ANNALES
POLONICI MATHEMATICI
XXIV (1971)

# On the asymptotic behavior of solutions of the second-order linear differential equation

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**1.** In his recent paper [1] A. C. Lazer has given a simple condition for all solutions of the differential equation

$$y''+p(x)y = 0 \tag{1}$$

to tend to zero as $x$ tends to infinity. Namely, he proved the following

THEOREM. *If $p(x) > 0$, $p(x) \in C^3_{[a,\infty)}$, $\lim\limits_{x\to+\infty} p(x) = +\infty$ and*

$$\int_a^\infty \left|\left(\frac{1}{\sqrt{p(x)}}\right)'''\right| dx < +\infty,$$

*then, for each solution $y(x)$ of* (1), $\lim\limits_{x\to\infty} y(x) = 0$.

In the present paper we will show that the same result may be obtained under more general assumptions than those of Lazer and we will prove a similar property of the derivative of every $y(x)$ satisfying equation (1). In the second part of this paper, returning to original Lazer's assumptions, we will strengthen considerably his theorem.

In the proof of his theorem Lazer considers the function

$$w(x) = \frac{[y'(x)]^2}{\sqrt{p(x)}} - \left(\frac{1}{\sqrt{p(x)}}\right)' y(x)y'(x) + \left[\sqrt{p(x)} + \tfrac{1}{2}\left(\frac{1}{\sqrt{p(x)}}\right)''\right] y^2(x).$$

As may be verified by differentiation, if $y(x)$ is a solution of (1) then

$$w(x) = w(a) + \tfrac{1}{2}\int_a^x \left(\frac{1}{\sqrt{p(t)}}\right)''' y^2(t)\, dt. \tag{2}$$

In the proofs of our theorems we will make use of the same function $w(x)$ and equality (2).

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[1] A. C. Lazer, *A stability condition for the differential equation $y''+p(x)y = 0$*, Michigan Math. Journ. 12 (1965), p. 193-196.

**2.** THEOREM 1. *If* $p(x) > 0$, $p(x) \in C^3_{[a,\infty)}$, $\lim_{x\to+\infty} p(x) = +\infty$, *and*

$$\limsup_{x\to+\infty} \frac{1}{\sqrt{p(x)}} \int_a^x \left| \left( \frac{1}{\sqrt{p(t)}} \right)''' \right| dt < 1,$$

*then, for each solution* $y(x)$ *of* (1),

$$\lim_{x\to+\infty} y(x) = 0 \tag{3}$$

*and*

$$\lim_{x\to+\infty} \frac{y'(x)}{\sqrt{p(x)}} = 0. \tag{4}$$

Proof. Let $y(x)$ be a non-trivial solution of (1). We shall first prove boundedness of $y(x)$ for $a \leqslant x < +\infty$. Since $p(x) \to +\infty$ as $x \to +\infty$, $y(x)$ is oscillatory. To show that $y(x)$ is bounded, it is therefore sufficient to prove that the absolute values of $y(x)$ at its relative maximum and minimum points are bounded. Suppose then that these values are unbounded. Then there exists a sequence $\{c_n\}$ such that

$$y'(c_n) = 0, \quad |y(c_n)| = \max\{|y(x)|: x \in [a, x_n]\} \quad (n = 1, 2, \ldots),$$
$$\lim_{n\to\infty} c_n = \infty \quad \text{and} \quad \lim_{n\to\infty} |y(c_n)| = \infty.$$

From equality (2), for $x = c_n$ $(n = 1, 2, \ldots)$, we obtain

$$w(c_n) \doteq \left[ \sqrt{p(c_n)} + \frac{1}{2} \left( \frac{1}{\sqrt{p(x)}} \right)''_{x=c_n} \right] y^2(c_n) = w(a) + \frac{1}{2} \int_a^{c_n} \left( \frac{1}{\sqrt{p(t)}} \right)''' y^2(t)\, dt.$$

Since

$$\left( \frac{1}{\sqrt{p(x)}} \right)'' = \left( \frac{1}{\sqrt{p(x)}} \right)''_{x=a} + \int_a^x \left( \frac{1}{\sqrt{p(t)}} \right)''' dt \tag{5}$$

setting

$$c = \left| \frac{1}{2} \left( \frac{1}{\sqrt{p(x)}} \right)''_{x=a} \right|$$

we have the inequality

$$\sqrt{p(c_n)}\, y^2(c_n) \leqslant c y^2(c_n) + |w(a)| + y^2(c_n) \int_a^{c_n} \left| \left( \frac{1}{\sqrt{p(t)}} \right)''' \right| dt.$$

Hence we obtain the inequality

$$y^2(c_n) \sqrt{p(c_n)} \left[ 1 - \frac{c}{\sqrt{p(c_n)}} - \frac{1}{\sqrt{p(c_n)}} \int_a^{c_n} \left| \left( \frac{1}{\sqrt{p(t)}} \right)''' \right| dt \right] \leqslant |w(a)|. \tag{6}$$

Since, by the assumption, $\lim_{n\to\infty} y^2(c_n)\sqrt{p(c_n)} = +\infty$, from (6) we obtain

$$\liminf_{x\to+\infty}\left[1-\frac{1}{\sqrt{p(x)}}\int_a^x\left|\left(\frac{1}{\sqrt{p(t)}}\right)'''\right|dt\right]\leqslant 0,$$

which is in contradiction with the assumption.

Thus we have

$$\alpha = \limsup_{x\to+\infty} y^2(x) < +\infty.$$

To complete the proof of (3) it is sufficient to prove that $\alpha = 0$. Suppose that $\alpha > 0$. Then for every $\varepsilon > 0$ there exists a number $M$ such that

$$y^2(x) < \alpha+\varepsilon \quad \text{for } x > M. \tag{7}$$

Let $x_1 < x_2 < \ldots$ be the successive relative maximum and minimum points of $y(x)$. Then $y'(x_n) = 0$ and $x_n \to +\infty$ as $n \to +\infty$. From (2) for $x = x_n$ we have

$$\sqrt{p(x_n)}\,y^2(x_n)+\tfrac{1}{2}\left(\frac{1}{\sqrt{p(x)}}\right)''_{x=x_n} y^2(x_n) = w(a)+\tfrac{1}{2}\int_a^{x_n}\left(\frac{1}{\sqrt{p(t)}}\right)''' y^2(t)\,dt. \tag{8}$$

Setting $b > M$ and

$$\begin{aligned} c &= \left|\tfrac{1}{2}\left(\frac{1}{\sqrt{p(x)}}\right)''_{x=a}+\tfrac{1}{2}\int_a^b\left(\frac{1}{\sqrt{p(t)}}\right)'''dt\right|,\\ d &= \left|w(a)+\tfrac{1}{2}\int_a^b\left(\frac{1}{\sqrt{p(t)}}\right)''' y^2(t)\,dt\right| \end{aligned} \tag{9}$$

from equalities (5) and (8) we obtain the inequality

$$\sqrt{p(x_n)}\,y^2(x_n)\leqslant cy^2(x_n)+d+\tfrac{1}{2}y^2(x_n)\int_b^{x_n}\left|\left(\frac{1}{\sqrt{p(t)}}\right)'''\right|dt+ \\ +\tfrac{1}{2}\int_b^{x_n}\left(\frac{1}{\sqrt{p(t)}}\right)''' y^2(t)\,dt.$$

From inequality (7) we obtain the futher inequality

$$y^2(x_n)\leqslant\frac{c}{\sqrt{p(x_n)}}(\alpha+\varepsilon)+\frac{d}{\sqrt{p(x_n)}}+\frac{\alpha+\varepsilon}{\sqrt{p(x_n)}}\int_a^{x_n}\left|\left(\frac{1}{\sqrt{p(t)}}\right)'''\right|dt \quad \text{for } x_n > M.$$

Hence

$$\limsup_{n\to\infty} y^2(x_n) \leqslant (a+\varepsilon)\limsup_{n\to\infty} \frac{1}{\sqrt{p(x_n)}} \int_a^{x_n} \left|\left(\frac{1}{\sqrt{p(t)}}\right)'''\right| dt. \tag{10}$$

Write

$$\beta = \limsup_{x\to+\infty} \frac{1}{\sqrt{p(x)}} \int_a^{x} \left|\left(\frac{1}{\sqrt{p(t)}}\right)'''\right| dt.$$

If $\beta = 0$, then from (7) it follows that $a = 0$ and we have the contradiction. If $\beta > 0$, then, for $\varepsilon < a(1-\beta)/\beta$, we have $\beta(a+\varepsilon) < a$, in contradiction with (10). This proves that $a = 0$ and, therefore, relation (3) is satisfied.

For to prove (4), consider the sequence $x_1 < x_2 < \ldots$ of the successive relative maximum and minimum points of the function $y'(x)/\sqrt{p(x)}$. It is sufficient to show that

$$\lim_{n\to\infty} \frac{y'(x_n)}{\sqrt{p(x_n)}} = 0. \tag{11}$$

Since

$$\left(\frac{y'(x)}{\sqrt{p(x)}}\right)'_{x=x_n} = 0 \quad (n = 1, 2, \ldots)$$

therefore

$$\left(\frac{1}{\sqrt{p(x)}}\right)'_{x=x_n} = \left(\frac{\sqrt{p(x)}\,y(x)}{y'(x)}\right)_{x=x_n} \quad (n = 1, 2, \ldots).$$

Hence and from equality (2), for $x = x_n$ $(n = 1, 2, \ldots)$, we have

$$\begin{aligned} w(x_n) &= \frac{[y'(x_n)]^2}{\sqrt{p(x_n)}} + \tfrac{1}{2}\left(\frac{1}{\sqrt{p(x)}}\right)''_{x=x_n} y^2(x_n) \\ &= w(a) + \tfrac{1}{2}\int_a^{x_n} \left(\frac{1}{\sqrt{p(t)}}\right)''' y^2(t)\,dt. \end{aligned} \tag{12}$$

By (3), for any given $\varepsilon > 0$ there exists a number $M$ such that

$$y^2(x) < \varepsilon \quad \text{for } x > M. \tag{13}$$

Setting $b > M$, from inequalities (12), (13) and equalities (9) we have

$$\left(\frac{y'(x)}{\sqrt{p(x)}}\right)^2_{x=x_n} \leqslant \frac{c\varepsilon+d}{\sqrt{p(x_n)}} + \varepsilon \frac{1}{\sqrt{p(x_n)}} \int_a^{x_n} \left|\left(\frac{1}{\sqrt{p(t)}}\right)'''\right| dt \quad \text{for } x_n > M. \tag{14}$$

Since the right-hand side of inequality (14) can be made arbitrarily small, we have (11) and the proof is completed.

**3.** Theorem 1 is a generalization of Lazer's theorem, since if the function $p(x)$ satisfies the assumptions of this theorem, then

$$\limsup_{x\to+\infty} \frac{1}{\sqrt{p(x)}} \int_a^x \left| \left( \frac{1}{\sqrt{p(x)}} \right)''' \right| dt = 0 .$$

If, however, the function $p(x)$ satisfies the Lazer's assumptions we can obtain a stronger result, namely the following

THEOREM 2. *If* $p(x) > 0$, $p(x) \in C^3_{[a,\infty)}$, $\lim_{x\to+\infty} p(x) = +\infty$, *and*

$$\int_a^\infty \left| \left( \frac{1}{\sqrt{p(x)}} \right)''' \right| dx < +\infty ,$$

*then, for each solution* $y(x)$ *of* (1),

$$\left[ y^2(x) + \frac{[y'(x)]^2}{p(x)} \right] = O\left( 1/\sqrt{p(x)} \right).$$

Proof. Since $\int_0^\infty \left| \left( 1/\sqrt{p(x)} \right)''' \right| dx < +\infty$ and for every solution $y(x)$ of (1) $y(x) \to 0$ as $x \to +\infty$, therefore from equality (2) the functions $w(x)$ and $\left( 1/\sqrt{p(x)} \right)'' y^2(x)$ are bounded for $x \in [a, \infty)$. Thus there exists a number $M$ such that

$$\left| \frac{1}{\sqrt{p(x)}} [y'(x)]^2 - \left( \frac{1}{\sqrt{p(x)}} \right)' y(x) y'(x) + \sqrt{p(x)}\, y^2(x) \right| \leqslant M \qquad \text{for } x \in [a, \infty).$$

From the above inequality we have

$$\sqrt{p(x)} \left[ y^2(x) + \frac{[y'(x)]^2}{p(x)} \right] - \left| \left( \frac{1}{\sqrt{p(x)}} \right)' \right| |y(x) y'(x)| \leqslant M .$$

Since

$$|y(x) y'(x)| \leqslant \tfrac{1}{2} \sqrt{p(x)} \left[ y^2(x) + \left( \frac{y'(x)}{\sqrt{p(x)}} \right)^2 \right]$$

we obtain

$$\sqrt{p(x)} \left[ y^2(x) + \frac{[y'(x)]^2}{p(x)} \right] \left( 1 - \tfrac{1}{2} \left| \left( \frac{1}{\sqrt{p(x)}} \right)' \right| \right) \leqslant M \qquad \text{for } x \in [a, \infty). \tag{15}$$

Since $\left| \left( 1/\sqrt{p(x)} \right)''' \right|$ is integrable, $\left( 1/\sqrt{p(x)} \right)''$ is bounded for $x \in [a, \infty)$. Therefore, since $\lim_{x\to+\infty} 1/\sqrt{p(x)} = 0$, it follows that

$$\lim_{x\to\infty} \left( \frac{1}{\sqrt{p(x)}} \right)' = 0 .$$

Thus from inequality (15) it follows that there exists a positive number $K$ such that

$$\sqrt{p(x)}\left[y^2(x)+\frac{[y'(x)]^2}{p(x)}\right]\leqslant K \quad \text{for } x\in[a,\infty).$$

This completes the proof of Theorem 2.

COROLLARY. *For any solution $y(x)$ of (1) there exists a number $M$ such that*

$$|y(x)|\leqslant\frac{M}{\sqrt[4]{p(x)}},\quad |y'(x)|\leqslant M\sqrt[4]{p(x)} \quad \textit{for } x\in[a,\infty).$$

*Reçu par la Rédaction le 6. 9. 1969*