ANNALES
POLONICI MATHEMATICI
XXV (1971)

q-fractional differentiation and basic hypergeometric transformations

by MANJARI UPADHYAY* (Lucknow, India)

1. Introduction. Recently Agarwal [1] and Al-Salam [2] defined certain fractional q-integral operators and developed their fundamental properties. Agarwal [1] defined the operation of fractional q-differentiation by means of the correspondence

$$D_q^a f(x) = \frac{1}{\Gamma_q(-a)} \int_0^x [x-tq]_{-a-1} f(t)\, d(t;q), \tag{1.1}$$

where the basic integral is defined through the relation (cf. Hahn [4])

$$\int_0^x f(x)\, d(x;q) = x(1-q) \sum_{j=0}^{\infty} q^j f(xq^j).$$

Using the series definition for the basic integral, we may write (1.1) as

$$D_q^a f(x) = (1-q)^{-a} x^{-a} \sum_{j=0}^{\infty} \frac{[q^{-a}]_j q^j}{[q]_j} f(xq^j). \tag{1.2}$$

The object of this paper is to give some interesting applications of the D_q^a-operator in the deduction of general expansions of basic hypergeometric functions.

2. Definitions and notation. For $|q| < 1$, let

$$[a]_r \equiv [q^a]_r = (1-q^a)(1-q^{a+1}) \ldots (1-q^{a+r-1}); \quad [a]_0 \equiv [q^a]_0 = 1.$$

Then the generalized basic hypergeometric series is defined as

$${}_A\Phi_B\left[\begin{matrix} q^{(a)}; x \\ q^{(b)} \end{matrix}\right] \equiv {}_A\Phi_B\left[\begin{matrix} (a); x \\ (b) \end{matrix}\right] = \sum_{r=0}^{\infty} \frac{[(a)]_r x^r}{[(b)]_r [q]_r}; \quad |x| < 1.$$

* I am grateful to Professor R. P. Agarwal for suggesting the problem to me and for his guidance throughout the preparation of this paper. This work is supported by a Junior Research Fellowship of the University Grants Commission, New Delhi, India.

As usual, (a_N) stands for the sequence of N parameters $a_1, a_2, \ldots, a_N$; when $N = A$, we shall simply write (a) instead of (a_A).

Now we define a basic double hypergeometric function of higher order as

$$\Phi\left[\begin{array}{c|c} x;\lambda_1 & \begin{array}{c}(a)\\(b);(c)\\(d)\\(e);(f)\end{array} \\ y;\lambda_2 & \end{array}\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[(a)]_{m+n}[(b)]_m[(c)]_n x^m y^n q^{\lambda_1 m(m-1)} q^{\lambda_2 n(n-1)}}{[q]_m[q]_n[(d)]_{m+n}[(e)]_m[(f)]_n},$$

where $\lambda_1, \lambda_2 \geqslant 0$. For $\lambda_1 = \lambda_2 = 0$ we shall simply write

$$\Phi\left[\begin{array}{c|c} x & \begin{array}{c}(a)\\(b);(c)\\(d)\\(e);(f)\end{array} \\ y & \end{array}\right]; \quad \text{valid for } |x| < 1,\ |y| < 1.$$

This includes the basic analogue of Appell's functions which are defined as

$$\Phi^{(1)}[a; b, b'; c; x, y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[a]_{m+n}[b]_m[b']_n x^m y^n}{[q]_m[q]_n[c]_{m+n}},$$

$$\Phi^{(2)}[a; b, b'; c, c'; x, y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[a]_{m+n}[b]_m[b']_n x^m y^n}{[q]_m[q]_n[c]_m[c']_n},$$

$$\Phi^{(3)}[a, a'; b, b'; c; x, y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[a]_m[a']_n[b]_m[b']_n x^m y^n}{[q]_m[q]_n[c]_{m+n}},$$

$$\Phi^{(4)}[a; b; c, c'; x, y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[a]_{m+n}[b]_{m+n} x^m y^n}{[q]_m[q]_n[c]_m[c']_n}.$$

The confluent forms defined by Jackson [6] are given below:

$$\gamma_1(a; b; c; x, y; \lambda) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[a]_{m+n}[b]_m x^m y^n q^{\lambda n(n-1)}}{[q]_m[q]_n[c]_{m+n}},$$

$$\Psi_1(a; b; c, c'; x, y; \lambda) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[a]_{m+n}[b]_m x^m y^n q^{\lambda n(n-1)}}{[q]_m[q]_n[c]_m[c']_n},$$

$$\Xi_1(a, a'; b; c; x, y; \lambda) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[a]_m[a']_n[b]_m x^m y^n q^{\lambda n(n-1)}}{[q]_m[q]_n[c]_{m+n}}.$$

We further denote by

$$\prod\begin{bmatrix} x^{(a)}; x \\ x^{(b)} \end{bmatrix} \quad \text{or simply} \quad \prod\begin{bmatrix} (a); x \\ (b) \end{bmatrix}$$

the product $\prod_{u=0}^{\infty}\left[\dfrac{(1-x^{(a)+u})}{(1-x^{(b)+u})}\right].$

Lastly, we shall use the notation $D_{q,x}^{\beta,\lambda-1}[f(x)]$ to denote $D_q^\beta[x^{\lambda-1}f(x)]$. The following basic functions will also be used:

$$e_q(x) = \left\{\prod_{n=0}^{\infty}(1-xq^n)\right\}^{-1} = \sum_{n=0}^{\infty}\frac{x^n}{[q]_n}, \quad |x|<1;$$

$$E_q(x) = \prod_{n=0}^{\infty}(1-xq^n) = \sum_{n=0}^{\infty}\frac{(-x)^n q^{n(n-1)/2}}{[q]_n},$$

$$[x-y]_\nu = x^\nu \frac{e_q(q^\nu y/x)}{e_q(y/x)} = x^\nu \sum_{k=0}^{\infty}\frac{[-\nu]_k}{[q]_k}\left\{\frac{q^\nu y}{x}\right\}^k,$$

$$\Gamma_q(\alpha) = \frac{e_q(q^\alpha)}{e_q(q)(1-q)^{\alpha-1}} \quad (\alpha \neq 0, -1, -2, \ldots),$$

$$\frac{e_q(x)}{e_q(y)} = e_q([x-y]) = \sum_{n=0}^{\infty}\frac{[x-y]_n}{[q]_n}.$$

3. Throughout the paper we shall have occasion to operate with D_q^α on certain absolutely convergent series.

In particular, let $f(x) = \sum_{r=0}^{\infty} a_r x^{r+\mu}$ be an absolutely convergent series. Using (1.2) we have

$$D_q^\alpha f(x) = x^{\mu-\alpha}(1-q)^{-\alpha}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\frac{[-\alpha]_k}{[q]_k}a_r q^{k(\mu+r+1)}x^r.$$

If $Rl(\mu+1)>0$, then we can always interchange the order of summation in the two series on the right-hand side, since $\sum_{r=0}^{\infty} a_r x^r$ is assumed to be an absolutely convergent series:

$$D_q^\alpha f(x) = x^{\mu-\alpha}(1-q)^{-\alpha}\sum_{r=0}^{\infty}a_r x^r\sum_{k=0}^{\infty}\frac{[-\alpha]_k}{[q]_k}q^{k(\mu+r+1)}$$

$$= x^{\mu-\alpha}(1-q)^{-\alpha}\prod\begin{bmatrix}\mu-\alpha+1; q \\ \mu+1\end{bmatrix}\sum_{r=0}^{\infty}\frac{[\mu+1]_r}{[\mu-\alpha+1]_r}a_r x^r.$$

To illustrate our technique let us start with the simple relation

$$\frac{e_q(t)}{e_q(xt)} = e_q([t-xt]) = \sum_{n=0}^{\infty} \frac{t^n}{[q]_n} [1-x]_n, \qquad |t|<1,$$

or,

$$e_q(t)E_q(xt) = \sum_{n=0}^{\infty} \frac{t^n}{[q]_n} \sum_{r=0}^{n} \frac{[-n]_r}{[q]_r} (q^n x)^r.$$

Multiplying both sides by the series equivalent of $[1-y]_{-b}$ and then replacing x and y by xu and yu respectively, we get

$$e_q(t) \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{[b]_r}{[q]_r [q]_s} (q^{-b}y)^r (-xt)^s q^{s(s-1)/2} u^{r+s}$$
$$= \sum_{n=0}^{\infty} \frac{t^n}{[q]_n} \sum_{r=0}^{n} \sum_{s=0}^{\infty} \frac{[-n]_r [b]_s}{[q]_r [q]_s} (xq^n)^r (yq^{-b})^s u^{r+s}.$$

Now apply $D_{q,u}^{a-d,a-1}$, $D_{q,y}^{f-c,f-1}$ and $D_{q,x}^{f'-c',f'-1}$ successively on both sides of above; we obtain

$$(3.1) \qquad e_q(t)\Phi\left[\begin{array}{l|c} yuq^{-b};\ 0 & a \\ & b, f; f' \\ -xut;\ \frac{1}{2} & d \\ & c;\ c' \end{array}\right] = \sum_{n=0}^{\infty} \frac{t^n}{[q]_n} \Phi\left[\begin{array}{l|c} yuq^{-b} & a \\ & b, f;\ -n, f' \\ xuq^n & d \\ & c;\ c' \end{array}\right],$$

provided $|t|<1$ and $|yuq^{-b}|<1$.

The transformation (3.1) yields numerous interesting particular cases, some of which are mentioned below:

Set $f=c$ and $f'=c'$ in (3.1) to get

$$(3.2) \qquad e_q(t)\gamma_1(a; b; d; yuq^{-b}, -xut; \tfrac{1}{2})$$
$$= \sum_{n=0}^{\infty} \frac{t^n}{[q]_n} \Phi^{(1)}[a; b, -n; d;\ yuq^{-b}, xuq^n],$$

provided $|t|<1$ and $|yuq^{-b}|<1$.

Letting f, f' and $d \to \infty$ in (3.1) we get

$$(3.3) \qquad e_q(t)\Psi_1(a; b; c, c'; yuq^{-b}, -xut; \tfrac{1}{2})$$
$$= \sum_{n=0}^{\infty} \frac{t^n}{[q]_n} \Phi^{(2)}[a; b,\ -n; c, c'; yuq^{-b}, xuq^n],$$

provided $|t|<1$ and $|yuq^{-b}|<1$.

Next, letting a, c and $c' \to \infty$ in (3.1) we get

$$(3.4)\qquad e_q(t)\,\Xi_1(f, f'; b; d; yuq^{-b}, -xut; \tfrac{1}{2})$$
$$= \sum_{n=0}^{\infty} \frac{t^n}{[q]_n} \Phi^{(3)}[f, f'; -b, -n; d; yuq^{-b}, xuq^n],$$

provided $|t| < 1$ and $|yuq^{-b}| < 1$.

Next, let us take another elementary identity

$$(3.5)\qquad \frac{1}{[1-x]_b[1-xq^b]b'} = \frac{1}{[1-x]_{b+b'}}.$$

Multiplying both sides of (3.5) by the series equivalent of $[1-y]_{-c}$ and then replacing x and y by xu and yu respectively, we get

$$\sum_{r=0}^{\infty} \frac{[b]_r}{[q]_r} x^r \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{[b']_s[c]_t(xq^b)^s(yq^{-c})^t u^{r+s+t}}{[q]_s[q]_t}$$
$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{[b+b']_r[c]_s(xu)^r(yuq^{-c})^s}{[q]_r[q]_s}.$$

Apply

$$\prod_{i=1}^{A} D_{q,u}^{a_i-d_i, a_i-1}, \quad \prod_{i=1}^{E} D_{q,x}^{e_i-f_i, e_i-1} \quad \text{and} \quad \prod_{i=1}^{E_1} D_{q,y}^{e'_i-f'_i, e'_i-1}$$

successively on both sides of the above, to obtain the general expansion

$$(3.6)\qquad \Phi\left[\begin{array}{l|c} xu & (a) \\ yuq^{-c} & b+b', (e); (e'_{E_1}), c \\ & (d_A) \\ & (f_E); (f'_{E_1}) \end{array}\right]$$
$$= \sum_{r=0}^{\infty} \frac{[(a)]_r[b]_r[(e)]_r(xu)^r}{[q]_r[(d_A)]_r[(f_E)]_r} \times \Phi\left[\begin{array}{l|c} xuq^b & (a)+r \\ yuq^{-c} & b',\ (e)+r;\ c,\ (e'_{E_1}) \\ & (d_A)+r \\ & (f_E)+r; (f'_{E_1}) \end{array}\right],$$

provided $|xu| < 1$, $|yuq^{-c}| < 1$, $|xuq^b| < 1$.

(3.6) yields the following interesting special cases:

$$(3.7)\qquad \Phi^{(1)}[a; b+b', c; d; xu, yuq^{-c}]$$
$$= \sum_{r=0}^{\infty} \frac{[a]_r[b]_r(xu)^r}{[q]_r[d]_r} \Phi^{(1)}[a+r; b', c; d+r; xuq^b, yuq^{-c}]$$

(for $A=1$, $E=E_1=0$ in (3.6));

(3.8) $\Phi^{(2)}[a; b+b', c; f, f'; xu, yuq^{-c}]$

$$= \sum_{r=0}^{\infty} \frac{[a]_r [b]_r (xu)^r}{[q]_r [f]_r} \Phi^{(2)}[a+r; b', c; f+r, f'; xuq^b, yuq^{-c}]$$

(for $A=E=E_1=1$ and $d, e, e' \to \infty$ in (3.6));

(3.9) $\Phi^{(3)}[b+b', c; e, e'; d; xu, yuq^{-c}]$

$$= \sum_{r=0}^{\infty} \frac{[b]_r [e]_r (xu)^r}{[q]_r [d]_r} \Phi^{(3)}[b', c; e+r, e'; d+r; xuq^b, yuq^{-c}]$$

(for $A=E=E_1=1$ and $a, f, f' \to \infty$ in (3.6));

(3.10) $\Phi^{(4)}[a_1; a_2; f_1, f_1'; xu, yuq^{-c}]$

$$= \sum_{r=1}^{\infty} \frac{[b]_r [a_1]_r [a_2]_r (xu)^r}{[q]_r [f_1]_r} \Phi^{(4)}[a_1+r; a_2+r; f_1+r, f_1'; xuq^b, yuq^{-c}]$$

(for $A=2$, $E=1$, $E_1=2$; $f_2'=c$ and $b', e, d_1, d_2, e_1', e_2' \to \infty$ in (3.6)).

The results (3.7-10) are true under the convergence conditions $|xu|<1$, $|yuq^{-c}|<1$ and $|xuq^b|<1$.

4. In this section we shall derive some general expansions of basic double hypergeometric functions of higher order with the help of certain known expansions due to Verma [8].

First of all we start with a special case of Theorem I of Verma [8], namely

(4.1) $${}_{E+1}\Phi_{F+2}\begin{bmatrix}(e), \sigma-\alpha-\beta; x\\ (f), \sigma-\alpha, \sigma-\beta\end{bmatrix}$$

$$= \sum_{r=0}^{\infty} \frac{[(e)]_r [\alpha]_r [\beta]_r (-x)^r q^{r(r-1)/2}}{[q]_r [\sigma-\alpha]_r [\sigma-\beta]_r [\sigma+r-1]_r [(f)]_r} q^{r(\sigma-\alpha-\beta)} {}_E\Phi_{F+1}\begin{bmatrix}(e)+r; x\\ (f)+r, \sigma+2r\end{bmatrix},$$

provided $|x|<1$.

Multiply both sides of (4.1) by the series equivalent of $[1-y]_{-\lambda}$, and replace x and y by xu and yu respectively; then if we apply $\prod_{i=1}^{A} D_{q,u}^{a_i-b_i, a_i-1}$ and $\prod_{i=1}^{C} D_{q,v}^{c_i-d_i, c_i-1}$ successively, we get the general expansion

$$(4.2)\quad \Phi\left[\begin{array}{c|c} xu & (a) \\ & (e),\, \sigma-a-\beta;\, \lambda,\, (c) \\ yuq^{-\lambda} & (b_A) \\ & (f),\, \sigma-a,\, \sigma-\beta;\, (d_C) \end{array}\right] = \sum_{r=0}^{\infty} \frac{[(e)]_r [a]_r [\beta]_r [(a)]_r (-xu)^r}{[(b_A)]_r [(f)]_r [\sigma-a]_r [\sigma-\beta]_r} \times$$

$$\times \frac{q^{r(\sigma-a-\beta)+r(r-1)/2}}{[\sigma+r-1]_r [q]_r} \Phi\left[\begin{array}{c|c} xu & (a)+r \\ & (e)+r;\, \lambda,\, (c) \\ yuq^{-\lambda} & (b_A)+r \\ & \sigma+2r,\, (f)+r;\, (d_C) \end{array}\right],$$

provided $|xu| < 1$, $|yuq^{-\lambda}| < 1$.

Next, if we start with Theorem II of Verma [8], for $A = E = D = M = 1$, $F = C = H = L = N = J = B = G = I = K = 0$; $d = m = \sigma$, $a = \sigma - e$, $\lambda = 1$, $\alpha = 2$ and $\mu = e+\gamma+\gamma'-1$ in it and then sum the inner well-poised ${}_6\Phi_5$-series on the right-hand side, we get

$$(4.3)\quad \sum_{r=0}^{\infty} \frac{[\sigma-e]_r (xq^{\gamma})^r (yq^{\gamma'})^r q^{r(r+e-1)}}{[q]_r [\sigma+r-1]_r [\sigma]_{2r}}\, {}_1\Phi_1\left[\begin{matrix} e+r;\, xq^{\gamma} \\ \sigma+2r \end{matrix}\right] {}_0\Phi_1\left[\begin{matrix} -;\, yq^{\gamma'} \\ \sigma+2r \end{matrix}\right]$$

$$= \Phi\left[\begin{array}{c|c} xq^{\gamma} & e \\ & -;\, - \\ yq^{\gamma'} & \sigma \\ & -;\, e \end{array}\right],$$

provided $|xq^{\gamma}| < 1$, $|yq^{\gamma'}| < 1$.

If we replace x, y in (4.3) by $xuq^{-\gamma}$ and $yuq^{-\gamma'}$ respectively and then apply $\prod_{i=1}^{A} D_{q,u}^{a_i-b_i,a_i-1}$, $\prod_{i=1}^{C} D_{q,x}^{c_i-d_i,c_i-1}$ and $\prod_{i=1}^{F} D_{q,y}^{f_i-g_i,f_i-1}$ successively on both sides, we get

$$(4.4)\quad \Phi\left[\begin{array}{c|c} xu & e,\, (a) \\ & (c);\, (f) \\ yu & \sigma,\, (b_A) \\ & (d_C);\, e,\, (g_F) \end{array}\right] = \sum_{r=0}^{\infty} \frac{[\sigma-e]_r [(a)]_{2r} [(c)]_r [(f)]_r (xu)^r (yu)^r}{[q]_r [\sigma+r-1]_r [\sigma]_{2r} [(b_A)]_{2r} [(d_C)]_r} \times$$

$$\times \frac{q^{r(r+e-1)}}{[(g_F)]_r} \Phi\left[\begin{array}{c|c} xu & (a)+2r \\ & e+r,\, (c)+r;\, (f)+r \\ yu & (b_A)+2r \\ & \sigma+2r,\, (d_C)+r;\, \sigma+2r,\, (g_F)+r \end{array}\right],$$

provided $|xu| < 1$ and $|yu| < 1$.

(4.4) is of particular importance to us in as much as it includes a number of expansions given earlier by Jackson [5] (1). I mention here

(1) There are some misprints in certain results of Jackson [5].

some of them:

(I) $A = 0$, $C = 1$, $F = 2$; σ, $g_2 \to \infty$ and $f_2 = e$, (4.4) gives ([5]; (33)).

(II) $A = 0, C = F = 2$; $e, d_1, d_2, g_1, g_2 \to \infty$, (4.4) gives ([5], (35)).

(III) $A = 1$, $C = F = 2$; $b = e = d_1$ and σ, a, d_2, g_1, $g_2 \to \infty$, (4.4) gives ([5], (36)).

(IV) $A = 0$, $C = 1$, $F = 2$; $f_2 = e$ and d, g_1, $g_2 \to \infty$, (4.4) gives ([5], (37)).

(V) $A = C = 1$, $F = 2$; $f_2 = e$ and σ, a, d, g_1, $g_2 \to \infty$, (4.4) gives ([5], (39)).

(VI) $A = C = F = 1$; b, e, d, $g \to \infty$, (4.4) gives ([5], (41)).

(VII) $A = C = F = 1; b = d = e$ and $\sigma, g \to \infty$, (4.4) gives ([5], (42)).

(VIII) $A = C = F = 1; f = e$ and σ, b, $c \to \infty$, (4.4) gives ([5], (43)).

Next, let us start with Theorem III of Verma [8], viz.,

$$(4.5) \quad {}_{A+B}\Phi_C\left[\begin{matrix}(a), (b); \lambda x\\ (c)\end{matrix}\right] = \sum_{r=0}^{\infty} \frac{[(d)]_r[(e)]_r(-x)^r q^{r(r-1)/2}}{[q]_r[(f)]_r} \times$$
$$\times {}_{A+B+F+1}\Phi_{D+E+C}\left[\begin{matrix}(a), (b), (f), -r; \lambda q\\ (d), (e), (c)\end{matrix}\right] {}_{D+E}\Phi_F\left[\begin{matrix}(d)+r, (e)+r; x\\ (f)+r\end{matrix}\right],$$

provided $|x| < q$, $|\lambda q| < 1$.

Multiply both sides by the series equivalent of $[1-y]_{-\alpha}$, and replace x and y by xu and yu respectively; then if we apply $\prod_{i=1}^{G} D_{q,u}^{g_i-h_i,g_i-1}$ and $\prod_{i=1}^{L} D_{q,v}^{l_i-m_i,l_i-1}$ successively, we get

$$(4.6) \quad \Phi\left[\begin{matrix}\lambda xu\\ yuq^{-\alpha}\end{matrix}\left|\begin{matrix}(g)\\ (a), (b); \alpha, (l)\\ (h_G)\\ (c); (m_L)\end{matrix}\right.\right] = \sum_{r=0}^{\infty} \frac{[(d)]_r[(e)]_r[(g)]_r(-xu)^r q^{r(r-1)/2}}{[q]_r[(f)]_r[(h_G)]_r} \times$$
$$\times {}_{A+B+F+1}\Phi_{D+E+C}\left[\begin{matrix}(a), (b), (f), -r; \lambda q\\ (d), (e), (c)\end{matrix}\right] \Phi\left[\begin{matrix}xu\\ yuq^{-\alpha}\end{matrix}\left|\begin{matrix}(g)+r\\ (d)+r, (e)+r; \alpha, (l)\\ (h_G)+r\\ (f)+r; (m_L)\end{matrix}\right.\right],$$

provided $|xu| < q$, $|\lambda q| < 1$ and $|yuq^{-\alpha}| < 1$.

Again, if we consider

$${}_{A+B}\Phi_C\left[\begin{matrix}(a), (b); \lambda x\\ (c)\end{matrix}\right] = \sum_{r=0}^{\infty} \frac{[d]_r(-x)^r q^{r(r-1)/2}}{[q]_r} {}_1\Phi_0[d+r; x] \times$$
$$\times {}_{A+B+1}\Phi_{C+1}\left[\begin{matrix}(a), (b), -r; \lambda q\\ (c), d\end{matrix}\right],$$

which is (4.5) with $D = 1$, $E = F = 0$ and use the fact that

$$ {}_1\Phi_0[d+r; x] = {}_1\Phi_0[e; x]\, {}_1\Phi_0[d-e+r; xq^e] $$

we get, on applying $D_{q,x}^{\alpha-\beta,\alpha-1}$

$$ (4.7) \qquad {}_{A+B+1}\Phi_{C+1}\begin{bmatrix} a, (a), (b); \lambda x \\ \beta, (c) \end{bmatrix} = \sum_{r=0}^{\infty} \frac{[d]_r [a]_r (-x)^r q^{r(r-1)/2}}{[q]_r [\beta]_r} \times $$

$$ \times \Phi^{(1)}[a+r; e, d-e+r; \beta+r; x, xq^e]\, {}_{A+B+1}\Phi_{C+1}\begin{bmatrix} (a), (b), -r; \lambda q \\ (c), d \end{bmatrix}. $$

Multiply both sides of (4.7) by the series equivalent of $[1-y]_{-\mu}$, replace λ and y by λu and yu respectively and then apply $\prod_{i=1}^{G} D_{q,u}^{g_i-h_i,g_i-1}$ and $\prod_{i=1}^{L} D_{q,y}^{l_i-m_i,l_i-1}$ successively, to get

$$ (4.8) \qquad \Phi\left[\begin{array}{l|c} \lambda x u & (g) \\ yuq^{-\mu} & (a), (b), a; \mu, (l) \\ & (h_G) \\ & \beta, (c); (m_L) \end{array}\right] = \sum_{r=0}^{\infty} \frac{[d]_r [a]_r (-x)^r q^{r(r-1)/2}}{[q]_r [\beta]_r} \times $$

$$ \times \Phi^{(1)}[a+r; e, d-e+r; \beta+r; x, xq^e]\, \Phi\left[\begin{array}{l|c} \lambda uq & (g) \\ yuq^{-\mu} & (a), (b), -r; \mu, (l) \\ & (h_G) \\ & d, (c); (m_L) \end{array}\right], $$

provided $|\lambda uq| < 1$, $|\lambda uq^{-\mu}| < 1$, $|x| < q$, $|xq^e| < 1$.

Once again considering (4.5) with $D = E = F = 1$ and using the q-analogue of Euler's identity (Slater [7], p. 97) in the form

$$ {}_2\Phi_1\begin{bmatrix} a, b; x \\ c \end{bmatrix} = {}_1\Phi_0[a+b-c; x]\, {}_2\Phi_1\begin{bmatrix} c-a, c-b; xq^{a+b-c} \\ c \end{bmatrix} $$

to replace the ${}_2\Phi_1$-series on the right-hand side and then applying $D_{q,x}^{\alpha-\beta,\alpha-1}$ on both sides, we get another expansion, namely

$$ {}_{A+B+1}\Phi_{C+1}\begin{bmatrix} (a), (b), a; \lambda x \\ \beta, (c) \end{bmatrix} = \sum_{r=0}^{\infty} \frac{[d]_r [e]_r [a]_r (-x)^r q^{r(r-1)/2}}{[q]_r [f]_r [\beta]_r} \times $$

$$ {}_{A+B+2}\Phi_{C+2}\begin{bmatrix} (a), (b), f, -r; \lambda q \\ d, e, (c) \end{bmatrix} \Phi\left[\begin{array}{l|c} x & a+r \\ xq^{d+e-f+r} & d+e-f+r; f-d, f-e \\ & \beta+r \\ & -; f+r \end{array}\right]. $$

If we further multiply both sides of the above relation by the series equivalent of $[1-y]_{-\mu}$, replace λ, y by λu and yu respectively and then apply $\prod_{i=1}^{G} D_{q,u}^{g_i-h_i,g_i-1}$ and $\prod_{i=1}^{L} D_{q,y}^{l_i-m_i,l_i-1}$ successively, we get another more general expansion, namely

$$(4.9)\qquad \Phi\left[\begin{array}{l|c}\lambda xu & (g)\\ & (a),(b),a;\mu,(l)\\ yuq^{-\mu} & (h_G)\\ & \beta,(c);(m_L)\end{array}\right]=\sum_{r=0}^{\infty}{}'\frac{[d]_r[e]_r[a]_r(-x)^r q^{r(r-1)/2}}{[q]_r[f]_r[\beta]_r}\times$$

$$\times\Phi\left[\begin{array}{l|c}x & a+r\\ & d+e-f+r;f-d,f-e\\ xq^{d+e-f+r} & \beta+r\\ & -;f+r\end{array}\right]\Phi\left[\begin{array}{l|c}\lambda uq & (g)\\ & (a),(b),f,-r;\mu,(l)\\ yuq^{-\mu} & (h_G)\\ & d,e,(c);(m_L)\end{array}\right],$$

provided $|\lambda uq|<1$, $|x|<q$ and $|yuq^{-\mu}|<1$.

Lastly, starting with Theorem IV of Verma [8], viz.,

$${}_{A+B}\Phi_C\left[\begin{matrix}(a),(b);\lambda x\\ (c)\end{matrix}\right]=\sum_{r=0}^{\infty}\frac{[d]_r[e]_r(-x)^r q^{r(r-1)/2}}{[q]_r[f+r-1]_r}\,{}_2\Phi_1\left[\begin{matrix}d+r,e+r;x\\ f+2r\end{matrix}\right]\times$$

$$\times\,{}_{A+B+2}\Phi_{C+2}\left[\begin{matrix}(a),(b),f+r-1,-r;\lambda q\\ d,e,(c)\end{matrix}\right],$$

provided $|x|<q$, $|\lambda q|<1$ and then following the same procedure as for obtaining (4.9), we get the following result:

$$(4.10)\qquad \Phi\left[\begin{array}{l|c}\lambda xu & (g)\\ & (a),(b),a;\mu,(l)\\ yuq^{-\mu} & (h_G)\\ & (c),\beta;(m_L)\end{array}\right]=\sum_{r=0}^{\infty}\frac{[d]_r[e]_r[a]_r(-x)^r q^{r(r-1)/2}}{[q]_r[\beta]_r[f+r-1]_r}\times$$

$$\times\Phi\left[\begin{array}{l|c}x & a+r\\ & d+e-f;f-d+r,f-e+r\\ xq^{d+e-f} & \beta+r\\ & -;f+2r\end{array}\right]\times$$

$$\times\Phi\left[\begin{array}{l|c}\lambda uq & (g)\\ & (a),(b),f+r-1,-r;\mu,(l)\\ yuq^{-\mu} & (h_G)\\ & d,e,(c);(m_L)\end{array}\right],$$

provided $|x|<q$, $|xq^{d+e-f}|<1$, $|\lambda uq|<1$ and $|yuq^{-\mu}|<1$.

5. In this section we shall derive some more general expansions of basic double hypergeometric functions of higher order by starting with some other selected results.

Let us start with the relation

$$(5.1)\qquad \Phi^{(1)}[a;b,b';c;x,y]$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[a]_m[c-a]_n[b]_m[b']_n x^m(-y)^n q^{n(a+m)+n(n-1)/2}}{[q]_m[q]_n[c]_{m+n}[1-y]_{b'+n}}.$$

To prove (5.1), we use the following result:

$$ {}_2\Phi_1\left[\begin{matrix} a, b; x \\ c \end{matrix}\right] = \sum_{r=0}^{\infty} \frac{[a]_r [c-b]_r (-x)^r q^{br+r(r-1)/2}}{[q]_r [c]_r [1-x]_{a+r}}, \tag{5.2} $$

the truth of which follows as below:

The right-hand side of (5.2)

$$ = \sum_{r=0}^{\infty} \frac{[a]_r [c-b]_r (-x)^r q^{r(r-1)/2+br}}{[q]_r [c]_r} \sum_{s=0}^{\infty} \frac{[a+r]_s x^s}{[q]_s} $$

$$ = \sum_{n=0}^{\infty} \frac{[a]_n x^n}{[q]_n} \sum_{r=0}^{n} \frac{[c-b]_r [-n]_r q^{r(b+n)}}{[q]_r [c]_r}. $$

Summing the inner ${}_2\Phi_1$-series, we get the left-hand side of (5.2). We now prove (5.1).

The left-hand side of (5.1)

$$ = \sum_{m=0}^{\infty} \frac{[a]_m [b]_m x^m}{[q]_m [c]_m} \sum_{n=0}^{\infty} \frac{[a+m]_n [b']_n y^n}{[q]_n [c+m]_n}; $$

using (5.2) to the inner ${}_2\Phi_1$-series, we get (5.1).

If we replace x and y by xu and yu respectively in (5.1) and replace $\frac{1}{[1-y]_{b'+n}}$ by its series equivalent, then on applying on both sides

$$ \prod_{i=1}^{E} D_{q,u}^{d_i-e_i, d_i-1}, \quad \prod_{i=1}^{F} D_{q,x}^{f_i-g_i, f_i-1} \quad \text{and} \quad \prod_{i=1}^{K} D_{q,y}^{k_i-l_i, k_i-1} $$

successively, we get the following result:

$$ \Phi\left[\begin{array}{c|c} xu & a, (d_E) \\ yu & b, (f); b', (k) \\ & c, (e) \\ & (g_F); (l_K) \end{array}\right] = \sum_{n=0}^{\infty} \frac{[c-a]_n [b']_n [(d_E)]_n [(k)]_n (-yu)^n q^{n(n-1)/2}}{[q]_n [c]_n [(e)]_n [(l_K)]_n} \times \tag{5.3} $$

$$ \times q^{an} \Phi\left[\begin{array}{c|c} xuq^n & (d_E)+n \\ yu & a, b, (f); b'+n, (k)+n \\ & (e)+n \\ & c+n, (g_F); (l_K)+n \end{array}\right], $$

provided $|xu| < 1$ and $|yu| < 1$.

Next, consider (5.1), in the form

$$\Phi^{(1)}[a;b,b';c;x,y] = \sum_{n=0}^{\infty} \frac{[c-a]_n [b']_n (-y)^n q^{an+n(n-1)/2}}{[q]_n [c]_n [1-y]_{b'+n}} \times$$
$$\times \sum_{m=0}^{\infty} \frac{[a]_m [b]_m (xq^n)^m}{[q]_m [c+n]_m},$$

provided $|x|<1$, $|y|<1$.

Using the transformation (5.2) to replace the inner ${}_2\Phi_1$-series on the right-hand side of the above relation and replacing x, y by xu and yu respectively, we obtain, on applying $\prod_{i=1}^{E} D_{q,u}^{d_i-e_i,d_i-1}$, $\prod_{i=1}^{F} D_{q,x}^{f_i-g_i,f_i-1}$ and $\prod_{i=1}^{K} D_{q,v}^{k_i-l_i,k_i-1}$ successively, the expansion

$$(5.4) \qquad \Phi\left[\begin{array}{c|c} & a, (d_E) \\ xu & b, (f); b', (k) \\ yu & c, (e) \\ & (g_F); (l_K) \end{array}\right]$$
$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{[c-a]_{m+n} [b]_m [b']_n [(d_E)]_{m+n} [(f)]_m}{[q]_m [q]_n [c]_{m+n} [(e)]_{m+n} [(g_F)]_m} \times$$
$$\times \frac{[(k)]_n (-xu)^m (-yu)^n q^{\frac{1}{2}(m+n)(m+n+2a-1)}}{[(l_K)]_n} \times$$
$$\times \Phi\left[\begin{array}{c|c} & (d_E)+m+n \\ xuq^n & b+m, (f)+m; b'+n, (k)+n \\ yu & (e)+m+n \\ & (g_F)+m; (l_K)+n \end{array}\right],$$

provided $|xu|<1$ and $|yu|<1$.

We start again with the following relation:

$$(5.5) \qquad \Phi^{(3)}[c-a, a; b, b'; c; x, y]$$
$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{[a]_{m+n} [b]_m [b']_n (-x)^m (y)^n}{[q]_m [q]_n [c]_{m+n} [1-x]_{b+m}} \times q^{\frac{1}{2}m(m-1)+m(c-a)},$$

provided $|x| < 1$ and $|y| < 1$, the truth of which follows easily by using the result (5.2).

If we write down the series for $\dfrac{1}{[1-x]_{b+m}}$ in (5.5), replace x, y by xu and yu respectively and then apply $\prod_{i=1}^{E} D_{q,u}^{d_i-e_i,d_i-1}$, $\prod_{i=1}^{F} D_{q,x}^{f_i-g_i,f_i-1}$ and $\prod_{i=1}^{K} D_{q,v}^{k_i-l_i,k_i-1}$ successively, we get

$$(5.6)\qquad \Phi\left[\begin{array}{c|c} & (d_E) \\ xu & c-a,\, b,\, (f);\, a,\, b',\, (k) \\ yu & c,\, (e) \\ & (g_F);\, (l_K) \end{array}\right]$$

$$= \sum_{m=0}^{\infty} \frac{[a]_m [b]_m [(d_E)]_m [(f)]_m (-xu)^m}{[q]_m [c]_m [(e)]_m [(g_F)]_m} \times$$

$$\times\, q^{\frac{1}{2}m(m-1)+m(c-a)}\, \Phi\left[\begin{array}{c|c} & (d_E)+m \\ xu & b+m,\, (f)+m;\, a+m,\, (k),\, b' \\ yu & (e)+m \\ & (g_F)+m;\, (l_K),\, c+m \end{array}\right],$$

provided $|xu| < 1$ and $|yu| < 1$.

In particular, for $E = K = 0$, $F = 1$; $g = c-a$, (5.6) gives an expansion for $\Phi^{(3)}$, viz.,

$$\Phi^{(3)}[b, a; f, b'; c; xu, yu] = \sum_{m=0}^{\infty} \frac{[a]_m [b]_m [f]_m (-xu)^m q^{\frac{1}{2}m(m-1)+m(c-a)}}{[q]_m [c]_m [c-a]_m} \times$$

$$\times\, {}_2\Phi_1\left[\begin{matrix} b+m,\, f+m;\, xu \\ c-a+m \end{matrix}\right] {}_2\Phi_1\left[\begin{matrix} a+m,\, b';\, yu \\ c+m \end{matrix}\right].$$

Lastly, if we start with relation (5.5) in the form

$$\Phi^{(3)}[c-a, a; b, b'; c; x, y]$$

$$= \sum_{m=0}^{\infty} \frac{[a]_m [b]_m (-x)^m q^{\frac{1}{2}m(m-1)+m(c-a)}}{[q]_m [c]_m [1-x]_{b+m}} \sum_{n=0}^{\infty} \frac{[a+m]_n [b']_n y^n}{[q]_n [c+m]_n}$$

and use the result (5.2) to replace the inner ${}_2\Phi_1$-series on the right-hand side of the above relation, we get as before

$$
(5.7) \qquad \Phi\left[\begin{array}{c|c} & (d_E) \\ xu & c-a, b, (f);\ a, b', (k) \\ yu & c, (e) \\ & (g_F);\ (l_K) \end{array}\right]
$$

$$
= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[a]_m [c-a]_n [b]_m [b']_n [(d_E)]_{m+n}}{[q]_m [q]_n [c]_{m+n} [(e)]_{m+n}} \times
$$

$$
\times \frac{[(f)]_m [(k)]_n (-xu)^m (-yu)^n q^{\frac{1}{2}(m+n)(m+n-1)+m(c-a)+na}}{[(g_F)]_m [(l_K)]_n} \times
$$

$$
\times \Phi\left[\begin{array}{c|c} & (d_E)+m+n \\ xu & b+m, (f)+m;\ b'+n, (k)+n \\ yu & (e)+m+n \\ & (g_F)+m;\ (l_K)+n \end{array}\right],
$$

provided $|xu| < 1$ and $|yu| < 1$.

6. We conclude this paper by generalizing certain results of Carlitz [3]. Consider the transformation ([3], (4.1))

$$
(6.1) \qquad e_q(t)\, e_q(xt) = \sum_{n=0}^{\infty} \frac{t^n}{q_n} H_n(x), \qquad |t| < 1, |x| < 1,
$$

where

$$
H_n(x) = \sum_{r=0}^{n} \frac{[-n]_r}{[q]_r} x^r q^{nr-\frac{1}{2}r(r-1)}.
$$

If we multiply both sides of (6.1) by the series equivalent of $[1+uy]_m$, m being a positive integer, and then replace t by tu, we obtain

$$
\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{p=0}^{m} \frac{[-m]_p t^r (xt)^s (-yq^m)^p u^{r+s+p}}{[q]_r [q]_s [q]_p}
$$

$$
= \sum_{n=0}^{\infty} \frac{t^n}{[q]_n} H_n(x) \sum_{p=0}^{m} \frac{[-m]_p (-yq^m)^p u^{n+p}}{[q]_p}.
$$

Applying $D_{q,u}^{a-b,a-1}$ on both sides, we get

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{p=0}^{m}\frac{[-m]_p[a]_{r+s+p}(ut)^r(xut)^s(-yuq^m)^p}{[q]_r[q]_s[q]_p[b]_{r+s+p}}$$
$$=\sum_{n=0}^{\infty}\frac{(ut)^n}{[q]_n}H_n(x)\sum_{p=0}^{m}\frac{[a]_{n+p}[-m]_p(-yuq^m)^p}{[q]_p[b]_{n+p}}.$$

Replacing t and y by tq^{-a} and yq^{-a} respectively and then passing to the limit as $a\to-\infty$, we get

(6.2) $$\sum_{n=0}^{\infty}\frac{(ut)^n q^{\frac{1}{2}n(n-1)}}{[q]_n[b]_{n+m}}H_n(x)L_{m,q}^{b+n-1}(yuq^{m+n})$$
$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{p=0}^{m}\frac{[-m]_p(-ut)^r(-xut)^s(yuq^m)^p q^{\frac{1}{2}(r+s+p)(r+s+p-1)}}{[q]_m[q]_r[q]_s[q]_p[b]_{r+s+p}},$$

where

(6.3) $$L_{n,q}^{a}(x)=\prod\begin{bmatrix}a+1;q\\a+n+1\end{bmatrix}\sum_{r=0}^{n}\frac{[-n]_r x^r q^{\frac{1}{2}r(-1)}}{[q]_n[q]_r[a+1]_r}$$

is a q-analogue of the Laguerre polynomial defined by Jackson [6].

Next, let us start with the relation

(6.4) $$[1+tq]_a E_q(xt)=\sum_{n=0}^{\infty}t^n q^{\frac{1}{2}n(n+1)}L_{n,q}^{a-n}(x),$$

the truth of which follows directly as below:

The left-hand side

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{[-a]_r(-tq^{1+a})^r(-xt)^s q^{\frac{1}{2}s(s-1)}}{[q]_r[q]_s}$$
$$=\sum_{n=0}^{\infty}\frac{[-a]_n}{[q]_n}(-q^{1+a}t)^n\sum_{s=0}^{n}\frac{[-n]_s x^s q^{\frac{1}{2}s(s-1)}}{[q]_s[1+a-n]_s}$$
$$=\sum_{n=0}^{\infty}t^n q^{\frac{1}{2}n(n+1)}L_{n,q}^{a-n}(x).$$

Replace t by ut in (6.4) and multiply both sides by the series equivalent of $[1+uy]_m$, m being a positive integer:

(6.5) $$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{p=0}^{m}\frac{[-a]_r[-m]_p(-tuq^{1+a})^r(-xut)^s(-yuq^m)^p q^{\frac{1}{2}s(s-1)}}{[q]_r[q]_s[q]_p}$$
$$=\sum_{n=0}^{\infty}t^n q^{\frac{1}{2}n(n+1)}L_{n,q}^{a-n}(x)\sum_{r=0}^{m}\frac{[-m]_r(-yq^m)u^{n+r}}{[q]_r},$$

now applying $D_{q,u}^{\mu-\lambda,\mu-1}$ on both sides of (6.5), we have

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{p=0}^{m}\frac{[\mu]_{r+s+p}[-a]_r[-m]_p(-tuq^{1+a})^r(-xut)^s(-yuq^m)^p q^{\frac{1}{2}s(s-1)}}{[q]_r[q]_s[q]_p[\lambda]_{r+s+p}}$$
$$=\sum_{n=0}^{\infty}t^n q^{\frac{1}{2}n(n+1)}L_{n,q}^{a-n}(x)\sum_{r=0}^{m}\frac{[\mu]_{r+n}[-m]_r(-yuq^m)^r u^n}{[q]_r[\lambda]_{r+n}}.$$

Replacing t and y by $tq^{-\mu}$ and $yq^{-\mu}$ respectively and then passing to the limit as $\mu\to-\infty$, we get

$$\sum_{n=0}^{\infty}\frac{[q]_m(-ut)^n q^{n^2}}{[\lambda]_{m+n}}L_{n,q}^{a-n}(x)L_{m,q}^{\lambda+n-1}(yuq^{m+n})$$
$$=\lim_{\mu\to-\infty}\sum_{r=0}^{\infty}\frac{[-a]_r[\mu]_r}{[q]_r[\lambda]_r}(-utq^{1+a})^r\gamma_1(\mu+r;\ -m:\lambda+r;\ yuq^m,\ -xut;\ \tfrac{1}{2})$$
$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{p=0}^{m}\frac{[-a]_r[-m]_p(utq^{1+a})^r(xut)^s(yuq^m)^p q^{\frac{1}{2}(r+s+p)(r+s+p-1)+\frac{1}{2}s(s-1)}}{[q]_r[q]_s[q]_p[\lambda]_{r+s+p}}.$$

References

[1] R. P. Agarwal, *Certain fractional q-integrals and q-derivatives*, Proc. Cambridge Philos. Soc. 66 (1969), p. 365—370.

[2] W. Al-Salam, *Some fractional q-integrals and q-derivatives*, Proc. Edinburgh Math. Soc. 15 (1969), p. 135—140.

[3] L. Carlitz, *Some q-polynomials in two variables*, Math. Nachr. 17 (1959).

[4] W. Hahn, *Beiträge zur theorie der Heineschen Reihen*, ibidem 2 (1949), p. 340—379.

[5] F. H. Jackson, *On basic double hypergeometric functions*, Quart. Journ. Math. (Oxford) 13 (1942), p. 69—82.

[6] — *Basic double hypergeometric functions II*, ibidem 15 (1944), p. 49—61.

[7] L. J. Slater, *Generalized hypergeometric functions*, Cambridge University Press (1966).

[8] A. Verma, *Certain expansions of generalized basic hypergeometric functions*, Duke Math. Journ. 31 (1964), p. 79—90.

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY, Lucknow, India

Reçu par la Rédaction le 12. 5. 1969