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ALGORITHM 13

Z. CYLKOWSKI (Wrocław)

APPROXIMATE FUNCTION EXPANSION INTO A CHEBYSHEV SERIES

1. Procedure declaration.

procedure *TChex*(p, q, x, f, n, a, ex);
  value p, q;
  integer n;
  real p, q, x, f;
  Boolean ex;
  array a;
  comment For a given natural number $N \geqslant 2$ the procedure *TChex* calculates the approximate values

$$(1)\qquad a_0^{(n)}, a_1^{(n)}, \ldots, a_n^{(n)} \quad (n = 2, 4, 8, \ldots, n \leqslant N)$$

of the coefficients of the expansion of $f(x)$ in the interval $\langle p, q\rangle$ into a series of Chebyshev polynomials of first kind, i.e. it calculates numbers (1) such that for $x \in \langle p, q\rangle$ holds

$$(2)\quad f(x) \approx \frac{1}{2} a_0^{(n)} + \sum_{k=1}^{n} a_k^{(n)} T_k\left(\frac{2x-p-q}{q-p}\right) \quad (T_k(t) = \cos(k \arccos t)).$$

This equality is accurate if $f(x)$ is a polynomial of degree at most n.
Data:
p, q — interval limits $\langle p, q\rangle$,
  f — arithmetical expression with value $f(x)$ depending upon x,
  n — natural number N.
Results:
    n — upper summation limit in (2), a number of the form 2^l (l — natural),
$a[0:n]$ — array of approximate values of (1) ($a[k] = a_k^{(n)}$ for $k = 0, 1, \ldots, n$).
Remark: During calculations the variable n assumes successively the values 2, 4, 8, ... and for each of them the coefficients of (1)

are found. The final value of n is always not greater than N but it may depend upon the value of ex.

Other parameters:

x — the variable occurring in the 4-th actual parameter,

ex — Boolean expression which may depend upon the actual values of n, $a[0:n]$. If $ex =$ **true** while doubling the value of n then the procedure calculates the coefficients (1) for the new n, if $ex =$ **false** the calculations are finished.

Remarks:

1° The coefficients $a_k^{(n/2)}$ $(k = 0, 1, \ldots, \frac{1}{2}n)$ may be calculated from

$$a_k^{(n/2)} = \begin{cases} a_0^{(n)} + 2a_n^{(n)} & (k = 0), \\ a_k^{(n)} + a_{n-k}^{(n)} & (k = 1, 2, \ldots, \frac{1}{2}n - 1), \\ a_{n/2}^{(n)} & (k = \frac{1}{2}n). \end{cases} \tag{3}$$

2° The coefficients (1) with lower indices being only either even or odd may be obtained from the fact that if we have in the interval $\langle -1, 1\rangle$ the formula

$$g(x) \underset{\mathrm{df}}{=} f\left(\tfrac{1}{2}(p+q) + \tfrac{1}{2}(q-p)x\right) = \tfrac{1}{2}a_0 + \sum_{k=1}^{n} a_k T_k(x), \tag{4}$$

then in the interval $\langle 0, 1\rangle$ holds

$$\begin{gathered} g(\sqrt{x}) + g(-\sqrt{x}) = a_0 + 2\sum_{k=1}^{n/2} a_{2k} T_k(2x-1), \\ \sqrt{x}\left(g(\sqrt{x}) - g(-\sqrt{x})\right) = a_1 + \sum_{k=1}^{n/2} (a_{2k-1} + a_{2k+1}) T_k(2x-1) \qquad (a_{n+1} = 0); \end{gathered} \tag{5}$$

```
begin
    integer h, h2, i0, i1, j, k, m, n1, n2;
    real a0, a1, a2, c, c2;
    m: = n;
    n: = n2: = h2: = 2;
    x: = p;
    a2: = f;
    x: = p: = .5 × (p + q);
    a1: = f;
    x: = q;
    a0: = f;
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```
q: = q-p;
c: = .5×(a0+a2);
a[0]: = c+a1;
a[1]: = .5×(a0-a2);
a[2]: = .5×(c-a1);
for n2: = n2+n2 while ex ∧ n2 ≤ m do
  begin
    n1: = h2+1;
    a0: = 3.14159265359/h2;
    for j: = h2 step -1 until 1 do
      begin
        c2: = cos(a0×(j-.5));
        x: = p+q×c2;
        a[h2+j]: = f
      end j;
    c2: = c2+c2;
    for h: = h2 while h > 1 do
      begin
        h2: = .5×h;
        a2: = c: = c2;
        c2: = c2×c2-2;
        for j: = 0, k while j < h2 do
          begin
            k: = j+1;
            i1: = n1-k;
            for i0: = n1+j step h until n2 do
              begin
                i1: = i1+h;
                a0: = a[i0];
                a1: = a[i1];
                a[i0]: = c×(a0-a1);
                a[i1]: = a0+a1;
                c: = -c
```

```
                end i0;
              a0: = a2;
              a2: = abs(c);
              c: = c2 × a2 − a0
            end j
        end h;
    k: = h: = .25 × n2;
    h2: = n1 − 3;
    for j: = 1 step 1 until h2 do
        begin
          if j < k
            then begin
                    i0: = n2 − j;
                    i1: = n2 − k;
                    a0: = a[i0];
                    a[i0]: = a[i1];
                    a[i1]: = a0
                 end j < k;
          i0: = h;
    aa: if i0 ⩽ k
            then begin
                    k: = k − i0;
                    i0: = .5 × i0;
                    go to aa
                 end i0 ⩽ k
            else k: = k + i0
          end j;
      for h2: = h while h > 1 do
          begin
            h: = .5 × h2;
            i0: = n2 − h;
            for k: = i0 − h + 1 step 1 until i0 do
```

```
                begin
                  a0: = a[k];
                  for j: = k-h2 step -h2 until n1 do
                    a0: = a[j]: = a[j]-a0
                end k
            end h2;
          a2: = 1/n2;
          a[0]: = .5×a[0];
          for j: = n1 step 1 until n2 do
            begin
              k: = n2-j;
              a0: = .5×a[k];
              a1: = a2×a[j];
              a[k]: = a0+a1;
              a[j]: = a0-a1
            end j;
          a[0]: = 2×a[0];
          n: = h2: = n2
        end n2
end TChex
```

2. Method used. Let

$$g(x) = f\left(\tfrac{1}{2}(p+q)+\tfrac{1}{2}(q-p)x\right) \quad (x \in \langle -1, 1\rangle),$$

$$x_j^{(n)} = \cos\frac{j\pi}{n} \quad (n = 2, 4, 8, \ldots; j = 0, 1, \ldots, n),$$

$$\delta_{kn} = \begin{cases} 0 & (k \neq n), \\ 1 & (k = n). \end{cases}$$

Procedure *TChex* uses the formulae given by Clenshaw [1]

$$a_k^{(n)} = \frac{2-\delta_{kn}}{n} \sum_{j=0}^{n}{}'' g(x_j^{(n)}) T_k(x_j^{(n)}) \quad (k = 0, 1, \ldots, n) \tag{6}$$

($\sum''$ denotes that the first and last summands should be halved) to calculate $a_0^{(2)}, a_1^{(2)}, a_2^{(2)}$. All other $a_k^{(2n)}$ are calculated after formulae which follow from (6):

$$a_k^{(2n)} = \tfrac{1}{2}(a_k^{(n)} + b_k^{(n)}) \quad (k = 0, 1, \ldots, n-1), \qquad a_n^{(2n)} = a_n^{(n)},$$

$$a_{2n-k}^{(2n)} = \tfrac{1}{2}(a_k^{(n)} - b_k^{(n)}) \quad (k = 1, 2, \ldots, n-1), \qquad a_{2n}^{(2n)} = \tfrac{1}{4}(a_0^{(n)} - b_0^{(n)}),$$

where $b_k^{(n)}$ are expressed by formulae given by Lanczos [3] (p. XVII):

$$(7) \qquad b_k^{(n)} = \frac{2}{n} \sum_{j=1}^{n} g(x_{2j-1}^{(2n)}) T_k(x_{2j-1}^{(2n)}) \quad (k = 0, 1, \ldots, n-1).$$

Since the numbers given by (7) are Chebyshev coefficients of the polynomial P satisfying

$$P(x_{2j-1}^{(2n)}) = g(x_{2j-1}^{(2n)}) \quad (j = 1, 2, \ldots, n),$$

degree of $P \leqslant n$, therefore, from (5), they are calculated from the coefficients of the polynomial

$$P_0(x) = P(\sqrt{\tfrac{1}{2}x + \tfrac{1}{2}}) + P(-\sqrt{\tfrac{1}{2}x + \tfrac{1}{2}})$$

and

$$P_1(x) = \sqrt{\tfrac{1}{2}x + \tfrac{1}{2}}\,(P(\sqrt{\tfrac{1}{2}x + \tfrac{1}{2}}) - P(-\sqrt{\tfrac{1}{2}x + \tfrac{1}{2}})).$$

The polynomials P_0 and P_1 are of degree at most $\tfrac{1}{2}n$ and satisfy the following interpolation conditions:

$$P_0(x_{2j-1}^{(n)}) = P(x_{2j-1}^{(2n)}) + P(x_{2(n-j)+1}^{(2n)}),$$

$$P_1(x_{2j-1}^{(n)}) = x_{2j-1}^{(2n)}\big(P(x_{2j-1}^{(2n)}) - P(x_{2(n-j)+1}^{(2n)})\big),$$

where $j = 1, 2, \ldots, \tfrac{1}{2}n$.

If $n = 2$, the coefficients of the polynomials P_0 and P_1 are already obtained (the coefficient of $T_n(x)$ is not needed); otherwise formulae (5) are used repeatedly which leads to n polynomials of degree at most one which satisfy one interpolation condition each.

3. Coefficients of the expansion into a series of Chebyshev polynomials of second kind. If, as before,

$$g(x) = \tfrac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k T_k(x) \quad (x \in \langle -1, 1 \rangle)$$

and, in addition,

$$g(x) = \sum_{k=0}^{\infty} a_k U_k(x) \quad \left(U_k(x) = \frac{\sin\big((k+1)\arccos x\big)}{\sqrt{1-x^2}}\right),$$

then it follows from the known relation $T_k(x) = \frac{1}{2}\big(U_k(x) - U_{k-2}(x)\big)$ that

$$a_k = \tfrac{1}{2}(\alpha_k - \alpha_{k+2}) \qquad (k = 0, 1, \ldots),$$

and Chebyshev coefficients relevant to Chebyshev polynomials of 2-nd kind may easily be obtained by procedure *TChex*.

In addition, the expansion of function g into a series of polynomials $U_k(x)$ may be obtained directly using a method similar to that described in Section 2. Thus, for $x \in \langle 0, 1\rangle$ the following formulae hold:

$$\text{(8)} \qquad \begin{aligned} g(\sqrt{x}) + g(-\sqrt{x}) &= 2\sum_{k=0}^{\infty} (\alpha_{2k} + \alpha_{2k+2})\, U_k(2x-1), \\ \frac{g(\sqrt{x}) - g(-\sqrt{x})}{\sqrt{x}} &= 4\sum_{k=0}^{\infty} \alpha_{2k+1}\, U_k(2x-1); \end{aligned}$$

they are used for calculating $\beta_k^{(n)}$, given by (10).

The approximate values

$$\text{(9)} \qquad \alpha_0^{(n)}, \alpha_1^{(n)}, \ldots, \alpha_n^{(n)} \qquad (n = 2^l - 2;\ l = 1, 2, \ldots)$$

of the Chebyshev coefficients are calculated from the recurrent formulae

$$\begin{aligned} \alpha_0^{(0)} &= g(0), \\ \left.\begin{aligned} \alpha_k^{(2n+2)} &= \tfrac{1}{2}(\beta_k^{(n)} + \alpha_k^{(n)}) \\ \alpha_{2n+2-k}^{(2n+2)} &= \tfrac{1}{2}(\beta_k^{(n)} - \alpha_k^{(n)}) \end{aligned}\right\} & \quad (k = 0, 1, \ldots, n), \\ \alpha_{n+1}^{(2n+2)} &= \tfrac{1}{2}\beta_{n+1}^{(n)}, \end{aligned}$$

where

$$\text{(10)} \qquad \beta_k^{(n)} = \frac{2}{n+2}\sum_{j=1}^{n+2} \big(1 - (x_{2j-1}^{(2n+4)})^2\big) g(x_{2j-1}^{(2n+4)})\, U_k(x_{2j-1}^{(2n+4)})$$

$$(k = 0, 1, \ldots, n+1).$$

These formulae follow from an application of a known quadrature (see e.g. [2], p. 111, formula (7.3.7)):

$$\begin{aligned} \alpha_k &= \frac{2}{\pi}\int_{-1}^{1} \sqrt{1-x^2}\, g(x)\, U_k(x)\, dx \\ &\approx \frac{2}{n+2}\sum_{j=1}^{n+1} \big(1 - (x_j^{(n+2)})^2\big)\, g(x_j^{(n+2)})\, U_k(x_j^{(n+2)}) = \alpha_k^{(n)}. \end{aligned}$$

The numbers $\beta_k^{(n)}$, similarly as $b_k^{(n)}$ before, are obtained by applying (8) to the polynomial

$$Q(x) = \sum_{k=0}^{n} \beta_k^{(n)} U_k(x) + \tfrac{1}{2}\beta_{n+1}^{(n)} U_{n+1}(x)$$

and to the polynomials which arise from Q in such a way.

The definition of Q and [2] (p. 111, formula (7.3.6)) yield

$$\beta_k^{(n)} = \frac{2+2\delta_{k,n+1}}{\pi} \int_{-1}^{1} \frac{(1-x^2)Q(x)\,U_k(x)}{\sqrt{1-x^2}}\,dx$$

$$= \frac{2}{n+2} \sum_{j=1}^{n+2} \left(1-(x_{2j-1}^{(2n+4)})^2\right) Q(x_{2j-1}^{(2n+4)})\, U_k(x_{2j-1}^{(2n+4)}) \qquad (k = 0, 1, \ldots, n+1),$$

and, because the interpolation polynomial is uniquely determined, there is

$$Q(x_{2j-1}^{(2n+4)}) = g(x_{2j-1}^{(2n+4)}) \qquad (j = 1, 2, \ldots, n+2).$$

The following list contains in the left column the lines or groups of consecutive lines which are to be removed from the body of procedure *TChex* and replaced by the lines from the right column in order to obtain a procedure calculating the coefficients (9), i.e. $a[k] = a_k^{(n)}$ $(k = 0, 1, \ldots, n)$.

integer *h, h2, i0, i1, j, k, m, n1, n2*;	**integer** *h, h2, i0, i1, j, k, m, n1, n11, n2*;
n: = *n2*: = *h2*: = *2*;	*n*: = *0*;
x: = *p*; *a2*: = *f*;	*n1*: = *1*;
a1: = *f*; *x*: = *q*; *a0*: = *f*;	
c: = .*5* × (*a0* + *a2*); *a*[*0*]: = *c* + *a1*; *a*[*1*]: = .*5* × (*a0* − *a2*); *a*[*2*]: = .*5* × (*c* − *a1*); **for** *n2*: = *n2* + *n2* **while** *ex* ∧ *n2* ≤ *m* **do**	*a*[*0*]: = *f*; **for** *n2*: = *n1* + *n1* **while** *ex* ∧ *n2* ≤ *m* **do**
n1: = *h2* + *1*;	*n11*: = *h2*: = *n1* + *1*;
for *j*: = *h2* **step** −*1* **until** *1* **do**	**for** *j*: = *n1* **step** −*1* **until** *0* **do**
c2: = *cos* (*a0* × (*j* − .*5*));	*c2*: = *cos* (*a0* × (*j* + .*5*));

```
a[h2+j]: = f
a[i0]: = c×(a0−a1);
k: = h: = .25×n2;
h2: = n1−3;
for j: = 1 step 1 until h2 do
i0: = n2−h;
for k: = i0−h+1 step 1 until i0 do
for j: = k−h2 step −h2 until n1 do
a2: = 1/n2;
a[0]: = .5×a[0];
for j: = n1 step 1 until n2 do
a[j]: = a0−a1
a[0]: = 2×a[0];
n: = h2: = n2
```

```
a[n1+j]: = f
a[i0]: = (a0−a1)/c;
k: = h: = .5×n11;
for j: = 1 step 1 until n do
i0: = n1+h2−1;
for k: = n1+h step 1 until i0 do
for j: = k+h2 step h2 until n2 do
a2: = .5/n11;
a[n1]: = a[n1]×(a2+a2);
for j: = n11 step 1 until n2 do
a[j]: = a1−a0
n1: = n2+1;
n: = n2
```

4. Certification. For the function $f(x) = \log x$ and the interval $\langle p, q\rangle = \langle \frac{1}{2}, \frac{3}{2}\rangle$ the results given in Table 1 were obtained on the ODRA 1204 computer (accuracy: 2^{-37}). To test the influence of rounding errors, the maximum deviations Δ_n of the approximate coefficients (1) from the appropriate accurate Chebyshev coefficients have been also calculated (Table 2).

TABLE 1

k	$a_k^{(2)}$	$a_k^{(4)}$	$a_k^{(8)}$
0	−.1438410362	−.1386862144	−.1386729286
1	.5493061443	.5359283009	.5358983852
2	−.0719205181	−.0719205181	−.0717967711
3		.0133778435	.0128252633
4		−.0025774109	−.0025774109
5			.0005525802
6			−.0001237470
7			.0000299156
8			−.0000066429

TABLE 2

n	Δ_n
2	.0134077595
4	.0005525858
8	.0000015822
16	.0000000000
32	.0000000000
64	.0000000004
128	.0000000004
256	.0000000044
512	.0000000177
1024	.0000000926

If the actual parameters of procedure *TChex* are not involved, the calculation time is nearly proportional to $\nu(\log_2\nu+1)$, where ν is the final value of parameter n.

References

[1] C. W. Clenshaw, *Chebyshev series for mathematical functions*, London 1962.
[2] В. И. Крылов, *Приближенное вычисление интегралов*, Москва 1959.
[3] *Tables of Chebyshev polynomials*, Washington 1952.

DEPT. OF NUMERICAL METHODS
MATHEMATICAL INSTITUTE
UNIVERSITY OF WROCŁAW

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Z. CYLKOWSKI (Wrocław ALGORYTM 13

PRZYBLIŻONE ROZWIJANIE FUNKCJI W SZEREG CZEBYSZEWA

STRESZCZENIE

Procedura *TChex* dla danej liczby naturalnej $N \geqslant 2$ oblicza przybliżone wartości (1) współczynników rozwinięcia funkcji $f(x)$ w przedziale $\langle p, q\rangle$ w szereg względem wielomianów Czebyszewa I rodzaju, tj. takie liczby (1), że dla $x \in \langle p, q\rangle$ zachodzi równość (2). Równość (2) jest dokładna, jeśli $f(x)$ jest wielomianem co najwyżej n-tego stopnia.

Dane:

p, q — końce przedziału $\langle p, q\rangle$,
f — wyrażenie arytmetyczne o wartości $f(x)$, zależne od parametru x,
n — liczba naturalna N.

Wyniki:

n — górna granica sumowania w (2), liczba postaci 2^l (l — liczba naturalna),
$a[0:n]$ — tablica przybliżonych wartości (1) ($a[k] = a_k^{(n)}$ dla $k = 0, 1, \ldots, n$).

Uwaga: W czasie obliczeń zmienna n przyjmuje kolejno wartości $2, 4, 8, \ldots$ i dla każdej z nich znajduje się współczynniki (1). Końcowa wartość tej zmiennej jest zawsze nie większa od N, ale może zależeć od wartości parametru *ex*.

Inne parametry:

x — zmienna występująca w czwartym parametrze aktualnym,
ex — wyrażenie boolowskie, które może zależeć od aktualnych wartości n i $a[0:n]$.

Jeśli przy podwojeniu wartości n zachodzi $ex =$ **true**, to procedura oblicza współczynniki (1) dla nowego n, a jeśli $ex =$ **false**, to obliczenia kończy się.

Uwagi:

1° Współczynniki $a_k^{(n/2)}$ ($k = 0, 1, \ldots, \frac{1}{2}n$) można odtworzyć z wzoru (3).

2° Współczynniki (1) o dolnych wskaźnikach tylko parzystych albo tylko nieparzystych można otrzymać, uwzględniając zależność, że jeśli w przedziale $\langle -1, 1\rangle$ zachodzi wzór (4), to w przedziale $\langle 0, 1\rangle$ zachodzą wzory (5).

Użyta w procedurze *TChex* metoda jest opisana w § 2. Metodę tę i treść procedury można zmodyfikować (§ 3) tak, żeby otrzymać współczynniki (9) rozwinięcia funkcji $f(x)$ w szereg względem wielomianów Czebyszewa II rodzaju. Obliczenia kontrolne (§ 4) wykonano na maszynie cyfrowej ODRA 1204, przy czym czas obliczeń był proporcjonalny do wielkości $\nu(\log_2\nu+1)$, gdzie ν jest końcową wartością parametru n.