\alpha}z+a_2z^2+\ldots$, mapping the circle $|z| < 1$ into a convex region. λ and α are real numbers.

One can prove [6] that the class L is a subclass of the family S and that it includes the class of starlike functions. In 1955 Reade [9] has proved that in the class L the hypothesis of Bieberbach is true. If $\tilde{\varphi}(z)\in S^0$, we denote this class by L_0. Krzyż [7], Szczepankiewicz and Zamorski [10] worked on extremal problems in this class. The class L was investigated by Bielecki and Lewandowski [2]. Z. Lewandowski has proved that the family L coincides with the family of linearly attainable functions which was introduced in 1936 by Biernacki [3].

A number of extremal problems in the class L have been solved by Aleksandrow and Gutlianski [1]. They have given the sets of values of many functionals defined in the family L. In this paper the modulus of the K-th derivative of a function of the class L has been estimated. On the basis of a structural formula the variation formulas have been deduced.

In the second chapter the basic theorem which gives a characterization of the boundary and extremal functions for an extensive class of functionals $F(f)$, $f\in L$ defined in Chaper II will be proved.

CHAPTER I

The estimates of the moduli of the derivatives of close-to-convex functions in the family L. The variation formulas in the classes L, $\tilde{C}$, $\tilde{S}^*$

1. DEFINITIONS AND NOTATION

The following notation will be applied in this paper:

$E = \{z\colon |z| < 1\}$;

S^* — the class of functions $f(z) = z + b_2 z^2 + \ldots$ regular in E and mapping the unit circle into a starlike region in relation to the point $w = 0$;

C — the class of functions of the form $h(z) = 1 + c_1 z + c_2 z^2 + \ldots$ regular in E and satisfying there the condition $\operatorname{re} h(z) > 0$ for $z \in E$;

M — the class of functions $\alpha(t)$ real in $(-\pi, \pi]$, non-decreasing and such that $\int\limits_{-\pi}^{\pi} d\alpha(t) = 1$;

$\tilde{S}^*$ — the class of functions $\tilde{f}(z) = f(z)/z$, where $f(z) \in S^*$;

$\tilde{C}$ — the class of functions of the form

$$\tilde{h}(z) = \int\limits_{-\pi}^{\pi} \frac{e^{it} + e^{i\varphi} z}{e^{it} - z}\, d\alpha(t), \qquad \text{where } |\varphi| < \pi \text{ and } \alpha(t) \in M;$$

$\tilde{S}^0$ — the class of functions of the form $\tilde{\varphi}(z) = \lambda e^{i\alpha} \varphi(z)$, where $\varphi(z) \in S^0$ and $\lambda > 0$;

C_1 — the class of functions of the form $h_1(z) = \gamma e^{i\beta} + a_1 z + a_2 z^2 + \ldots$ satisfying the conditions $\operatorname{re} h_1(z) > 0$ and $\gamma > 0$.

From the definition of the family L the following equality holds:

$$f'(z) = \tilde{\varphi}'(z) \cdot h_1(z), \qquad \text{where } f(z) \in L \text{ and } \tilde{\varphi}(z) \in \tilde{S}^0,\ h_1(z) \in C_1. \tag{2}$$

Making use of the expansions of the functions $f(z)$, $\tilde{\varphi}(z)$, $h_1(z)$ into a power series and of equality (2), we find that $\lambda\gamma e^{i\alpha} e^{i\beta} = 1$. Thus $\beta = -\alpha$ and $\lambda = 1/\gamma$. Without loss of generality of our considerations we may assume that $\lambda = 1$. From the definition of the class C_1 the inequality $|\alpha| < \frac{1}{2}\pi$ follows.

The following theorem [10] is true

THEOREM 1. *If $f(z) \in L$, then*

$$|f^{(k)}(z)| \leqslant k!\, \frac{k+r}{(1-r)^{k+2}}, \qquad |z| = r,\ k = 0, 1, \ldots \tag{3}$$

The bound is sharp, being attained by

$$f(z) = \frac{z}{(1-\varepsilon z)^2}, \qquad |\varepsilon| = 1.$$

2. PROPERTIES OF THE CLASS L: COMPACTNESS AND CONNECTEDNESS

We prove

THEOREM 2. *The class L is a compact family.*

Proof. From the estimate of the modulus of a function of the class L it follows that L is a close-to-uniformly bounded family in E, and thus is a normal family. Let $\{f_n(z)\}$, where $f_n(z) \in L$ is a convergent-sequence. We denote the limit of that sequence by $f_0(z)$. We know from the definition of the class L that $f_n(0) = 0$ and $f'_n(0) = 1$.

By the theorem of Weierstrass we find that $f_0(z)$ is a regular function and that $f_0(0) = 0$ and $f'_0(0) = 1$. Thus the limit function is finite and different from a constant. By $\tilde{\varphi}_n(z)$ and $h_{1,n}(z)$ we denote functions satisfying equation (2):

$$f'_n(z) = \tilde{\varphi}'_n(z)\cdot h_{1,n}(z). \tag{4}$$

It follows from the simple relationships between the functions of the classes S^0 and $\tilde{S}^0$ on the hand those of the classes C and C_1 on the other that the families $\tilde{S}^0$ and C_1 are compact. Thus from the sequences $\{\tilde{\varphi}_n(z)\}$ and $\{h_{1,n}(z)\}$ one may take out subsequences $\{\varphi_{n_k}(z)\}$ and $\{h_{1,n_k}(z)\}$ which tend to $\varphi_0(z) \in \tilde{S}^0$ and to $h_{1,0} \in C_1$ respectively and satisfy the equation: $f'_{n_k}(z) = \tilde{\varphi}'_{n_k}(z)\cdot h_{1,n_k}(z)$.

For $n_k \to \infty$ we obtain

$$\lim_{n_k\to\infty} \varphi'_{n_k}(z)\cdot h_{1,n_k}(z) = \lim_{n_k\to\infty} f'_{n_k}(z) = \lim_{n\to\infty} f'_n(z) = f'_0(z).$$

Hence

$$f'_0(z) = \tilde{\varphi}'_0(z)\cdot h_{1,0}(z). \tag{5}$$

Thus we have proved that $f_0(z)$ in E is a regular function $f_0(0) = 0$, and $f'_0(0) = 1$ and that there exists a function $\tilde{\varphi}(_0z) \in S^0$ satisfying the condition

$$\operatorname{re}\frac{f'_0(z)}{\tilde{\varphi}'_0(z)} > 0.$$

Thus $f_0(z)$ belongs to the family L, which proves its compactness.

A family R is called *connected* if for arbitrary functions $f_1(z)$ and $f_2(z)$ of this family there exists a class $W \subset R$ of functions $f(z, t)$, $z \in D$,

$t\epsilon[a, b]$ satisfying the following equalities:

1. $f(z, t)$ is close-to-uniformly continuous in D in relation to t;

2. $f(z, t)$ converges close-to-uniformly to $f_1(z)$, as $t \to a$;

3. $f(z, t)$ converges almost uniformly to $f_2(z)$ as $t \to b$.

THEOREM 3. *The class L is connected.*

Proof. Let $f_1(z)$ and $f_2(z)$ be arbitrary functions from the family L. The class W mentioned in the definition will be defined as follows:

$$f(z, t) = \begin{cases} \frac{1}{1-t} f_1[z(1-t)] & \text{for } 0 \leqslant t < 1, \\ z & \text{for } \quad t = 1, \\ \frac{1}{t-1} f_2[z(t-1)] & \text{for } 1 < t \leqslant 2. \end{cases} \tag{6}$$

For every $t\epsilon[0, 2]$ the function $f(z, t)$ belongs to L. It suffices to put for $\tilde{\varphi}(z)$, in equation (1),

$$\tilde{\varphi}(z, t) = \begin{cases} \frac{1}{1-t} \tilde{\varphi}_1[z(1-t)] & \text{for } 0 \leqslant t < 1, \\ z & \text{for } \quad t = 1, \\ \frac{1}{t-1} \tilde{\varphi}_2[z(t-1)] & \text{for } 1 < t \leqslant 2, \end{cases}$$

where $\tilde{\varphi}_1(z)$ and $\tilde{\varphi}_2(z)$ satisfy the conditions

$$\operatorname{re} \frac{f_1'(z)}{\tilde{\varphi}_1'(z)} > 0 \quad \text{and} \quad \operatorname{re} \frac{f_2'(z)}{\tilde{\varphi}_2'(z)} > 0.$$

We estimate the difference

$$h = |f(z, t_1) - f(z, t_2)|, \quad t_1, t_2 \epsilon [0, 1).$$

For an arbitrary z, $|z| < r < 1$ we have

$$h = \left| \frac{1}{1-t_1} f_1[z(1-t_1)] - \frac{1}{1-t_2} f_1[z(1-t_2)] \right|,$$

$$\begin{aligned} h &= \Big| \sum_{k=2}^{\infty} A_k (1-t_1)^{k-1} z^k - \sum_{k=2}^{\infty} A_k (1-t_2)^{k-1} z^k \Big| \\ &\leqslant \sum_{k=2}^{\infty} |A_k| \, |(1-t_1)^{k-1} - (1-t_2)^{k-1}| \, |z|^k \\ &\leqslant \sum_{k=2}^{\infty} k |t_1 - t_2| (k-1) r^k \leqslant |t_1 - t_2| \cdot B(r), \end{aligned}$$

where

$$B(r) = \sum_{k=2}^{\infty} k(k-1)r^k = \frac{2r^2}{(1-r)^3}.$$

From the estimate $h \leqslant |t_1-t_2|\cdot B(r)$ it follows that the functions $f(z, t)$ are close-to-uniformly continuous in E. Putting $t_2 = 0$ and $t_2 = 1$ respectively, we find that $f(z, t)$ converges close-to-uniformly to $f_1(z)$ as $t \to 0$ and that $f(z, t)$ converges close-to-uniformly to z as $t \to 1$. In a similar way we may prove that

$$|f(z, t_1)-f(z, t_2)| \leqslant |t_1-t_2|\cdot B(r) \quad \text{for } t_1, t_2 \in [1, 2].$$

Assuming $t_2 = 1$ and $t_2 = 2$ respectively, we observe that $f(z, t)$ converges close-to-uniformly to z as $t \to 1$ and that $f(z, t)$ converges close-to-uniformly to $f_2(z)$ as $t \to 2$.

We have proved that the class W defined by equality (6) satisfies conditions 1–3. Thus the family L is connected.

3. THE STRUCTURAL FORMULA AND THE VARIATION FORMULAS IN THE CLASS L

3.1. The structural formula. It can easily be proved that $f(z) \in L$ if and only if

$$(7) \qquad f(z) = \int_0^z \left[\int_{-\pi}^{\pi} \frac{e^{it}+e^{i\varphi}z}{e^{it}-z}\, d\alpha(t)\right] \cdot e^{-2\int_{-\pi}^{\pi} \log(1-e^{-it}z)d\mu(t)} dz$$

for some functions $\alpha(t)$ and $\mu(t)$ of the family M and for an arbitrary φ, $|\varphi| < \pi$ (comp. e.g. [1]) and $f(z) \in S^*$ if and only if

$$f(z) = z\exp \int_{-\pi}^{\pi} \log(1-e^{-it} z)\, d\mu(t)$$

for some function $\mu(t) \in M$.

Equality (7) may be written in the form

$$(8) \qquad f(z) = \int_0^z h(z)\cdot \tilde{f}(z)\, dz,$$

where

$$(8') \qquad \tilde{h}(z) = \int_{-\pi}^{\pi} \frac{e^{it}+e^{i\varphi}z}{e^{it}-z}\, d\alpha(t), \qquad \tilde{h}(z) \in \tilde{C},$$

$$(8'') \qquad \tilde{f}(z) = \exp\left[-2\int_{-\pi}^{\pi} \log(1-e^{-it}z)\, d\mu(t)\right], \qquad \tilde{f}(z) \in S^*.$$

3.2. Variation formulas in the class $\tilde{C}$. I can easily be observed (comp. [4]) that the function

$$(9)\qquad \tilde{h}_\lambda(z) = \tilde{h}(z)+\lambda \int_{t_1}^{t_2} g_1(z, t)|a(t)-c|\,dt$$

belongs to the class $\tilde{C}$ together with the function $\tilde{h}(z)$, where

$$\lambda\in(-1, 1), \quad t_1, t_2\in(-\pi, \pi], \quad c = \lim_{t\to t_1^-} a(t) \quad \text{or} \quad c = \lim_{t\to t_2^+} a(t)$$

and

$$(9')\qquad g_1(z, t) = -i\frac{(1+e^{i\varphi})e^{it}z}{(e^{it}-z)^2}.$$

If $a(t)$ is a step function, then the function

$$(10)\qquad \tilde{h}_\lambda(z) = \tilde{h}(z)+\lambda[g(z, t_1)-g(z, t_2)]$$

belongs to the class $\tilde{C}$, where

$$g(z, t) = \frac{e^{it}+e^{i\varphi}z}{e^{it}-z},$$

t_1, t_2 are the discontinuity points of $a(t)$, $\lambda\in(-\varepsilon, \varepsilon)$ for $\varepsilon>0$ and sufficiently small.

3.3. The variation formulas in the class $\tilde{S}^*$. Let $f(z)\in S^*$. Basing ourselves on the variation formulas in the class C, one may prove (comp. [4]) that the function

$$(11)\qquad f_\lambda(z) = f(z)+\lambda\int_{t_1}^{t_2} f(z)\frac{2ie^{-it}z}{1-e^{-it}z}|\mu(t)-c|\,dt+o(\lambda)$$

also belongs to S^*

If $\mu(t)$ has at least two steps, the function of the from

$$(12)\qquad f_\lambda(z) = f(z)+\lambda f(z)[\log(1-e^{-it_2}z)-\log(1-e^{-it_1}z)]+o(\lambda)$$

belongs to S^* together with $f(z)$, where $\lambda\in(-\varepsilon, \varepsilon)$ and t_1, t_2 are discontinuity points of $\mu(t)$.

On dividing both sides of inequalities (11) and (12) by z we obtain the variation formulas in $\tilde{S}^*$:

$$(13)\qquad \tilde{f}_\lambda(z) = \tilde{f}(z)+\lambda\int_{t_1}^{t_2}\tilde{f}(z)\frac{2ie^{-it}z}{1-e^{-it}z}|\mu(t)-c|\,dt+o(\lambda),$$

$$(14)\qquad \tilde{f}_\lambda(z) = \tilde{f}(z)+\lambda\tilde{f}(z)[\log(1-e^{-it_2}z)-\log(1-e^{it_1}z)]+o(\lambda).$$

3.4. Variation formulas in the class L. Let us come back to equality (8). Replacing $\tilde{h}(z)$ in that formula by $\tilde{h}_\lambda(z)$ defined by formula (9), we obtain the first variation formula in the class L:

$$(15)\qquad f_\lambda(z) = f(z)+\lambda U_1(z),$$

where

$$(15') \qquad U_1(z) = \int_0^z \int_{t_1}^{t_2} g_1(z,t)\,|\alpha(t)-c|\,dt\,dz.$$

Putting in (8) $\tilde{h}_\lambda(z)$ for $\tilde{h}(z)$, we get the second variation formula:

$$(16) \qquad f_\lambda(z) = f(z) + \lambda U_2(z),$$

where

$$(16') \qquad U_2(z) = \int_0^z \tilde{f}(z)[g(z,t_1) - g(z,t_2)]\,dz.$$

The next two formulas can be obtained, by replacing in (8), $\tilde{f}(z)$ by the function $\tilde{f}_\lambda(z)$ defined by formula (13) or by formula (14). Making use of (13), we have

$$(17) \qquad f_\lambda(z) = f(z) + \lambda U_3(z) + o(\lambda),$$

where

$$U_3(z) = \int_0^z \int_{t_1}^{t_2} \tilde{h}(z)\tilde{f}(z)\frac{2ie^{-it}z}{1-e^{it}z}\,|\mu(t)-c|\,dt\,dz.$$

If $\mu(t)$ is a step function, we obtain from (8) and (14)

$$(18) \qquad f_\lambda(z) = f(z) + \lambda U_4(z) + o(\lambda),$$

where

$$U_4(z) = \int_0^z \tilde{h}(z)\cdot\tilde{f}(z)[\log(1-e^{-it_2}z) - \log(1-e^{-it_1}z)]\,dz.$$

Formulas (15), (16), (17) and (18) are the four basic variation formulas in the class L.

CHAPTER II

Boundary functions in the family L in relation to some functionals

1. DEFINITIONS AND NOTATION

Let a given function

$$(19) \qquad F(f) = F(f', \overline{f'}, \ldots, f^{(n)}, \overline{f^{(n)}})$$

be differentiable in a region D defined by the inequalities

$$(19') \qquad \frac{1-r}{(1+r)^3} \leqslant |f'| \leqslant \frac{1+r}{(1-r)^3}; \qquad \frac{1-r}{(1+r)^3} \leqslant |\overline{f'}| \leqslant \frac{1+r}{(1-r)^3};$$

$$|f^{(k)}| \leqslant k!\,\frac{k+r}{(1-r)^{k+2}}; \qquad |\overline{f^{(k)}}| \leqslant k!\,\frac{k+r}{(1-r)^{k+2}}, \qquad k = 2, 3, \ldots, n.$$

Consider the functional

$$F(f(\mathfrak{z})) = F\left(f'(\mathfrak{z}), \bar{f}'(\mathfrak{z}), \ldots, f^{n}(\mathfrak{z}), \overline{f^{(n)}}(\mathfrak{z})\right) \tag{20}$$

defined in the family L. $\mathfrak{z}$ is a fixed point of the circle $|z| < 1$ and $|\mathfrak{z}| = r$.

We denote by Δ the set of values of the functional $F(f)$. It follows from Theorem 2 and 3 that Δ is a closed region. We denote by Γ the boundary of the region Δ. One may prove (comp. [5]) that for every point $\dot{F} \in \Delta$ there exists a point $F_0 \in \Gamma$ such that, for $F \in \Delta$ and sufficiently close to F_0, the inequality

$$|F_0 - F| \leqslant |F - \dot{F}| \tag{21}$$

holds.

The point F_0 is called a *regular point* of the boundary Γ.

We denote the set of these points by Γ'. One may prove (see [5]) that Γ' is dense in Γ. A function $f(z) \in L$ is called a *boundary function* in relation to the functional $F(f)$ if $F(f(\mathfrak{z})) \in \Gamma$. A boundary function which satisfies the condition $F(f(\mathfrak{z})) \in \Gamma'$ will be denoted by $f^*(z)$. The functions $\tilde{f}(z)$, $\tilde{h}(z)$, $\alpha(t)$, $\mu(t)$, corresponding to the function $f^*(z)$ will be denoted by $\tilde{f}^*(z)$, $\tilde{h}^*(z)$, $\alpha^*(t)$ and $\mu^*(t)$.

2. THE FUNCTIONS $\alpha^*(t)$ AND $h^*(z)$ AND THEIR PROPERTIES

Preserving the notation of (9′) (15′) (16′) and (20), we assume

$$a_k = \beta \frac{\partial F}{\partial f^{(k)}} + \overline{\beta} \frac{\partial \overline{F}}{\partial \bar{f}^{(k)}}, \quad k = 1, 2, \ldots, n; \ |\beta| = 1.$$

Let $\Phi_1(t)$, $-\pi < t \leqslant \pi$, be a function defined by the formula

$$\Phi_1(t) = \sum_{k=1}^{n} a_k \frac{d^{k-1}}{d\mathfrak{z}^{k-1}} [\tilde{f}^*(\mathfrak{z}) g_1(\mathfrak{z}, t)] + \overline{a}_k \overline{\frac{d^{k-1}}{d\mathfrak{z}^{k-1}} [\tilde{f}^*(\mathfrak{z}) g_1(\mathfrak{z}, t)]}. \tag{22}$$

We shall prove

LEMMA 1. *The function $\Phi_1(t)$ satisfies the equation*

$$\int_{t_1}^{t_2} \Phi_1(t) |\alpha(t) - c| \, dt = 0, \tag{23}$$

where $t_1, t_2 \in (-\pi, \pi]$.

Proof. Making use of variation formula (15), we obtain

$$F\left(f^{*\prime}(\mathfrak{z}) + \lambda U_1'(\mathfrak{z}); \ldots; \bar{f}^{*(n)}(\mathfrak{z}) + \lambda \overline{U}_1^{(n)}(\mathfrak{z})\right) - F\left(f^{*\prime}(\mathfrak{z}) \ldots \bar{f}^{*(n)}(\mathfrak{z})\right)$$
$$= \lambda \sum_{k=1}^{n} \left(p_k U_1^{(k)}(\mathfrak{z}) + q_k \overline{U}_1^{(k)}(\mathfrak{z})\right) + o(\lambda),$$

where

$$p_k = \frac{\partial F}{\partial f^{*(k)}}, \quad q_k = \frac{\partial F}{\partial \bar{f}^{*(k)}}.$$

Hence

$$\Delta F = \lambda \sum_{k=1}^{n} (p_k U_1^{(k)}(\mathfrak{z}) + q_k \bar{U}_1^{(k)}(\mathfrak{z}) + o(\lambda).$$

Inequality (21) can be written in the form

$$|F - F_0|^2 + |F_0 - \dot{F}|^2 + 2\operatorname{re}(F - F_0)\overline{(F_0 - \dot{F})} \geqslant |F_0 - \dot{F}|^2. \tag{24}$$

Because of $F - F_0 = \Delta F$ we have

$$\lambda \operatorname{re}\beta \sum_{k=1}^{n} \big(p_k U_1^{(k)}(\mathfrak{z}) + q_k \bar{U}_1^{(k)}(\mathfrak{z})\big) + o(\lambda) \geqslant 0,$$

where $\beta = e^{i \arg \overline{(F_0 - \dot{F})}}$.

Since λ takes arbitrary values from the interval $(-1, 1)$, we obtain

$$\operatorname{re}\beta \sum_{k=1}^{n} \big(p_k U_1^{(k)}(\mathfrak{z}) + q_k \bar{U}_1^{(k)}(\mathfrak{z})\big) = 0. \tag{25}$$

By successive transformations we have

$$\sum_{k=1}^{n} \big(\beta p_k U_1^{(k)}(\mathfrak{z}) + \beta q_k \bar{U}_1^{(k)}(\mathfrak{z})\big) + \sum_{k=1}^{n} \big(\bar{\beta}\bar{p}_k \bar{U}_1^{(k)}(\mathfrak{z}) + \bar{\beta}\bar{q}_k U_1^{(k)}(\mathfrak{z})\big) = 0$$

or

$$\sum_{k=1}^{n} \big(a_k U_1^{(k)}(\mathfrak{z}) + \bar{a}_k \bar{U}_1^{(k)}(\mathfrak{z})\big) = 0.$$

By (15′) and the theorem on the sum of integrals we obtain

$$\int_{t_1}^{t_2} \sum_{k=1}^{n} \Big\{ a_k \frac{d^{k-1}}{d\mathfrak{z}^{k-1}} [\bar{f}^*(\mathfrak{z}) g_1(\mathfrak{z}, t)] + \bar{a}_k \overline{\frac{d^{k-1}}{d\mathfrak{z}^{k-1}} [\bar{f}^*(\mathfrak{z}) g_1(\mathfrak{z}, t)]} \Big\} |a(t) - c|\, dt = 0. \tag{26}$$

By (22) the lemma follows.

LEMMA 2. *If at every point $w = (f', \bar{f}', \ldots, f^{(n)}, \overline{f^{(n)}})$ of the region D defined by (19′) and for every β, $|\beta| = 1$ the condition*

$$\sum_{k=1}^{n} |a_k| > 0, \quad a_k = \beta \frac{\partial F}{\partial f^{(k)}} + \bar{\beta} \frac{\partial \bar{F}}{\partial \overline{f^{(k)}}}, \quad k = 1, 2, \ldots, n$$

is satisfied, then the function $\Phi_1(t)$ has no more than $2n$ roots in the interval $(-\pi, \pi)$ if $\mathfrak{z} \neq 0$ and $2n-3$ roots if $\mathfrak{z} = 0$.

Proof. By the formula of Leibniz and equality (22) we obtain

$$\Phi_1(t) = \sum_{k=1}^{n} \sum_{j=0}^{k-1} \Big[a_k \binom{k-1}{j} \bar{f}^{*(k-1-j)}(\mathfrak{z}) g_1^{(j)}(\mathfrak{z}, t) + \bar{a}_k \binom{k-1}{j} \overline{\bar{f}^{*(k-1-j)}(\mathfrak{z}) g_1^{(j)}(\mathfrak{z}, t)} \Big].$$

Grouping the coefficients at the successive derivatives of the function $g_1(\mathfrak{z}, t)$, we have

$$\Phi_1(t) = H(t) + \overline{H(t)},$$

where

$$H(t) = \Big[\sum_{k=0}^{n-1} \binom{k}{0} a_{k+1} \bar{f}^{*(k)}(\mathfrak{z})\Big] g_1(\mathfrak{z}, t) + \Big[\sum_{k=1}^{n-1} \binom{k}{1} a_{k+1} \bar{f}^{*(k-1)}(\mathfrak{z})\Big] g_1'(\mathfrak{z}, t) + \ldots + \Big[\sum_{k=n-1}^{n-1} \binom{k}{n-1} a_{k+1} \bar{f}^{*(k-n+1)}(\mathfrak{z})\Big] g_1^{(n-1)}(\mathfrak{z}, t).$$

The constants appearing in the square brackets will be denoted by $A_1, A_2, \ldots, A_n$. Thus

$$\Phi_1(t) = \sum_{k=1}^{n} [A_k g_1^{(k-1)}(\mathfrak{z}, t) + \bar{A}_k \bar{g}_1^{(k-1)}(\mathfrak{z}, t)]. \tag{26'}$$

We shall prove that the coefficients $A_1, A_2, \ldots, A_n$ do not vanish simultaneously. Suppose that the contrary is the case, i.e. that

$$\begin{aligned} &\sum_{k=0}^{n-1} \binom{k}{0} a_{k+1} \bar{f}^{*(k)}(\mathfrak{z}) = 0, \\ &\sum_{k=1}^{n-1} \binom{k}{1} a_{k+1} \bar{f}^{*(k-1)}(\mathfrak{z}) = 0, \\ &\ldots \\ &\sum_{k=n-1}^{n-1} \binom{k}{n-1} a_{k+1} \bar{f}^{*(k-n+1)}(\mathfrak{z}) = 0. \end{aligned} \tag{27}$$

We treat $a_1, a_2, \ldots, a_n$ as unknowns. System (27) is homogeneous. It is easily seen that the determinant of this system is different from zero. By the theorem of Cramer it has one solution $a_1 = 0,\ a_2 = 0, \ldots, a_n = 0$. These equalities contradict the assumption of Lemma 2. Thus $\sum_{k=1}^{n} |A_k| > 0$.

By (9′) and (26′) the function $\Phi_1(t)$ can be written in the form

$$\Phi_1(t) = \sum_{k=1}^{n} A_k \frac{d^{k-1}}{d\mathfrak{z}^{k-1}} \frac{-i(1+e^{i\varphi})e^{it}\mathfrak{z}}{(e^{it}-\mathfrak{z})^2} + \overline{A_k \frac{d^{k-1}}{d\mathfrak{z}^{k-1}} \frac{-i(1+e^{i\varphi})e^{it}\mathfrak{z}}{(e^{it}-\mathfrak{z})^2}}.$$

It may easily be proved that

$$\frac{d_{k-1}}{d\mathfrak{z}^{k-1}} \cdot \frac{-i(1+e^{i\varphi})e^{it}\mathfrak{z}}{(e^{it}-\mathfrak{z})^2} = a_k \frac{(k-1)e^{2it}+\mathfrak{z}e^{it}}{(e^{it}-\mathfrak{z})^{k+1}},$$

where $a_k = -i(1+e^{i\varphi})(k-1)!$. Thus

$$(28) \qquad \Phi_1(t) = \sum_{k=1}^{n} B_k \frac{(k-1)e^{2it}+\mathfrak{z}e^{it}}{(e^{it}-\mathfrak{z})^{k+1}} + \bar{B}_k \overline{\frac{(k-1)e^{2it}+\mathfrak{z}e^{it}}{(e^{it}-\mathfrak{z})^{k+1}}}, \quad B_k = A_k \cdot a_k.$$

Consider the function

$$(29) \qquad G(v) = \sum_{k=1}^{n} B_k \frac{(k-1)v^2+\mathfrak{z}v}{(v-\mathfrak{z})^{k+1}} + \bar{B}_k \frac{(k-1)v^{k-2}+\bar{\mathfrak{z}}v^{k-1}}{(1-v\bar{\mathfrak{z}})^{k+1}}.$$

Comparing (28) and (29), we find that

$$(30) \qquad G(e^{it}) = \Phi_1(t).$$

The function $G(v)$ is rational in v and regular for $v \neq \mathfrak{z}$ and $v \neq 1/\mathfrak{z}$. We write it in the form

$$G(v) = \sum_{k=1}^{n} \bar{B}_k \frac{(k-1)v^{k-2}+\bar{\mathfrak{z}}v^{k-1}}{(1-v\bar{\mathfrak{z}})^{k+1}} +$$

$$+ [B_1\mathfrak{z}v(v-\mathfrak{z})^{n-1} + B_2(v^2+\mathfrak{z}v)^{n-2} + \ldots + B_n(n-1)v^2+\mathfrak{z}v)] \frac{1}{(v-\mathfrak{z})^{n+1}}.$$

To fix attention, suppose that B_n is the non-vanishing coefficient. As $v \to \mathfrak{z}$, $G(v) \to \infty$. Thus $G(v) \not\equiv 0$.

The function $G(v)$, being regular, must have a finite number of roots on the circumference $v = e^{it}$. By equality (30) we have

$$(31) \qquad \Phi_1(t) \not\equiv 0.$$

By successive transformations of equation (28) we obtain

$$\Phi_1(t) = \frac{e^{it}W_{n-1}(e^{it})(e^{-it}-\bar{\mathfrak{z}})^{n+1} + e^{-it}W_{n-1}(e^{it})(e^{it}-\mathfrak{z})^{n+1}}{(e^{it}-\mathfrak{z})^{n+1}(e^{-it}-\bar{\mathfrak{z}})^{n+1}},$$

where $W_{n-1}(e^{it})$ is a polynomial whose degree is $k \leqslant n-1$,

$$\Phi_1(t) = \frac{e^{it}W_{n-1}(e^{it})(1-e^{it}\mathfrak{z})^{n+1}e^{-it(n+1)}+e^{-it}W_{n-1}(e^{-it})(e^{it}-\mathfrak{z})^{n+1}}{|e^{it}-\mathfrak{z}|^{2(n+1)}},$$

$$\Phi_1(t) = \frac{W_{n-1}(e^{it})(1-e^{it}\mathfrak{z})^{n+1}+W_{n-1}(e^{it})(e^{it}-\mathfrak{z})^{n+1}}{|e^{it}-\mathfrak{z}|^{2(n+1)}e^{int}},$$

$$\Phi_1(t) = \frac{W_{2n}(e^{it})}{|e^{it}-\mathfrak{z}|^{2(n+1)}e^{int}}. \tag{32}$$

It follows from the last equality and from (31) that $\Phi_1(t)$ has in the interval $(-\pi, \pi]$ no more than $2n$ roots. It can easily be observed that if $\mathfrak{z} = 0$ the number of roots of the function $\Phi_1(t)$ does not exceed $2n-2$.

We denote the roots $\Phi_1(t)$ by $t_1, t_2, \ldots, t_{2n}$. We shall prove

LEMMA 3. *In every interval* (t_i, t_{i+1}), $i = 0, 1, \ldots, 2n$, $t_0 = -\pi$, $t_{2n+1} = \pi$, *we have* $|a(t)-c| = 0$ *or* $a(t) = \text{const}$.

Proof. Suppose $|a(t)-c| \not\equiv 0$. Thus there exists a point t_i^0 such that $|a(t_i^0)-c| > 0$. To fix our attention, assume that $\Phi_1(t) > 0$ for $t \in (t_i, t_{i+1})$ and $c = \lim_{t\to t_i^+} a^*(t)$. Since $a(t)$ is a non-decreasing function, $|a(t)-c| > 0$ in the entire interval $[t_i^0, t_{i+1}]$. Thus

$$\int_{t_i}^{t_{i+1}} \Phi_1(t)\,|a^*(t)-c|\,dt > 0.$$

The above inequality contradicts equation (23). Thus $a^*(t) = \text{const}$ in every interval (t_i, t_{i+1}).

LEMMA 4. *The function* $\tilde{h}^*(z)$ *has the form*

$$\tilde{h}^*(z) = \sum_{k=1}^{N} \lambda_k \frac{e^{it_k}+e^{i\varphi}z}{e^{it_k}-z},$$

where $\lambda_k \geqslant 0$, $\sum_{k=1}^{N} \lambda_k = 1$, $t_k = (-\pi, \pi]$, $|\varphi| < \pi$, $N \leqslant n$ *if* $\mathfrak{z} \neq 0$ *and* $N \leqslant n-1$ *if* $\mathfrak{z} = 0$.

Proof. We have found that $a^*(t)$ is in interval (t_i, t_{i+1}) continuous function. In this case we may apply the second variation formula in the class L. By a similar procedure to that followed in the proof of Lemma 1 one may prove that the boundary function satisfies the equation

$$\sum_{k=1}^{n} a_k \frac{d^{k-1}}{d\mathfrak{z}^{k-1}}[\tilde{f}^*(\mathfrak{z})g(\mathfrak{z}, t_l)]+\bar{a}_k \frac{\overline{d^{k-1}}}{d\mathfrak{z}^{k-1}}[\tilde{f}^*(\mathfrak{z})g(\mathfrak{z}, t_l)] \tag{33}$$

$$= \sum_{k=1}^{n} a_k \frac{d^{k-1}}{d\mathfrak{z}^{k-1}}[\tilde{f}^*(\mathfrak{z})g(\mathfrak{z}, t_{l+1})]+\bar{a}_k \frac{\overline{d^{k-1}}}{d\mathfrak{z}^{k-1}}[\tilde{f}^*(\mathfrak{z})g(\mathfrak{z}, t_{l+1})],$$

where t_l and t_{l+1} are the adjacent step-points of $\alpha^*(t)$. Consider the function

$$(34)\qquad \Psi_1(t) = \sum_{k=1}^{n} a_k \frac{d^{k-1}}{d\mathfrak{z}^{k-1}}[\tilde{f}^*(\mathfrak{z})g(\mathfrak{z}, t)] + \bar{a}_k \overline{\frac{d^{k-1}}{d\mathfrak{z}^{k-1}}[\tilde{f}^*(\mathfrak{z})g(\mathfrak{z}, t)]}.$$

By (33) we have

$$\Psi_1(t_l) = \Psi_1(t_{l+1}).$$

Thus in the interval (t_l, t_{l+1}) there exists a point (t_l^0) such that $\Psi_1'(t_0) = 0$.

Comparing (9), (10), (22) and (34) we find that $\Psi_1'(t) = \Phi_1(t)$. The function $\alpha^*(t)$ is a step-function in the interval $(-\pi, \pi]$ and the number of its steps is not greater than n. In fact, supposing that $\alpha^*(t)$ has more than n step-points we find that $\Phi_1(t)$ has more than $2n$ roots. This is impossible by Lemma 2. By (8') the result of the lemma follows. From the above Lemmas we infer the following theorem:

THEOREM 4. *If at every point $w = (f', \overline{f'}, \ldots, f^{(n)}, \overline{f^{(n)}})$ of the region D defined by* (19') *and for every β, $|\beta| = 1$ the condition*

$$\sum_{k=1}^{n} |a_k| > 0, \qquad a_k = \beta \frac{\partial F}{\partial f^{(k)}} + \bar{\beta} \overline{\frac{\partial F}{\partial f^{(k)}}}, \qquad k = 1, 2, \ldots, n,$$

is satisfied, then the function $\tilde{h}^*(z)$ is of the from

$$\tilde{h}(z) = \sum_{k=1}^{N} \lambda_k \frac{e^{it_k} + e^{i\varphi} z}{e^{it_k} - z},$$

where $\lambda_k \geqslant 0$, $\sum_{k=1}^{N} \lambda_k = 1$, $t_k \in (-\pi, \pi)$, $N \leqslant n$ for $\mathfrak{z} \neq 0$ and $N \leqslant n-1$ for $\mathfrak{z} = 0$.

3. THE FUNCTIONS $\mu^*(t)$ AND $\tilde{f}^*(z)$ AND THEIR PROPERTIES

We shall prove the following

THEOREM 5. *If an any point $w = (f', \overline{f'}, \ldots, f^{(n)}, \overline{f^{(n)}})$ of the region D defined by* (19') *and for every $|\beta| = 1$ the condition*

$$\sum_{k=1}^{n} |a_k| > 0, \qquad a_k = \beta \frac{\partial F}{\partial f^{(k)}} + \bar{\beta} \overline{\frac{\partial F}{\partial f^{(k)}}}, \qquad k = 1, 2, \ldots, n,$$

is satisfied, then the function $\tilde{f}^*(z)$ is of the form

$$\tilde{f}^*(z) = \prod_{j=1}^{M} (1 - e^{-i\theta_j} z)^{-2\mu_j},$$

where

$$\theta_j \in (-\pi, \pi], \quad \mu_j \geqslant 0, \quad \sum_{j=1}^{M} \mu_j = 1,$$

$$M \leqslant n \text{ for } \mathfrak{z} \neq 0 \quad \text{and} \quad M \leqslant n-1 \text{ for } \mathfrak{z} = 0.$$

Proof. By a similar procedure to that followed in the proof of Lemma 1 and by formula (17) one may prove that the function $\tilde{f}^*(z)$ satisfies the equation

$$\int_{t_1}^{t_2} \Phi_2(t)\,|\mu^*(t) - c|\,dt = 0, \tag{35}$$

where

$$\Phi_2(t) = \sum_{k=1}^{n} a_k \frac{d^{k-1}}{d\mathfrak{z}^{k-1}}[H(\mathfrak{z})g_2(\mathfrak{z}, t)] + \bar{a}_k \overline{\frac{d^{k-1}}{d\mathfrak{z}^{k-1}}[H(\mathfrak{z})g_2(\mathfrak{z}, t)]},$$

$$H(\mathfrak{z}) = \tilde{h}^*(\mathfrak{z}) \cdot \tilde{f}^*(\mathfrak{z}), \quad g_2(\mathfrak{z}, t) = \frac{2i\mathfrak{z}}{e^{it} - \mathfrak{z}}.$$

The function $\Phi_2(t)$ is real continuous and has in the interval $(-\pi, \pi]$ no more than $2n$ roots. We denote them by $\theta_1, \theta_2, \ldots, \theta_{2n}$. In every interval (θ_i, θ_{i+1}), $\mu^*(t)$ is constant. The fourth variation formula together with the property of the boundary points leads to the equation

$$\sum_{k=1}^{n} a_k \frac{d^{k-1}}{d\mathfrak{z}^{k-1}}[H(\mathfrak{z})\log(1 - e^{-i\theta_l}\mathfrak{z})] + \bar{a}_k \overline{\frac{d^{k-1}}{d\mathfrak{z}^{k-1}}[H(\mathfrak{z})\log(1 - e^{-i\theta_l}\mathfrak{z})]} \tag{36}$$
$$= \sum_{k=1}^{n} a_k \frac{d^{k-1}}{d\mathfrak{z}^{k-1}}[H(\mathfrak{z})\log(1 - e^{-i\theta_{l+1}}\mathfrak{z})] + \bar{a}_k \overline{\frac{d^{k-1}}{d\mathfrak{z}^{k-1}}[H(\mathfrak{z})\log(1 - e^{-i\theta_{l+1}}\mathfrak{z})]},$$

where θ_l, θ_{l+1} are adjacent points of discontinuity of $\mu^*(t)$. Consider the function

$$\Psi_2(t) = 2\sum_{k=1}^{n} a_k \frac{d^{k-1}}{d\mathfrak{z}^{k-1}}[H(\mathfrak{z})\log(1 - e^{-it}\mathfrak{z})] + \tag{37}$$
$$+ \bar{a}_k \overline{\frac{d^{k-1}}{d\mathfrak{z}^{k-1}}[H(\mathfrak{z})\log(1 - e^{-it}\mathfrak{z})]}.$$

From (36) the equality

$$\Psi_2(\theta_l) = \Psi_2(\theta_{l+1})$$

follows.

By an argument analogous to that contained in the proof of Lemma 4 we find that $\mu^*(t)$ is a step function in the interval $(-\pi, \pi)$ and that the number of its steps does not exceed n. If $\mathfrak{z} = 0$, then $M \leqslant n-1$.

By equality (8″) the result of Theorem 5 follows.

4. THE BASIC THEOREM

Theorem (4) and (5) and inequality (8) imply

THEOREM 6. *If the functional $F(f)$ defined in the class L is of the form* (20) *and at every point the region D defined by* (19′) *and for every β, $|\beta| = 1$ the condition*

$$\sum_{k=1}^{n} |a_k| > 0, \qquad a_k = \beta \frac{\partial F}{\partial f^{(k)}} + \bar{\beta} \frac{\overline{\partial F}}{\partial \bar{f}^{(k)}}, \qquad k = 1, 2, \ldots, n,$$

is satisfied, then the boundary function $f^*(z)$ is of the form

$$f^*(z) = \int_0^z \prod_{j=1}^{M} (1 - e^{-i\theta_j} z)^{-2\mu_j} \sum_{k=1}^{N} \lambda_k \frac{e^{it_k} + e^{i\varphi} z}{e^{it_k} - z}\, dz, \tag{38}$$

where $\mu_j \geqslant 0$, $\lambda_k \geqslant 0$, $\sum_{j=1}^{M} \mu_j = 1$, $\sum_{k=1}^{N} \lambda_k = 1$, $\theta_j, t_k \in (-\pi, \pi]$, $|\varphi| < \pi$, $M \leqslant n$ *and* $N \leqslant n$ *if* $\mathfrak{z} \neq 0$ *and* $M \leqslant n-1$ *and* $N \leqslant n-1$ *if* $\mathfrak{z} = 0$.

Using another method Aleksandrow and Gutianski [1] have proved the above theorem.

5. EXTREMAL FUNCTIONS IN RELATION TO REAL FUNCTIONALS

If $F(f)$ is a real functional, the assumptions of Theorem 6 may be weakened. Proceeding in the same way as in sections 2 and 3 of Chapter II we may prove.

THEOREM 7. *If the functional $F(f)$ is of form* (20) *and for and arbitrary $w \in D$, $w = (f', \overline{f'}, \ldots, f^{(n)}, \overline{f^{(n)}})$,*

$$\sum_{k=1}^{n} \left| \frac{\partial F}{\partial f^{(k)}} \right| > 0,$$

then every extremal function is of form (38).

The assumption of Theorem 7 is weaker than the assumption of Theorem 6. This follows from the implication

$$\left(\sum_{k=1}^{n} \left| \beta \frac{\partial F}{\partial f^{(k)}} + \bar{\beta} \frac{\overline{\partial F}}{\partial f^{(k)}} \right| > 0 \right) \Rightarrow \left(\sum_{k=1}^{n} \left| \frac{\partial F}{\partial f^{(k)}} \right| > 0 \right). \tag{39}$$

Proof. Suppose that $\sum_{k=1}^{n}\left|\frac{\partial F}{\partial f^{(k)}}\right| = 0$; hence

$$\frac{\partial F}{\partial f^{(k)}} = 0 \quad \text{for } k = 1, 2, \ldots, n. \tag{40}$$

Since the functional $F(f)$ is real, we have

$$\frac{\partial F}{\partial f^{(k)}} = \overline{\frac{\partial F}{\partial \bar{f}^{(k)}}}. \tag{41}$$

From inequalities (40) and (41) it follows that

$$\sum_{k=1}^{n}\left|\beta\frac{\partial F}{\partial f^{(k)}} + \bar{\beta}\,\overline{\frac{\partial F}{\partial f^{(k)}}}\right| = 0.$$

This equality contradicts the assumption of (39). Thus (39) holds.

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