On the infinite subsemigroups of a compact semigroup

by

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In this paper we are concerned with the possibility of embedding a semigroup in a compact semigroup [1].

In 1939, in his study of the representations of the Poincaré group, Wigner [28] showed that the Lorentz group had no finite dimensional unitary representations, i.e. as an abstract group, the Lorentz group cannot be found as a subgroup of any compact group. The following year, von Neumann and Wigner [25], using a technique reminiscent of that of Wigner's for the Lorentz group, showed that the group $SL(2, Q)$, where Q denotes the rationals, also has no finite dimensional unitary representations. Recalling that the closure of a group in a compact semigroup is a compact group, we obtain two examples of abstract semigroups which cannot be embedded in any compact semigroup. Another example of such a semigroup, equally as important from our standpoint, is the bicyclic semigroup [3]. Here, it should be noted, all the subgroups and, indeed, all ot the associated Schützenberger groups are trivial. Hence, the question of immersing a semigroup in a compact semigroup cannot, in general, be reduced to the corresponding problem for its subgroups.

Among the results in this paper we give a necessary and sufficient condition for a completely simple semigroup to be embedded in a compact semigroup. Also, a number of examples are given defeating a number of tempting conjectures.

Notation. [2] For a fixed group G, following von Neumann and Wigner, we denote by G_0 the set of points of G which cannot be separated from the identity by a finite dimensional unitary representation. If $G_0 = G$,

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[1] Following the usual conventions, by a semigroup we will always mean an *abstract* semigroup. By a compact semigroup we will always mean a topological semigroup whose underlying space is compact Hausdorff.
[2] The reader may well note that the material here could have been framed in categorical settings. However, the traditional and perhaps cerebral nomenclature seemed the more appropriate.

the group is said to be minimally almost periodic. If G_0 is trivial, G is called maximally almost periodic. Thus, a group is maximally periodic if, and only if, G is a subgroup of some compact group. A group is minimally almost periodic if, and only if, no nontrivial homomorphic image of G is a subgroup of a compact group.

Following P. Hall, we shall say that a semigroup enjoys a certain property $\mathfrak{F}$ *residually*, if any pair of points can be separated by a homomorphism onto a semigroup having property $\mathfrak{F}$. According to a result of Numakura [20], a compact totally disconnected semigroup is the inverse limit of finite semigroups. Thus, a semigroup is embeddable in a compact zero dimensional semigroup if, and only if, it is residually finite.

Since some of the properties of interest to us are not hereditary, it is convenient to say that a semigroup has a property $\mathfrak{F}$ *subresidually* if any pair of points can be separated by a homomorphism into some semigroup having property $\mathfrak{F}$. It might be appropriate to remark that a maximally almost periodic semigroup, i.e. one which has enough finite dimensional unitary representations to separate points, is certainly subresidually compact. However, a subresidually compact semigroup may have only degenerate homomorphisms into compact groups.

Compactifications. A number of properties can be more conveniently treated by the use of compactifications. So, following Holm [12], we regard the Bohr compactification of a semigroup S as a pair $(\delta, \hat{S})$ where δ is a dense homomorphism of S into the compact semigroup $\hat{S}$ with the property that the diagram

$$
\begin{array}{c}
\hat{S} \\
\delta \uparrow \\
S \xrightarrow{\gamma} T
\end{array}
$$

where γ is a dense homomorphism of S into the compact semigroup T, completes to the diagram

$$
\begin{array}{c}
\hat{S} \\
\delta \uparrow \quad \searrow^{\xi} \\
S \xrightarrow{\gamma} T
\end{array}
$$

where ξ is a continuous homomorphism of $\hat{S}$ onto T.

As Holm has shown, $\hat{S}$ may be viewed as the separated completion of S with respect to the finest uniform structure $\mathfrak{U}$ which is precompact, compatible with the operation(s) in S, and defines a topology coarser than the initial topology. It might be remarked that if G is a group the uniform structure $\mathfrak{U}$ need only be taken compatible with the multiplica-

tion in $\hat{G}$. For in this case $\hat{G}$ will, at any rate, be a compact semigroup containing a dense subgroup. It follows that $\hat{G}$ is, in fact, a compact (topological) group.

Now an abelian group, since it has enough characters, goes faithfully into its Bohr compactification. For an abelian group A the group $\hat{A}$ can be calculated as $\mathrm{ch}\{(\mathrm{ch}\,A)_{\text{discrete}}\}$.

PROPOSITION 1. *A semigroup S is residually finite if and only if no two distinct points of S are sent, under δ, into the same component of $\hat{S}$.*

Let x and y be distinct points of S such that $\delta(x)$ does not belong to the component of $\delta(y)$. Form the decomposition of $\hat{S}$ which shrinks each component to a point. The hyperspace $\hat{S}/\mathcal{C}$ will be a totally disconnected compact semigroup. By the result of Numakura [20], $\hat{S}/\mathcal{C}$ is residually finite.

On the other hand, if $\delta(x)$ belongs to the component of $\delta(y)$ then clearly x cannot be separated from y by a homomorphism onto a finite semigroup, since such a homomorphism would have to factor through $\hat{S}$ by δ.

Let G be a group. Define the subgroup G_{00} as the set of all elements which cannot be separated from the identity by a homomorphism onto some finite semigroup. Now the groups G_0 and G_{00} are both normal. It follows readily that G/G_0 is maximally almost periodic and that G/G_{00} is residually finite. In general, of course, $G_0 \subseteq G_{00}$. The inequality may be proper as, for example, the group $G = Z(p^\infty)$. Here $G_{00} = G$ while $G_0 = \{1\}$.

We do have the following observation however;

PROPOSITION 2. *Let G be a finitely generated group. Then*

$$ G_0 = G_{00} . $$

For let x be an element of G_{00} and suppose that x is not in G_0 so that G/N_x can be embedded in a compact group where N_x is a normal subgroup missing x. By the Peter–Weyl theorem, the group G/N_x is seen to be residually a matrix group. Since Malcev [17] has shown that a finitely generated matrix group is residually finite, the result follows.

In [16], Maak extends almost periodic compactifications to semigroups by proving that an almost periodic function on a semigroup can be approximated appropriately by unitary representations. Thus the Maak compactification of a semigroup S clearly will not often coincide with the Bohr compactification since the former is always a group. Let S_M be the quotient semigroup obtained from S by identifying points which cannot be separated by a homomorphism into some unitary group. Then $\hat{S}_M$, compactification of S_M, is the Maak compactification of S and the diagram

$$
\begin{array}{c}
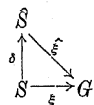\hat{S}_M \\
\uparrow \\
S \longrightarrow G
\end{array}
$$

where G is a compact group, completes to the diagram

$$\begin{array}{ccc} & \hat{S}_M & \\ & \uparrow & \nwarrow \\ & S & \rightarrow G \end{array}$$
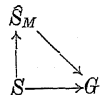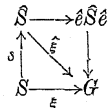
Following Thierrin [23] we shall call a semigroup S a *homogroup* if it has an ideal K which is a subgroup. It is to be noted that K is necessarily a minimal ideal. A compact semigroup will become a homogroup if, for example, its idempotents commute. If S is a homogroup and e the identity of its minimal ideal K then $x \rightarrow xe$ is a retracting homomorphism of S onto K.

PROPOSITION 3. *Let S have the property that if $f: S \rightarrow T$ is a homomorphism into a compact semigroup T then the closure of $f(S)$ is a homogroup. (This is the case, for example, if S is abelian, normal, or is itself a homogroup.) Then the Maak compactification of S is the minimal ideal of the Bohr compactification.*

Proof. Let $K(\hat{S})$ be the minimal ideal of $\hat{S}$. Now $K(\hat{S})$ is of the form $\hat{e}\hat{S}\hat{e}$ where $\hat{e}$ is the identity of $K(\hat{S})$. If $\xi: S \rightarrow G$ is a dense representation of S into the compact group G, there is a diagram

$$\begin{array}{ccc} & \hat{S} & \\ \delta \uparrow & \searrow^{\hat{\xi}} & \\ & S \xrightarrow{\xi} & G \end{array}$$

But this diagram extends to

$$\begin{array}{ccc} \hat{S} & \longrightarrow & \hat{e}\hat{S}\hat{e} \\ \delta\uparrow \nearrow^{\hat{\xi}} & & \downarrow \\ S & \xrightarrow{\xi} & G \end{array}$$

This shows that $\hat{e}\hat{S}\hat{e}$ is the Maak compactification.

We recall that a semigroup S has a group G as maximal group image if any homomorphism of S onto a group T factors through G [6]:

$$\begin{array}{ccc} & G & \\ & \uparrow & \searrow \\ & S & \longrightarrow T \end{array}$$

Of course, a semigroup need not have a maximal group image but a reasonably wide class of semigroups such as completely simple semigroups do have such images.
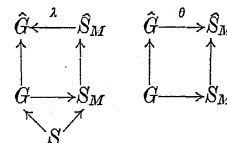
It is natural to expect that for certain semigroups the Maak compactification, being a group, can be given as the Bohr compactification of the maximal group image. Now if a semigroup S were to have the above property then any homomorphic copy $f(S)$ in a compact group would have to be a subgroup. To see this, note that $f = \alpha\gamma\beta$ in the diagram where T is a compact group, and G the maximal group image.

$$\begin{array}{ccc} & \hat{G} & \\ \gamma\uparrow & & \searrow^{\alpha} \\ G & & \\ \beta\uparrow & & \\ S & \xrightarrow{f} & T \end{array}$$
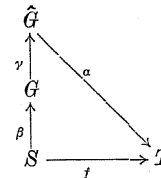
PROPOSITION 4. *Let S be semigroup with the property that any homomorphic copy of S embeddable in a compact group is a group. Then if S has a maximal group image G, we have*

$$\hat{G} = \hat{S}_M.$$

As above, we let S_M be the quotient semigroup obtained from S by identifying points which cannot be separated by a homomorphism into some unitary group. We now have two diagrams, the unlabeled maps being natural.

$$\begin{array}{ccc} \hat{G} \xleftarrow{\lambda} \hat{S}_M & & \hat{G} \xrightarrow{\theta} \hat{S}_M \\ \uparrow \qquad \uparrow & & \uparrow \qquad \uparrow \\ G \longrightarrow S_M & & G \longrightarrow S_M \\ \nwarrow \;\; \nearrow & & \\ S & & \end{array}$$

The usual diagram arguments show that $\theta\lambda$ leaves the image of S pointwise fixed in $\hat{S}_M$ so that θ is the inverse of the topological isomorphism λ.

As is well known, any abelian group can be embedded in a compact group. As we shall see, the abelian law is the only law which can be satisfied by certain maximally almost periodic groups.

For a discussion of laws in groups we refer the reader to [19]. An analogous discussion can appropriately be carried out for semigroups. The nilpotent laws L_c can be given in a semigroup setting, L_1 being the abelian law, L_2 being $xyzyx = yxzxy$ and so on (see [19]).

PROPOSITION 5. *Let the group G satisfy some non-trivial law and let $\xi: G \rightarrow C$ be a dense representation of G into the compact group C. Then C_1, the component of the identity of C, is abelian and*

$$\xi(G_{00}) \subseteq C_1.$$

Proof. If L is a law satisfied by G then L is satisfied by $\xi(G)$ and, by continuity, also by C. Now if C_1 were not abelian it would contain a free group on two generators [5]. Hence, C_1 must be abelian. Finally if $x \in G_{00}$ and $\xi(x) \notin C_1$ then the quotient group C/C_1 would be a zero dimensional compact group and there would be a homomorphism of G onto a finite separating x from the identity.

COROLLARY. *Let the group G satisfy some non-trivial law and suppose, moreover, that G has no normal subgroup of finite index. Then any compact group containing a dense homomorphic image of G, must be connected and abelian. Moreover, if G' is the derived group of G then*

$$\widehat{G/G'} = \hat{G} \, .$$

Proof. It is clear that $\delta(G)$ is abelian so that

$$\mathrm{Ker}\, \delta \supseteq G' .$$

Now if x were a point of $\mathrm{Ker}\, \delta$ and $x \notin G'$ then x could be sent to a point different from the identity in the compact group $\mathrm{ch}\{\mathrm{ch}(G/G')_d\}$. But this would yield a homomorphism into a compact group which could not be factored through $\hat{G}$.

Apparently, if G satisfies some law and has no finite homomorphic image then any homomorphism of G into a compact group can be factored through G/G'. Moreover, since G/G' is abelian and itself has no finite homomorphic image, it must be a divisible group. Hence the Bohr compactification of G will be the Bohr compactification of the direct sum of rationals and groups $Z(p^\infty)$. That is to say,

PROPOSITION 6. *If the group G satisfies some non-trivial law and has no finite homomorphic images then G/G' is the direct sum of rational groups and groups $Z(p^\infty)$. Moreover, the Bohr compactification of G is that the group G/G'.*

In particular, if G is a non-abelian nilpotent divisible group then G is not maximally almost periodic. Of course, a nilpotent group cannot be minimally almost periodic. For example, the last term but one of the upper central series yields an abelian factor group.

As an application of these results let us consider the case of a nilpotent semigroup: That is to say, a semigroup satisfying the same nilpotent law as given by Neumann [19]. Now if T is a compact nilpotent semigroup it follows readily that T is a homogroup. In effect, let K be the minimal ideal of T. Now it follows readily from the structure theory of K that for $x, y \in K$ the sets xTx and yTy either coincide or have no common part. (They are in fact the $\mathcal{H}$-classes of x and y respectively). A glance at the nature of the nilpotent laws in question completes the argument. Thus, if T is a nilpotent semigroup, its Bohr compactification,

$\hat{T}$, is a homogroup whose minimal ideal $K(\hat{T})$ is a nilpotent group which is, in turn, the Maak compactification of T by Proposition 2. If, moreover, T has no finite image then $K(\hat{T})$ is a compact connected nilpotent group and is therefore abelian and divisible. *Thus the Maak compactification of a divisible nilpotent semigroup is the compactification of a group which is at worst the cartesian product of rationals and groups $Z(p^\infty)$.*

Simple semigroups. We recall, at this time, the equivalences of Green: As is now customary, for a semigroup S let S^1 denote S if S has an identity and S with identity adjoined if S has no identity

$$
\begin{aligned}
a &\equiv b\,(\mathcal{L})\,, & S^1 a &= S^1 b\,, \\
a &\equiv b\,(\mathcal{R})\,, & a S^1 &= b S^1\,, \\
a &\equiv b\,(\mathcal{J})\,, & S^1 a S^1 &= S^1 b S^1\,, \\
\mathcal{H} &= \mathcal{L} \cap \mathcal{R}\,, & \mathcal{D} &= \mathcal{L} \circ \mathcal{R}\,(= \mathcal{R} \circ \mathcal{L})\,.
\end{aligned}
$$

The notion of stability has been introduced in [14] as an algebraic analogue of compactness. It will be particularly useful in our context. A semigroup S is called *stable* if for any $a, b \in S$ one has

(1) $$Sa \subseteq Sab \rightarrow Sa = Sab\,,$$

(2) $$aS \subseteq baS \rightarrow aS = baS\,.$$

Among the most natural semigroups to be considered in terms of residual properties are perhaps the simple semigroups. As we shall see, this question gives rise to two distinct problems in the case of stability. The completely simple case will be considered separately. The second case, dealing with $\mathcal{D}$-triviality seems to be more formidable.

A semigroup is called *completely simple* if it is simple (has no proper ideal) and contains a primitive idempotent (the idempotent e is primitive is eSe has no idempotent save e). This class of semigroups is rather well known (see [6]).

PROPOSITION 7. *Let S be a simple subsemigroup of a stable semigroup T. If S contains an idempotent, it is completely simple. If S contains no idempotent, the $\mathcal{D}$-classes of S are degenerate.*

Proof. If S contains an idempotent which is not primitive, then S contains a copy of the bicyclic semigroup. But this is impossible since the bicyclic semigroup admits no stable embeddings. Now suppose S contains no idempotents and a non-degenerate $\mathcal{D}$-class. Since S is simple, it is entirely contained in some $\mathcal{J}$-class of T. From the fact that T is stable we know that this $\mathcal{J}$-class is necessarily a $\mathcal{D}$-class, D, of T and D is regular. Now S contains a non-trivial $\mathcal{D}$-class, hence S must contain a non-trivial $\mathcal{L}$- or $\mathcal{R}$-class. Assuming the latter we have that there is

an element $a \in S$ such that $a \in aS = aS^2$, i.e., $a = abb'$ for some $b, b' \in S$. For $t \in T$, let R_t and L_t denote the $\mathfrak{L}$- and $\mathfrak{R}$-classes in T which contain t. Since T is stable and a, b and ab belong to D we have $ab \in L_b \cap R_a$ [3]. Thus by Green's theorem [10] we have $(L_{ab})b' = (L_b)b' = L_a$ and so $bb' \in L_a$. Now if $x \in L_a$, then $x = ta$ for some $t \in T^1$ so $xbb' = tabb' = ta = x$. Thus we have $(bb')bb' = bb'$ which is impossible since S contains no idempotent.

COROLLARY. *Let S be a left simple semigroup having no proper homomorphic images. Then either S is a group or S has no idempotent and no non-constant homomorphism into a stable semigroup. In particular, $\hat{S}$ is degenerate.*

Proof. If S has an idempotent then S is the direct product of a group and a left zero semigroup. The latter is therefore trivial. If S has no idempotent then there remains only to show that S cannot be embedded in a stable semigroup. By Proposition 2 if S had such an embedding $\mathfrak{D}$ would be trivial. The existence of semigroups which are left simple and have no proper homomorphic images has been well established (see for example [6]).

The Baer–Levi semigroups which are right cancellative and right simple also furnish examples. Specifically let S be the set of all one-to-one mapping α of X into itself such that $X \setminus \alpha(X)$ is infinite and X is countable. (See [24].) Later on we shall construct a $\mathfrak{D}$-trivial simple semigroup which is a subsemigroup of a compact group. At this point however let us note the following.

An unstable subsemigroup of a zero dimensional compact group. As Green [10] has remarked, the semigroup S generated by a, b, c, d, x, y subject to $axb = y$ and $cyd = x$ is such that $x \equiv y(\mathfrak{J})$ but $x \not\equiv y(\mathfrak{D})$. There is a natural image of S in the group generated by the same generators and relations. The latter, however, is free since we may eliminate successively as follows:

$$\langle a, b, c, d, x, y \colon axb = y,\ cyd = x \rangle, \quad \langle a, b, c, d, x \colon caxbd = x \rangle,$$

$$\langle a, b, c, d, x \colon c = xd^{-1}b^{-1}x^{-1}a^{-1} \rangle, \quad \langle a, b, d, x \colon \rangle.$$

The image of S is thus in a free group and is, therefore, residually finite.

The bicyclic semigroup furnishes us with a $\mathfrak{D}$-simple semigroup, having degenerate subgroups, and which is not residually finite or even residually stable. However, even in the case of a completely simple semigroup with finite structural group one need not have residual finiteness.

We now fix our attention on completely simple semigroups. We recall again a number of pertinent facts: A completely simple semigroup S can be characterized as a four tuple (X, Y, G, p). Here X and Y are respectively left and right trivial semigroups, G is a group and p is

a function from $X \times Y$ to G. The multiplication is given by $(x, y, g)(\bar{x}, \bar{y}, \bar{g}) = (x, \bar{y}, gp(\bar{x}, y)\bar{g})$. The sets X and Y are, in point of fact, the quotient semigroups $S/\mathfrak{L}$ and $S/\mathfrak{R}$. The structural group G is taken as H_e for any idempotent e. These are all mutually isomorphic. It is well known, and not difficult to see, that a homomorphic image of a completely simple semigroup is again completely simple and a compact semigroup which contains a dense completely simple subsemigroup is again completely simple. The above characterization of a completely simple semigroup carries over, in the appropriate manner, to the compact case (Wallace [27]). The group H_e in this case is compact, the quotient semigroups $S/\mathfrak{L}$ and $S/\mathfrak{R}$ are compact and the sandwich function as it were, $p \colon S/\mathfrak{L} \times S/\mathfrak{R} \to H_e$ is continuous. One further item of importance is that the canonical mappings $S \to S/\mathfrak{L}$ and $S \to S/\mathfrak{R}$ are open. (This can be noted by an analysis of the decomposition.) In fact, if D is a $\mathfrak{D}$-class of a compact semigroup then the canonical maps $D \to D/\mathfrak{L}$ and $D \to D/\mathfrak{R}$ are open [2].

Let us consider the nature of the Bohr compactification of a completely simple semigroup. First of all, we assert that if (X', Y', G', p') is the representation according to Rees of $\hat{S}$ as a four-tuple, then we must have $X' = \beta(X)$ and $Y' = \beta(Y)$. (Here, $\beta(T)$ is the Stone–Čech compactification of T.) To see this note that $S/\mathfrak{L}$ is a left trivial semigroup so that any continuous mapping f of $S/\mathfrak{L}$ to any compact space T can be viewed as a continuous homomorphism by endowing T with the left trivial multiplication. By the nature of the Bohr compactification, the following diagram commutes.

$$
\begin{array}{ccc}
\hat{S} & \longrightarrow & \hat{S}/\mathfrak{L} \\
\uparrow & \searrow & \downarrow \\
S & \longrightarrow S/\mathfrak{L} \longrightarrow & T
\end{array}
$$

Thus $\widehat{S/\mathfrak{L}}$ is, by uniqueness, $\beta(X) = \beta(S/\mathfrak{L})$. Likewise $\beta(Y) = \beta(S/\mathfrak{R}) = \widehat{S/\mathfrak{R}}$. Furthermore G' is a compact group having a dense continuous homomorphic image of G. Now G' is, of course, a homomorphic image of $\hat{G}$ and it may be proper.

Let us consider the following question: Given a dense representation $G \to G'$, when can the sandwich function p be "extended" to $\beta(X) \times \beta(Y)$? This, of course, includes the question of when the subresidual compactness of G extends to that of S. Otherwise said, when can one complete the diagram below?

$$
\begin{array}{ccc}
(\beta X) \times \beta(Y) & & G' \\
i_1 \uparrow \qquad \uparrow i_2 & & \nearrow i_3 \\
X \quad \times \quad Y & \xrightarrow{\ p\ } & G
\end{array}
$$

where i_1 and i_2 are canonical maps. To this end, we establish the following proposition which will give us a criterion for the subresidual compactness of certain completely simple semigroups.

LEMMA. *Let X, Y be complete regular T_2, C compact T_2, $f: X \times Y \to C$ continuous. Then f has a continuous extension to $\beta X \times \beta Y$ iff V, an entourage on C, implies there are open sets $U_1, ..., U_n$ in X and open sets $V_1, ..., V_n$ in Y such that $\{U_i \times V_i\}$ covers $X \times Y$ and if x and y are in $U_i \times V_i$ $(i = 1, ..., n)$ then $(f(x), f(y)) \in V$.*

Proof. If f has a continuous extension, say $\hat{f}$, to $\beta X \times \beta Y$ then $\hat{f}$ is uniformly continuous and the condition follows. To prove the condition is also sufficient, recall the theorem of Frolík [9]:

$f: X \times Y \to I \ (= [0, 1])$ has a continuous extension to $\beta X \times \beta Y$ iff $\varepsilon > 0$ implies there is $U_1 \times V_1, U_2 \times V_2, ..., U_n \times V_n$ a cover of $X \times Y$ (U_i open in X, V_i open in Y) such that $x, y \in U_i \times V_i$ implies $|f(x) - f(y)| < \varepsilon$. Now let $g: C \to I$ be any continuous function. We will show that $g \circ f$ satisfies the condition of Frolík, hence has an extension $g \circ f$, to $\beta X \times \beta Y$. Let $\varepsilon > 0$, then since g is uniformly continuous, there is an entourage V in C such that $(x, y) \in V$ implies $|g(x) - g(y)| < e$. Now for V we have a cover $\{U_i \times V_i\}$ of $X \times Y$ with open sets such that $x, y \in U_i \times V_i$ implies $(f(x), f(y)) \in V$ thus $|g \circ f(x) - g \circ f(y)| < \varepsilon$. Now let $e: C \to I^{C(C, I)}$ be the evaluation map and define

$$\hat{f}: \beta X \times \beta Y \to e(C) \quad \text{by} \quad \hat{f}(x)(g) = g \circ f(x).$$

Clearly $\hat{f}$ is continuous since $g \circ f$ is continuous and for $x \in X \times Y$, $f(x)(g) = g \circ \hat{f}(x) = g \circ f(x) = f(x)(g)$ hence $\hat{f}|_{X \times Y} = e \circ f$.

We shall say that the matrix function p of a completely simple semigroup $S = (X, Y, G, p)$ satisfies the condition of Frolík if δP satisfies the conditions of the previous lemma, with δ one to one.

PROPOSITION 8. *A completely simple semigroup (X, Y, G, p) is subresidually compact if and only if it satisfies the condition of Frolík.*

Now from the above it follows, as in the examples, that there exist completely simple semigroups having finite structural groups which are not residually finite, and indeed not subresidually compact.

The following gives at least one condition which suffices to imply residual finiteness for such semigroups.

PROPOSITION 9. *Let S be a completely 0-simple semigroup. If the structural group eSe, $e^2 = e$, is residually finite and if $S/\mathcal{L}$ (or $S/\mathcal{R}$) is finite then S is residually finite.*

Proof. Suppose $x, y \in S$ and x and y do not belong to the same maximal subgroup of S. Then x is not $\mathcal{R}$-equivalent to y or x is not $\mathcal{L}$-equivalent to y. Suppose the former case obtains. The natural mapping of S onto $S/\mathcal{R}$ is a homomorphism and $S/\mathcal{R}$ is a left trivial semigroup.

Now consider the two point set $\{0, 1\}$ endowed with left trivial multiplication. Map $S/\mathcal{R}$ onto $\{0, 1\}$ by $R_x \to 1$, $R_z \to 0$, $R_x \neq R_z$. This is clearly a homomorphism distinguishing R_x and R_y. Now the composition maps S homomorphically onto $\{0, 1\}$ distinguishing x and y. Now suppose x and y are contained in the same maximal subgroup G of S. First, recall that G is isomorphic to eSe and S is isomorphic to $G^0 \times I \times \Lambda$ where multiplication is given by $(g, i, \lambda)(g', i', \lambda') = (gp(i', \lambda)g', i, \lambda')$ where $p: I \times \Lambda \to G^0$ is a function and I is $S/\mathcal{R}$, Λ is $S/\mathcal{L}$. Since G is residually finite, there is a homomorphism $\Pi: G \to F$ where F is a finite group and $\Pi(x) \neq \Pi(y)$. Now let $q: I \times \Lambda \to F^0$ be given by $q = \Pi^0 p$ then $F^0 \times I \times \Lambda$ is a completely simple semigroup where multiplication is given by $(f, i, \lambda)(f', i', \lambda') = (fq(i', \lambda)f', i, \lambda')$ and the function $\overline{\Pi}^0: G \times I \times \lambda \to F^0 \times I \times \lambda$ defined by $\Pi^0(g, i, \lambda) = (\Pi^0(g), i, \lambda)$ is a homomorphism such that $\overline{\Pi}^0(x) \neq \Pi^0(g)$. We now define a congruence, $\sim$, upon $F^0 \times I \times \Lambda$ by $(f, i, \lambda) \sim (f', i', \lambda')$ if $f = f'$, $i = i'$, and $q(i, \lambda) = q(i', \lambda')$ for all $i \in I$. By a result of Tamura [22] $\sim$ is a congruence. One easily sees that $(F^0 \times I \times \Lambda)/\sim$ is a finite semigroup distinguishing x and y.

PROPOSITION 10. *Let S be a regular semigroup such that each $\mathcal{D}$-class has only finitely many $\mathcal{H}$-classes. If each maximal subgroup of S is residually finite then S is residually finite.*

Proof. Since S is regular, we know from Theorem 3.21 of [6] that the direct sum of all of the Schützenberger representations and their duals yields a faithful representation of S. Thus, if x and y are distinct elements of S there is a $\mathcal{D}$-class, D, such that either the Schützenberger representation $s \to M_D(s)$ or its dual, $s \to M_D^*(s)$, separates x from y. We may assume then, without loss, that $M_D(x) \neq M_D(y)$. We recall from section 3.5 of [6] that the representation, $s \to M_D(s)$, is by matrices over $\Gamma(H)^0$ — the Schützenberger group of any $\mathcal{H}$-class of D with a zero element adjoined. Since, by hypothesis, D has only finitely $\mathcal{H}$-classes, it follows, from the construction of these representations, that the matrices in question are n by n, i.e. finite. Moreover, since S is regular, $\Gamma(H)$ is isomorphic to some maximal subgroup H_e contained in the $\mathcal{D}$-class D. By hypothesis then, $\Gamma(H) \cong H_e$ is residually finite. Thus, any two points of $\Gamma(H)$ can be separated by some homomorphism onto some finite group G. Now if f is a homomorphism from $\Gamma(H)$ onto G then f induces a homomorphism from any semigroup of matrices over $\Gamma(H)^0$ to a semigroup of matrices over G^0. (The group G with a zero adjoined.) Thus, f induces a homomorphism from $M_D(S)$, which is a semigroup of n by n matrices over $\Gamma(H)^0$, to a semigroup of n by n matrices over G^0.

In particular, the element $M_D(x)$ and $M_D(y)$ of $M_D(S)$ can be separated by some homomorphism into a finite semigroup — a semigroup of n by n matrices over a finite group with zero.

As shown earlier, the question of a simple semigroup S which is subresidually compact has two parts. If S has an idempotent then it is already completely simple and so Proposition 9 applies. If, on the other hand, S has no idempotent, it may or may not be subresidually compact. A simple semigroup which is cancellative, without idempotent, and $\mathfrak{D}$-trivial which is not subresidually compact is given in Example 6. That a simple semigroup which is $\mathfrak{D}$-trivial and without idempotent can be a subsemigroup of a compact semigroup is given by the following:

PROPOSITION 11. *Any compact connected non-abelian group contains a subsemigroup which is simple and without idempotent. (It is necessarily $\mathfrak{D}$-trivial.)*

We shall make use of a construction due originally to Croisot. Let S be a free semigroup generated by $s_1, s_2, s_3, \ldots$, and let X be the free semigroup generated by a_{ij} and b_{ij} where i and j run over the positive integers. Let S' be the free product of S and X subject to the relations $s_i = a_{ij} s_j b_{ij}$. Now let F be the group generated by these same generators and relations. It follows readily that F is a free group on the generators a_{ij}, b_{ij}, s_1. Moreover, S is embedded in F under $a_{ij} \to a_{ij}$, $b_{ij} \to b_{ij}$ and $s_j \to a_{j1} s_1 b_{j1}$.

We now define S_n inductively by

$$S_1 = S ,$$
$$S_2 = S_1' ,$$
$$\cdots\cdots\cdots$$
$$S_n = S_{n-1}' .$$

Finally, we set $T = \bigcup_1^\infty S_n$. As Croisot has noted, T is cancellative, countable, simple, $\mathfrak{D}$-trivial, and without idempotent. From the nature of the construction, each S_n is embedded in the appropriate free group F_n in the same way that S was embedded in F. Just as $S_1 \subset S_2 \subset S_3 \subset \ldots$ so is $F_1 \subset F_2 \subset F_3 \ldots$ It follows that T is embedded in a free group on a (countably) infinite set of generators. Balcerzyk and Mycielski [5] have shown that any compact connected non-abelian group contains a copy of a free group on $2^{\aleph_0}$ generators.

From the proof of Proposition 11 one can conclude that every compact connected non-abelian group contains a copy of a particular simple semigroup without idempotent, namely T.

Example 2 provides an idempotent semigroup which is not subresidually compact. It is easy to see than an idempotent abelian semigroup is embeddable in a cartesian product of two element semigroups.

Lallement [15] has considered the class of regular semigroups subject to the conditions that each $\mathfrak{D}$-class be a completely simple subsemigroup and that for idempotents e, f, g, the conditions $f \leqslant e$, $g \leqslant e$ and $f \mathfrak{D} g$

imply $f = g$. Howie [13], has noted that the second condition imposed on the class of idempotent semigroups yields a variety. Now the law Howie uses is in fact equivalent to the law

$$xyzx = xzyx .$$

An idempotent semigroup belonging to this variety is called naturally ordered. In this context one should see Yamada [29], who earlier considered this class of idempotent semigroups.

PROPOSITION 12. *A naturally ordered idempotent semigroup is residually finite.*

Since $S/\mathfrak{D}$ is a semi-lattice, and since a semi-lattice is well known to be residually finite we may assume that the points to be separated by a finite congruence, say x and y, lie in the same $\mathfrak{D}$-class. (In this situation $\mathfrak{D} = \mathfrak{J}$.) We then have homomorphisms

$$S \xrightarrow{\alpha} S/\mathfrak{D} \xrightarrow{\beta} T' \xrightarrow{\gamma} T''$$

where α is the natural homomorphism. Let the homomorphism β be defined by the retracting endomorphism of $S/\mathfrak{D}$ which is multiplication by $\alpha(x)$. Thus $\beta(q) = \alpha(x) q = \alpha(y) q$. Now the set

$$\alpha(x)(S/\mathfrak{D}) \backslash \{\alpha(x)\}$$

is an ideal Q of $S/\mathfrak{D}$. Hence we may form the Rees quotient by S by the ideal

$$\alpha^{-1}\beta^{-1}(Q) .$$

Denote this Rees quotient by S' and let $s \to s'$ the natural homomorphism. Note that D_x is sent faithfully into S'. Now if s_0' is any non-zero element of S' we must have

$$s_0' D_x \cup D_{x'} s_0' \subseteq D_{x'} ,$$

and $D_{s_0'}$ may be identified with D_{s_0}. Now if $D_{s'}$ is an arbitrary non-zero $\mathfrak{D}$-class of S' we map $D_{s'}$ into $D_{x'}$ by sending $p \in D_{s'}$ to that unique element t in $D_{x'}$ such that $t \leqslant p$. This mapping yields an endomorphism since $p \to t$, $q \to y$ implies $t \leqslant p$, $y \leqslant q$ which gives $xy \leqslant pq$ so $pq \to xy$. Thus S is residually a rectangular band with zero (= a completely simple idempotent semigroup with zero). Hence S is residually finite.

Indeed S is residual with respect to semigroups of orders 2 and 3, where the former is with zero and identity the latter left or right trivial with zero.

Let us note now that some condition is necessary for a completely simple semigroup to be residually finite even though one keeps a finite structural group.

EXAMPLE 1. There exists a completely simple semigroup whose structural group is finite but which is not itself residually finite. Let G be a non-trivial finite group and let $I = \Lambda$ be the set of positive integers. Define the function

$$p\colon I \times \Lambda \to G$$

subject to $p(1, \lambda) = p(i, 1) = e =$ the identity of G, $p(n, n) = e$, $p(m, n) \neq e$ for $m \neq 1$, $n \neq 1$, $m \neq n$.

For example, if $G = Z_2 = \{x, e\}$ the matrix function could be viewed as the matrix

$$
\begin{array}{lllll}
e & e & e & e & e \ldots \\[4pt]
e & e & x & x & x \ldots \\[4pt]
e & x & e & x & x \ldots \\[4pt]
e & x & x & e & x \ldots \\[4pt]
e & x & x & x & e \ldots \\[4pt]
e & x & x & x & x & e \\
\end{array}
$$

$$\vdots$$

Now if $f\colon (G, I, \Lambda, p) \to (G^*, I^*, \Lambda^*, p^*)$ is a homomorphism onto and G^* is not trivial, we must have $G \cong G^*$ since G is simple. However, in this case we would also have $I = I^*$ and $\Lambda = \Lambda^*$ since no two rows coincide and no two columns coincide.

In point of fact, a semigroup such as the above with say $I = \Lambda =$ the positive integers, p as before and $G = Z_2$, is not even subresidually compact. Indeed, δ is constant on each $\mathcal{H}$-class.

EXAMPLE 2. Šain [21], has constructed the following semigroup which is idempotent and is not residually finite. We shall note that this semigroup is not even subresidually compact. The semigroup is as follows: Within the full transformation semigroup on the positive integers, let A be composed of those mappings which are constant; $a_i(x) = i$ all x. Let B be the set of all mappings b such that $b(1) = 1$, $b(2) = 2$ and $b(n) \in \{1, 2\}$ for all $n = 3, 4, \ldots$ Let $C = A \cup B$ be the desired semigroup. We need first, the following

LEMMA. *Let T be a compact semigroup and let A_i and B_i where $i = 1, 2, 3, \ldots$ be compact subsets of T such that $A_i \supset A_{i+1}$ and $B_i \supset B_{i+1}$ for all i. If for each i the product $A_i B_i$ contains X, $X \subset T$, then AB contains X where A is the common part of the A_i and B is the common part of the B_i.*

As a corollary to the above, we note that the semigroup of Šain is not a subsemigroup of a compact semigroup. In effect, using the same notation as before, let X consist of the elements a_1 and a_2. Let a_{n_i} converge to say α and let b_{m_i} converge to say β. Now it follows readily that

$$a_{n_i}\{b_{n_i}, b_{n_{i+1}}, \ldots\} \supset X.$$

From this we obtain an immediate contradiction, namely $\alpha\beta \supset X$. This is immediate from the lemma for example 2.

EXAMPLE 3. There exists an abelian semigroup which is not subresidually compact. The semigroup A will be generated by the elements $x, y, a_1, a_2, \ldots b_1, b_2, \ldots$ subject to the following relations: $a_i b_j = x$ whenever $i \leqslant j$ and y when $i > j$.

Now suppose A is a subsemigroup of a compact semigroup. Let a_{n_i} be a sequence of a_i's and b_{m_i} be a sequence of b_i's converging to say α and β respectively. Moreover suppose things have been arranged so that

$$n_i \leqslant m_i \quad \text{for all } i.$$

It follows that $\alpha\beta$ must be both x and y yielding a contradiction.

One thing to observe here is that although $x^2 = xy = y^2$ we do not have $x = y$. That is to say, A is not separative.

EXAMPLE 4. Let L be a left trivial semigroup (that is to say, $xy = x$ all x, y). Clearly the Stone–Čech compactification coincides with $\hat{L}$. Note that any mapping from L to a compact space X can be viewed as a homomorphism by endowing X with the left trivial multiplication. The same remarks hold for $L \times L$ so that

$$\beta(L \times L) = \widehat{L \times L}.$$

Suppose that the set for L is, say, the set of integers. It is well known that $\beta(L) \times \beta(L)$ and $\beta(L \times L)$ do not coincide. Thus Bohr compactification does not commute with cartesian products even in the finite case. The same remarks can be made by using a zero trivial semigroup (i.e. all products equal to some fixed element). This example will then be abelian.

EXAMPLE 5. The semigroup S of 2×2 matrices

$$\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}$$

where x and y are positive integers. The semigroup S is not algebraically embeddable in a compact group. To see this one has only to note that in S,

$$\begin{bmatrix} N & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^N \begin{bmatrix} N & 1 \\ 0 & 1 \end{bmatrix}.$$

From [25], it follows that

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

could not be separated from the identity with a unitary representation.

It should be noted however that S is residually finite nil. To see this one has only to examine the Rees quotients of certain ideals.

EXAMPLE 6. There exists a simple, $\mathcal{D}$-trivial cancellative semigroup without idempotent which is not subresidually compact. Let S be the semigroup of matrices

$$\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}$$

where x and y are positive and rational. To see that S is not a subsemi-group of a compact semigroup suppose on the con   that it were. Since S is simple it follows that $\overline{S}$ is completely   . The natural mapping $\overline{S} \to \overline{S}/\mathcal{L}$ defines a homomorphism onto a rig]   ial semigroup. It follows readily that the homomorphism must be c   nt on the matrices of the form

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

and from this that it must be constant on all of S. In the same way the map $\overline{S} \to \overline{S}/\mathcal{R}$ is trivial so that $\overline{S}$ is contained in a single $\mathcal{K}$-class and is thus a group. From the previous example it is immediate that this is impossible.

EXAMPLE 7. The bicyclic semigroup $\mathcal{C}(p,q)$. We recall that $\mathcal{C}(p,q)$ is defined as the monoid on two generators p and q subject to the relation $pq = $ identity. (For a detailed description of this semigroup, see [5].) Now (p,q) has $Z = $ integers as maximal group image. Moreover any proper homomorphic image of $\mathcal{C}(p,q)$ is a cyclic group. Finally $\mathcal{C}(p,q)$ can not be embedded in a compact semigroup. Now let T be a compact semigroup and γ a dense representation of $\mathcal{C}(p,q)$ into T. From what has been said above, it follows that there is a commutative diagram

$$
\begin{array}{ccc}
Z & & \\
\uparrow & \searrow & \\
\mathcal{C}(p,q) & \longrightarrow & T
\end{array}
$$
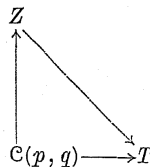
where Z denotes the integers. This diagram extends to

$$
\begin{array}{ccc}
\hat{Z} & & \\
\uparrow & \searrow & \\
Z & & \searrow \\
\uparrow & & \searrow \\
\mathcal{C}(p,q) & \longrightarrow & T
\end{array}
$$
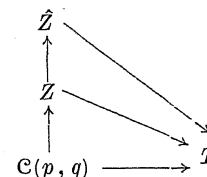
and by the uniqueness of the compactification we see that $\widehat{\mathcal{C}(p,q)} = \hat{Z}$.

EXAMPLE 8. There exists a 2-soluble (meta abelian) group which is not maximally almost periodic. As noted in [26] the group of mappings $x \to ax + b$ of the line where $a > 0$, a, b rational is the semi-direct product of the multiplicative non-zero rationals and the additive rationals. This group has no finite images and, as predicted by Proposition 5, its Bohr compactification is that of the additive rationals. Because of a result of Hall [11], a 2-soluble non-maximally almost periodic group can not be finitely generated.

EXAMPLE 9. As Fluch [8] has observed, there is a finitely presented group which is minimal   most periodic. This is the example of Higman on four generators and   tions. Each quotient group of this group is infinite.

EXAMPLE 10. There exists a three-step soluble group (i.e. $G''' = \{1\}$) which is finitely generated and not maximally almost periodic. As before, it suffices to show that G is not residually finite. This is precisely the group that is constructed in [11].

As is shown in [11] any two-step finitely generated soluble group is residually finite.

EXAMPLE 11. There exists a two-nilpotent group which is not maximally almost periodic. Indeed from above, any non-abelian, divisible nilpotent group will suffice. Of course, such a group can not be finitely generated.

EXAMPLE 12. If S is an ordered idempotent semigroup then $\hat{S}$ is zero dimensional. Thus, a curious application of the Bohr compactification is the existence of dimension raising homomorphisms. (See [1].) To see that $\hat{S}$ is zero dimensional note first that the order of S is carried over to $\hat{S}$. Now it follows that the only possible component of S would be an arc. Hence if $[\hat{s}_1, \hat{s}_2]$ were some non-degenerate component of S we must necessarily have $\delta(s) \in (\hat{s}_1, \hat{s}_2)$ for some $s \in S$, and indeed for infinitely many elements of S since S is dense in $\hat{S}$. However S is obviously residually finite so that Proposition 1 yields a contradiction.

Lawson [15] has constructed a one dimensional compact connected semi lattice with identity which has no homomorphisms onto an interval and consequently is not the continuous image of a zero dimensional compact semi lattice. It follows then that the discrete version of that semi lattice cannot have a Bohr compactification which is zero dimensional.

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