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Reçu par la Rédaction le 27. 4. 1971

On Helling cardinals (1)

by

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§ 0. Introduction. The cardinal number $\varkappa > \omega_1$ is called a *Helling cardinal* if every relational structure with a $\varkappa$-like ordering has an elementary substructure with a λ-like ordering for an arbitrary $\omega_0 < \lambda < \varkappa$. (If the language L of the structure under consideration is uncountable, then we can modify this notion, requiring $\lambda > |L|$.)

The existence of Helling cardinals is incompatible with the Axiom of Constructibility: it is easy to see that if there exists a Helling cardinal then $P(\omega_0) \cap L$ is countable. By the method of Silver [9] one can obtain the existence of $O^{\#}$ provided that there exists a Helling cardinal.

In 1966 Helling [2] proved that every measurable cardinal is a Helling cardinal, and Silver [8] has shown the same for Ramsey cardinals. These results seem to imply that Helling cardinals are very large. In Section 2 there is given a sufficient condition for a cardinal to be a Helling cardinal. This condition is satisfied by — among others — real-valued measurable cardinals. Prikry's result [6] entails that cardinals fulfilling our condition may be less than the continuum.

From the example of Fuhrken [1] we know that every Helling cardinal $\varkappa$ must be regular. If there is a limit (strongly limit) cardinal $\omega_0 < \lambda < \varkappa$, then $\varkappa$ is weakly (strongly) inaccessible — see Keisler [3]. From these remarks it is clear that every Helling cardinal which is larger than ω_ω must be weakly inaccessible. The question arises: can ω_n be a Helling cardinal? A partial solution of this is given in Section 3.

In Section 4 we give an application of previous results to a language with a generalized quantifier (2).

(1) The results of this paper are contained in the doctoral dissertation prepared by the authoress under the guidance of Professor C. Ryll-Nardzewski at Wrocław University. See also [11].

(2) The Corollary from [11] is not stated here. I am indebted to Professor A. Mostowski, who pointed out a gap in that proof. If this theorem is still true, the proof will be published.

§ 1. Preliminaries. The ordinal α is identified with the set of all ordinals less than α, i.e. $\alpha = \{\xi:\ \xi < \alpha\}$. Cardinals are initial ordinals. For a cardinal number $\varkappa$, by $[\varkappa]^n$ we denote the set of all n-element subsets of $\varkappa$, and by $[\varkappa]^{<\omega}$ the set of all finite subset of $\varkappa$.

$f * X$ denotes the image of a set X by a function f. By $|X|$ we denote the cardinality of X.

An ideal $\mathfrak{J}$ of subsets of X is said to be *ω_1-saturated* if there is no family of ω_1 disjoint subsets of X, not belonging to $\mathfrak{J}$.

We say that a cardinal number $\varkappa$ *has the property* $i(\varkappa, \omega_1)$ if $\varkappa$ carries an ω_1-saturated $\varkappa$-complete non-trivial ideal. (An ideal is non-trivial if it is proper and contains all points.)

LEMMA 1.1. *A cardinal number $\varkappa$ has the property $i(\varkappa, \omega_1)$ iff it satisfies the following condition* $(\ast)$:

$(\ast)$ *There is a $\varkappa$-complete, non-trivial filter $\mathcal{F}$ on $\varkappa$ such that for each cardinal $\lambda < \varkappa$ and each function f: $[\varkappa]^{<\omega} \to \lambda$ there is a set $U \in \mathcal{F}$ such that $|f * [U]^{<\omega}| < \omega_1$.*

The proof that $i(\varkappa, \omega_1)$ implies $(\ast)$ is due to Solovay [10], Theorem 5. Conversely, if $\varkappa$ satisfies $(\ast)$, then the family of complements of sets in $\mathcal{F}$ establishes the required ideal.

Remarks. If in $(\ast)$ we required $|U| = \varkappa$ instead of $U \in \mathcal{F}$, then we would obtain the condition of Rowbottom [7]. If we required $\mathcal{F}$ to be uniform (not necessarily $\varkappa$-complete), then $(\ast)$ would be transformed into the condition $\big(\varkappa, R(\omega_1)\big)$ in the paper of Prikry [6]. Neither Rowbottom's nor Prikry's condition implies the regularity of $\varkappa$, but the condition $(\ast)$ does.

An ordering $<$ of a set A is called *$\varkappa$-like* if $|A| = \varkappa$ and every initial segment of $<$ is of the power less than $\varkappa$.

For the cardinal numbers $\varkappa$, λ we write $\varkappa \twoheadrightarrow \lambda$ if every relational structure with a $\varkappa$-like ordering has an elementary substructure which is λ-like in the same ordering.

We say that $\varkappa$ is a *Helling cardinal* if $\varkappa \twoheadrightarrow \lambda$ holds for every $\omega_0 < \lambda < \varkappa$.

We are going to show that if $\varkappa$ satisfies $i(\varkappa, \omega_1)$ then it is a Helling cardinal.

§ 2. The main theorem. Let L be an arbitrary elementary language with a binary predicate $<$. By L_α we denote the language L with an additional quantifier Q_α ("there is at least α of ..."). Let λ be a cardinal. $L(\lambda)$ denotes the language L with an added sequence of constants $\{e_\xi:\ \xi < \lambda\}$ and a unary relation C.

Subsets of the sets of constants are treated as sequences with the ordering given by the indices. If $\boldsymbol{x}, \boldsymbol{y}$ are two sequences of constants, we say that they are *congruent up to the level* $\mu < \lambda$ if $\boldsymbol{x} \cap \{e_\xi:\ \xi < \mu\} = \boldsymbol{y} \cap \{e_\xi:\ \xi < \mu\}$ (notation: $\boldsymbol{x} \| \boldsymbol{y}(\mu)$).

We use the following classification of the terms of $L(\lambda)$: a term of L has level zero; otherwise a term τ of $L(\lambda)$ has level $\alpha+1$ (notation: $l(\tau) = \alpha+1$), if α is the largest index ξ such that e_ξ occurs in τ.

In $L_{\omega_1}(\lambda)$ — just as Helling and Keisler (¹) have done — we introduce a theory $T(\lambda)$ with axioms of the following forms:

(1) "*$<$ is a linear ordering*",

(2) "*$e_\xi < e_\eta$*" *for* $\xi < \eta < \lambda$,

(3) "*$C(e_\xi)$*" *for* $\xi < \lambda$,

(4) "*there exist at most countably many values of $\tau(\boldsymbol{x}')$ where $\tau(\boldsymbol{x}') < \tau_1$ and $\boldsymbol{x}' \| \boldsymbol{x}\ (l(\tau_1))$*", *for each n-placed term τ and each constant term τ_1 of $L(\lambda)$ and each n-element sequence $\boldsymbol{x} = \langle e_{\xi_1}, \ldots, e_{\xi_n}\rangle$.*

The scheme (4) may be formally written in $L_{\omega_1}(\lambda)$ as follows:

$$(4)\qquad \forall x_1, \ldots, x_n\Big(C(x_1) \wedge \ldots \wedge C(x_n) \to \Big(\neg Q_{\omega_1} y \exists x_1', \ldots, x_n'\ C(x_1') \wedge \ldots \wedge C(x_n') \wedge \forall z\big(C(z) \wedge z < e_{l(\tau_1)} \to (z = x_1 \leftrightarrow z = x_1' \wedge \ldots \wedge z = x_n \leftrightarrow z = x_n')\big) \wedge y = \tau(x_1', \ldots, x_n') \wedge y < \tau_1\Big)\Big).$$

LEMMA 2.1. *Let $\varkappa$ satisfy the condition $i(\varkappa, \omega_1)$. Let $\mathfrak{A} = \langle A, <, \ldots\rangle$ be a $\varkappa$-like model (of a theory in L). Then there are a sequence $\{z_\xi:\ \xi < \varkappa\}$ of elements of A and a subset $C \subseteq A$ such that the expansion $\mathfrak{A}' = \langle A, <, \ldots, C, z_\xi\rangle_{\xi<\varkappa}$ of $\mathfrak{A}$ is a model of $T(\varkappa)$, with $C^{\mathfrak{A}'} = C$ and $e_\xi^{\mathfrak{A}'} = z_\xi$.*

Proof. We introduce an additional relation $\precsim$ which well-orders A in the type $\varkappa$. Let $\mathcal{F}$ be a filter of subsets of A satisfying $(\ast)$ in Lemma 1.1. By $\mathrm{Tm}(\beta, n)$ we denote the set of all n-placed terms of level β.

We show that there are a decreasing sequence of sets $\{X_\xi:\ \xi < \varkappa\} \subseteq \mathcal{F}$ and a strictly increasing (in the sense of $<$) sequence $\{z_\xi:\ \xi < \varkappa\} \subseteq A$ such that:

(i) *z_ξ is the minimal element in the sense of $\precsim$ in X_ξ,*

(ii) *$z_\xi \precsim x$, for $x \in X_{\xi+1}$;*

(iii) *in the structure $\mathfrak{A}_\xi = \langle A, <, \ldots, \{z_\eta:\ \eta < \xi\}, z_\eta\rangle_{\eta<\xi}$ hold the axioms* (1)–(3) *of* $T(\xi)$,

(iv) *if τ is an n-placed term, τ_1 — a constant term and $l(\tau) \leqslant l(\tau_1) \leqslant \xi$, then for $\boldsymbol{x}$ running over $[X_{l(\tau_1)}]^n$ there are at most countably many values of $\tau(\boldsymbol{x})$ with $\mathfrak{A}_\xi \models \tau(\boldsymbol{x}) < \tau_1$.*

The proof of this is by induction on $\xi < \varkappa$.

Step 1. $\xi = 0$. Since $|\mathrm{Tm}(0, 0)| < \varkappa$ and $\varkappa$ is regular, there is a $y_0 \in A$ such that $\tau^{\mathfrak{A}} < y_0$ for every term $\tau \in \mathrm{Tm}(0, 0)$.

For an arbitrary $n < \omega_0$, $\tau \in \mathrm{Tm}(0, n)$, we define a function f_τ: $[A]^n \to \{y:\ y < y_0\}$ putting $f_\tau(\boldsymbol{a}) = \tau^{\mathfrak{A}}(\boldsymbol{a})$ if $\tau^{\mathfrak{A}}(\boldsymbol{a}) < y_0$ and $f_\tau(\boldsymbol{a}) = \min\{y:\ y < y_0\}$ (in the sense of $\precsim$) otherwise.

(¹) A similar method was used firstly by Silver in [8].

Since $|\{y\colon y < y_0\}| < \varkappa$, by (∗) for every f_τ there is a set $U_\tau \in \mathcal{F}$ such that $|f_\tau * [U_\tau]^n| < \omega_1$. We define $X_0 = \bigcap_{n<\omega_0} \bigcap_{\tau \in \mathrm{Tm}(0,n)} U_\tau$. Then $X_0 \in \mathcal{F}$ and for an arbitrary term τ of level zero there are at most countably many values of $\tau(\boldsymbol{a}) < y_0$ in $\mathfrak{A}$ (where $\boldsymbol{a}$ is a sequence of elements of X_0 of an appropriate length).

Then we set $z_0 = \min X_0$ in the sense of $\lessdot$.

Now we assume that for $\eta < \xi$, X_η and z_η have been defined.

Step 2. *ξ is a limit ordinal.* We set $X_\xi = \bigcap_{\eta<\xi} X_\eta$ and $z_\xi = \min X_\xi$ in the sense of $\lessdot$.

The satisfaction of (iv) results from the fact that if $l(\tau) \leqslant \xi$ then $l(\tau) < \xi$. Conditions (i)–(iii) are easy to verify.

Step 3. $\xi = \eta+1$. As in Step 1, we choose $y_\xi \in A$ such that $\tau^{\mathfrak{A}} < y_\xi$ for every term $\tau \in \mathrm{Tm}(\xi, 0)$. For $n < \omega_0$, $\tau \in \mathrm{Tm}(\xi, n)$ we define a function $f_\tau\colon [A]^n \to \{y\colon y < y_\xi\}$ such that $f_\tau(\boldsymbol{a}) = \tau^{\mathfrak{A}}(\boldsymbol{a})$ if $\tau^{\mathfrak{A}}(\boldsymbol{a}) < y_\xi$ and $f_\tau(\boldsymbol{a}) = \min\{y\colon y < y_\xi\}$ (in the sense of $\lessdot$) otherwise.

Then there is a set $U_\tau \in \mathcal{F}$ such that $|f_\tau * [U_\tau]^n| < \omega_1$. Let $Z_\xi = \bigcap_{n<\omega_0} \bigcap_{\tau \in \mathrm{Tm}(\xi,n)} U_\tau$. Then $Z_\xi \in \mathcal{F}$ and for every term τ of level $\leqslant \xi$ there are at most countably many values of $\tau(\boldsymbol{a}) < y_\xi$ for $\boldsymbol{a} \subseteq Z_\xi$.

We set $X_\xi = X_\eta \cap (Z_\xi - \{y\colon y \leqslant z_\eta\})$; then $X_\xi \in \mathcal{F}$ because $|\{y\colon y \leqslant z_\eta\}| < \varkappa$. Put $z_\xi = \min X_\xi$ in the sense of $\lessdot$. Of course, conditions (i)–(iv) are then satisfied.

Now we define $C = \{z_\xi\colon \xi < \varkappa\}$. The satisfaction of axioms (1)–(3) in $\mathfrak{A}'$ results immediately from (i)–(iv) holding for every $\xi < \varkappa$.

It remains to show that axiom (4) holds in $\mathfrak{A}'$. For this purpose we take τ — an n-placed term, and τ_1 — a constant term of $\boldsymbol{L}(\varkappa)$. Put $\xi = l(\tau_1) < \varkappa$.

If $l(\tau) \geqslant \xi$, then we substitute successive free variables which have not been used for all constants with indices $\geqslant \xi$ in τ. Thus we obtain a term τ' of $n+k$ places (for some k) with $l(\tau') < \xi$. If $\boldsymbol{x}$ is an n-tuple of z's, then we extend it to an $(n+k)$-tuple $\boldsymbol{x}'$, adding the constants with indices $\geqslant \xi$. Of course, if $\boldsymbol{x} \| \boldsymbol{y}(\xi)$, then $\boldsymbol{x}' \| \boldsymbol{y}'(\xi)$ and we choose a sequence $\boldsymbol{y}' \| \boldsymbol{x}(\xi)$ such that $\tau'(\boldsymbol{y}') = \tau(\boldsymbol{x})$.

We are going to show that there are at most countably many values of $\tau'(\boldsymbol{y}') < \tau_1$ such that $\boldsymbol{y}' \| \boldsymbol{x}'(\xi)$.

In order to do this, define $\boldsymbol{y}_1 = \boldsymbol{y}' \cap \{z_\eta\colon \eta < \xi\} = \boldsymbol{x}_1$ and $\boldsymbol{y}_2 = \boldsymbol{y}' \cap \{z_\eta\colon \eta \geqslant \xi\}$. Then $\tau'(\boldsymbol{y}') = (\tau'(\boldsymbol{y}_1))(\boldsymbol{y}_2)$ and $\boldsymbol{y}_2 \subseteq X_\xi$. Applying (iv) to the term $\tau'' = \tau'(\boldsymbol{y}_1) = \tau'(\boldsymbol{x}_1)$, we know that substituting an arbitrary sequence $\subseteq X_\xi$ for $\boldsymbol{y}_2$ we can obtain at most countably many values of τ'' less than τ_1. So every sequence $\boldsymbol{y}_1 \| \boldsymbol{x}(\xi)$ has only countably many expansions $\boldsymbol{y}'$ with distinct values of $\tau(\boldsymbol{y}') < \tau_1$. Thus there are at most countably many distinct values of $\tau(\boldsymbol{y})$ for $\boldsymbol{y} \| \boldsymbol{x}(\xi)$.

The case where $l(\tau) < \xi$ follows immediately from (iv).

LEMMA 2.2. *If $\mathfrak{A}$ can be expanded to a model of $\boldsymbol{T}(\varkappa)$, $\omega_0 < \lambda \leqslant \varkappa$, then $\mathfrak{A}$ has an λ-like elementary substructure.*

Proof. Let $\{z_\xi\}_{\xi<\varkappa}$ be a realization in $\mathfrak{A}$ of constants in $\boldsymbol{T}(\varkappa)$ and $\mathfrak{A}' = (\mathfrak{A}, z_\xi)_{\xi<\varkappa}$. Denote by $\boldsymbol{T}$ the elementary theory of the structure $\mathfrak{A}'$ with all Skolem functors and put $B = \{\tau^{\mathfrak{A}'}\colon \tau$ is a constant term of $\boldsymbol{T}$ and $l(\tau) < \lambda\}$. Of course we have $|B| = \lambda$, and by the definition of $\boldsymbol{T}$, the substructure of $\mathfrak{A}'$ with the universe B is an elementary substructure of $\mathfrak{A}'$.

To show that $\langle B, <\rangle$ is λ-like one needs to examine the power of the set $B_x = \{y \in B\colon y < x\}$ for an arbitrary $x \in B$. Let τ_1 be a term of level $<\lambda$ such that $x = \tau_1^{\mathfrak{A}'}$. We define a function $f\colon B_x \to \bigcup_{n<\omega_0} \mathrm{Tm}(0, n) \times \{z_\xi\colon \xi < l(\tau_1)\}$. For $y \in B_x$ let $f(y)$ be a pair $\langle \tau, \boldsymbol{a}^*\rangle$ where τ is a term of level zero, $\boldsymbol{a}$ is a sequence of z's of an appropriate length, $\tau^{\mathfrak{A}'}(\boldsymbol{a}) = y$ and $\boldsymbol{a}^* = \boldsymbol{a} \cap \{z_\xi\colon \xi < l(\tau_1)\}$. Of course, the sets $\bigcup_{n<\omega_0} \mathrm{Tm}(0, n)$ and $\{z_\xi\colon \xi < l(\tau_1)\}$ have a cardinality less than λ. By axiom (4) of $\boldsymbol{T}(\varkappa)$ the counterimage of an arbitrary pair $\langle \tau, \boldsymbol{a}^*\rangle$ is at most countable, and so $|B_x| \leqslant \omega_0 \cdot l(\tau_1) < \lambda$.

THEOREM 2.3. *If $\varkappa$ has the property $i(\varkappa, \omega_1)$, then $\varkappa$ is a Helling cardinal.*

Proof. By Lemmas 2.1 and 2.2.

§ 3. Concerning two-cardinal models. Before we state the next results, we introduce some auxiliary notions.

DEFINITION. A relational structure $\mathfrak{A} = \langle A, U, \ldots\rangle$ is called a *two-cardinal model of type* $\langle \varkappa, \lambda\rangle$ if $\varkappa, \lambda$ are cardinals, $U \subseteq A$, $|A| = \varkappa$, and $|U| = \lambda$. We write $\langle \varkappa, \lambda\rangle \twoheadrightarrow \langle \mu, \nu\rangle$ if every two-cardinal model of type $\langle \varkappa, \lambda\rangle$ has an elementary substructure of type $\langle \mu, \nu\rangle$.

Proposition 3.1 below gives a condition for ω_n to be a Helling cardinal in the terms of two-cardinal models.

PROPOSITION 3.1. *For a natural number $n > 1$, the cardinal ω_n is a Helling cardinal iff the following conditions are satisfied:*

$$(\ast\ast)\qquad \begin{array}{l} \langle \omega_n, \omega_{n-1}\rangle \twoheadrightarrow \langle \omega_1, \omega_0\rangle, \\ \langle \omega_n, \omega_{n-1}\rangle \twoheadrightarrow \langle \omega_2, \omega_1\rangle, \\ \dots\dots\dots\dots\dots\dots \\ \langle \omega_n, \omega_{n-1}\rangle \twoheadrightarrow \langle \omega_{n-1}, \omega_{n-2}\rangle. \end{array}$$

Proof. (Outline) Let $\mathfrak{A} = \langle A, U, <, f, \ldots\rangle$, where $U \subseteq A$, $|A| = \mu$, $|U| = \nu$, $f\colon A^2 \to A$, $<$ orders A, and for $x \in A$ the function $f(x, \cdot)$ is an embedding of $\{y \in A\colon y < x\}$ into U. Then there is an elementary sentence σ which is satisfied in such a structure $\mathfrak{A}$ iff (for $\mu \neq \nu$) $\mu = \nu^+$.

Suppose the conditions (⁂) hold. If $\langle A, <, \ldots\rangle$ is an ω_n-like structure, then, choosing a $U \subseteq A$ of power ω_{n-1} and f as desired, we obtain a model $\mathfrak{A}$ of σ. By (⁂) this model has an elementary submodel of type $\langle\omega_k, \omega_{k-1}\rangle$ for every $0 < k < n$. The order $<$ in this submodel is ω_k-like, which is ensured by σ being satisfied in it.

Conversely, let $\langle A, U, \ldots\rangle$ be a model of type $\langle\omega_n, \omega_{n-1}\rangle$, ω_n being a Helling cardinal. Then $\langle A, <, \ldots\rangle$ is an ω_n-like model, $<$ being a well-ordering of A in the type ω_n, and a function f may be defined such that $\mathfrak{A} = \langle A, U, <, f, \ldots\rangle$ is a model of σ. Now, A has an elementary ω_k-like submodel for $0 < k < n$. By the properties of σ this submodel is of the type $\langle\omega_k, \omega_{k-1}\rangle$.

COROLLARY 3.2. *The sentence "ω_2 is a Helling cardinal" is consistent with the Zermelo–Fraenkel Set Theory under the assumption that there is a Ramsey cardinal. In symbols*:

$$\mathrm{Con}(\mathbf{ZFC} + \exists\varkappa \text{ Ramsey}) \rightarrow \mathrm{Con}(\mathbf{ZFC} + \omega_2 \text{ is Helling}).$$

Proof. By Proposition 3.1, ω_2 is a Helling cardinal iff $\langle\omega_2, \omega_1\rangle \twoheadrightarrow \langle\omega_1, \omega_0\rangle$ holds. This condition is known an *Chang's Conjecture*. Silver in [9] has shown the consistency of Chang's Conjecture provided that a Ramsey cardinal exists.

Remarks. Let us recall that $\varkappa$ is a *Rowbottom cardinal* (see [5]) if $\forall\lambda < \varkappa[\langle\varkappa, \lambda\rangle \twoheadrightarrow \langle\varkappa, \omega_0\rangle]$. From this definition it follows that formally Rowbottom cardinals play a similar role in the theory of two-cardinal models to that played by Helling cardinals in the theory of models with ordering. So, a natural question arises: Do those classes of cardinals coincide?

(1) Corollary 3.2 states that ω_2 may be a Helling cardinal. On the other hand, in **ZFC** one can prove that ω_2 is not a Rowbottom cardinal.

(2) Prikry [6] has shown that there may be a Rowbottom cardinal which is cofinal with ω_0. No such a cardinal is Helling (Helling cardinals must be regular).

(3) W. Marek asked me if it is consistent with **ZFC** that Helling and Rowbottom cardinals coincide (under some additional assumptions, e.g. that they exist). We can give the following partial answer, suggested to me by B. Węglorz:

PROPOSITION 3.3. *Let $\mathfrak{M}$ be a model of* **ZFC**, *$\varkappa \in \mathfrak{M}$ and $\mathfrak{M} \models$ "$\varkappa = \bigcup_{n<\omega} \varkappa_n$, where $\varkappa_n$ are measurable". Then, in every mild* (1) *Cohen extension of $\mathfrak{M}$, $\varkappa$ is a Rowbottom but not a Helling cardinal.*

(1) That is, in generic extension where a set of conditions has a power less than $\varkappa$ in $\mathfrak{M}$.

Proof. Suppose C is a set of conditions and $|C| < \varkappa$. So, there is an $n < \omega$ such that $|C| < \varkappa_n$. Thus, by the result of Lévy and Solovay [4], in the resulting Cohen extension $\mathfrak{M}[G]$ all $\varkappa_m$ for $m > n$ are measurable. Hence, $\varkappa$ in $\mathfrak{M}[G]$ is a sum of countably many measurable cardinals, and by a theorem of Prikry [6], it is a Rowbottom cardinal in $\mathfrak{M}[G]$. On the other hand, $\varkappa$ is obviously singular in $\mathfrak{M}[G]$, and so it is not a Helling cardinal in $\mathfrak{M}[G]$.

(4) In connection with our Theorem 2.3, it is easy to see that the condition $i(\varkappa, \omega_1)$ is not necessary for $\varkappa$ to be a Helling cardinal. Indeed, by Corollary 3.2 ω_2 may be Helling cardinal, but (as Solovay showed in [10]) if $\varkappa$ fulfils $i(\varkappa, \omega_1)$, then it must be very large (e.g. larger than the first weakly inaccessible cardinal).

§ 4. Application to languages $L_\varkappa$.

The relation between models of theories in $L_\varkappa$ and $\varkappa$-like ordered models is given in the following theorem of Fuhrken [1]:

THEOREM (Fuhrken). *For any theory T in a language $L_\varkappa$ there is a theory T^* in L where the predicate $<$ and some functors are added such that every model $\mathfrak{A}$ of T with $|A| = \varkappa$ can be expanded to a $\varkappa$-like model of T^* and conversely — every $\varkappa$-like model of T^* is a model of T.*

COROLLARY 4.1. *Suppose that the condition $i(\varkappa, \omega_1)$ holds. Let T be a theory in $L_\varkappa$ and T_λ — a L_λ-counterpart of T for an uncountable $\lambda < \varkappa$. If $\mathfrak{A}$ is a model of T, then $\mathfrak{A}$ has an elementary* (*in the sense of L*) *submodel which is a model of T_λ.*

Proof. Obvious from 2.3.

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Reçu par la Rédaction le 5. 5. 1971

Collectionwise normality and the extension of functions on product spaces

by

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1. Introduction and preliminary results. Let S be a nonempty subset of a topological space X. The subset S is said to be P-*embedded* in X if every continuous pseudometric on S extends to a continuous pseudometric on X. The subset S is C-*embedded* (respectively C^*-*embedded*) in X if every continuous (respectively bounded continuous) real valeud function on S extends to a continuous (respectively bounded continuous) real valued function on X. It is clear that every C-embedded subset is a C^*-embedded subset; moreover every P-embedded subset is a C-embedded one (see Theorem 2.4 of [7]). The concept of P-embedding characterizes collectionwise normal spaces in the same way as C-embedding (and also C^*-embedding) characterize a normal spaces. Specifically a topological space X is collectionwise normal if and only if every closed subset of X is P-embedded in X (see [12]).

Since a pseudometric on a space X is a function on the product set $X\times X$, it is of interest to relate the extension of pseudometrics to the extension of functions on $X\times X$ (without the triangle inequality). In [1] we showed that a subset S is P-embedded in X if and only if every continuous function from S into a bounded, closed convex subset of a Banach space extends to a continuous function on X with values in the convex subset. Using results developed in [1] to demonstrate this result, we also showed in [1] that if L is any Fréchet space, then every uniformly continuous function from S into L can be extended to a continuous function on X. Also uniform continuity of the extended function is shown not to be attainable.

We now turn our attention to relating P-embedding to the extension of functions from product sets. By utilizing results from [1] we show that a subspace S is P-embedded in the Tichonov space X if and only if for all locally compact hemicompact Hausdorff spaces A, the product set $S\times A$ is C^*-embedded in the product space $X\times A$ if and only if the