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Closed mappings and the Freudenthal compactification

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The main purpose of this paper is to give a characterization of closed mappings of locally compact weakly paracompact spaces into compact spaces and to apply this characterization in a study of the problem of extending closed mappings over the Freudenthal compactification. In the first section we state Theorem 1, giving a necessary condition for the closedness of a mapping $f\colon X\to Y$ from a weakly paracompact space X into a compact space Y, and give some applications of this theorem. The above-mentioned characterization of closed mappings is given in Theorem 2. The second section contains results about extensions of closed mappings over some compactifications. The main theorem of this part is Theorem 5, an essential generalization of a result of Morita ([7], Theorem 5). Lastly, the third section contains some facts on the Freudenthal compactification. In particular, Theorem 7 gives a characterization of the Freudenthal compactification of some subsets of manifolds.

All notions and notations are taken from [1] with a small modification: if rX is a compactification of X then we regard X as lying in rX and we write shortly $rX\setminus X$ instead of $rX\setminus r(X)$. All spaces are assumed to be $T_{3\frac{1}{2}}$ and all mappings are assumed to be continuous. The weakly paracompact (metacompact) spaces are called shortly WPC spaces.

We define, moreover, some useful notation: if $\mathcal{A}$ is a collection of disjoint subsets of the space X, then $X/\mathcal{A}$ denotes the quotient space X/R_A, where the equivalence relation R_A is defined as follows:

$$xR_A y \quad \text{iff} \quad x = y \quad \text{or} \quad x, y \in A \quad \text{for some } A \in \mathcal{A}.$$

1. Closed mappings.

Definition 1. A mapping $f\colon X\to Y$ is *closed* iff for every closed subset A of X its image $f(A)$ is closed in Y.

Let us notice the following obvious

Proposition 1. *If there exists a compact subset Z of X such that $f(X\setminus Z)$ is finite, then the mapping $f\colon X\to Y$ is closed.*

Proposition 2. *Let $f\colon X\to Y$ be a closed mapping into the compact space Y and let $\mathcal{A} = \{A_s\}_{s\in S}$ be a covering of X satisfying one of the following conditions:*

(i) *$\mathcal{A}$ is locally finite,*

(ii) *$\mathcal{A}$ is open and point-finite.*

Then there exists a finite subset S' of S such that the set $f(X\setminus \bigcup_{s\in S'} A_s)$ is finite.

Proof. We assume that the set S is infinite and our assertion does not hold, i.e. that for every finite subset S_1 of S the set $f(X\setminus \bigcup_{s\in S_1} A_s)$ is infinite. We shall define inductively a discrete set C closed in X. Let x_1 be a point of X and let $S_1 = \{s\colon x_1 \in A_s\}$. Since the covering $\mathcal{A}$ is point-finite in both cases (i) and (ii), the set S_1 is finite, and hence, by our assumption, the set $f\big(X\setminus \bigcup_{s\in S_1} A_s\big)$ is infinite. Now we assume that we have defined, for some n, points $\{x_1, \ldots, x_n\}$ such that $f(x_i)\neq f(x_j)$ if $i\neq j$. Let $S_n = \{s\in S\colon x_i\in A_s$ for some $i\leqslant n\}$. The set S_n is finite and hence the set $f\big(X\setminus \bigcup_{s\in S_n} A_s\big)$ is infinite. Take as x_{n+1} a point from the set

$$X\setminus\Big(\bigcup_{s\in S_n} A_s \cup f^{-1}\big(f(\{x, \ldots, x_n\})\big)\Big).$$

Let $C = \{x_1, x_2, \ldots\}$. For any $s\in S$ the set A_s contains at most one point from the set C. By (i) or (ii) the family $\big\{\{x_i\}\big\}_{i=1}^{\infty}$ is locally finite and hence the set C is closed and discrete in X. The mapping $f|C$ is closed and one-to-one, hence it is a homeomorphism of C onto a closed subset of Y. But the compact space Y cannot contain an infinite closed and discrete subspace, and this contradiction establishes our proposition.

Theorem 1. *Let X be a* WPC *space and let Y be compact. If the mapping $f\colon X\to Y$ is closed, then for every open covering $\mathfrak{U} = \{U_s\}_{s\in S}$ of X there exists a finite subset S' of S such that $f(X\setminus \bigcup_{s\in S'} U_s)$ is finite.*

Proof. As X is WPC, $\mathfrak{U}$ has a point-finite refinement $\mathfrak{V} = \{V_t\}_{t\in T}$. Applying Proposition 2, we obtain a finite subset T' of T such that $f(X\setminus\bigcup_{t\in T'} V_t)$ is finite. Selecting for every $t\in T'$ such an $s\in S$ that $V_t\subset U_s$, we obtain a finite set $S'\subset S$ satisfying the required conditions.

We now give two important corollaries to Theorem 1.

Definition 2. A $T_{3\frac{1}{2}}$ space X is called *rim-compact* iff it has a base $\mathcal{B} = \{B_s\}_{s\in S}$ such that $\operatorname{Fr}(B_s)$ is compact for every $s\in S$.

Morita has proved ([5], Theorem 1) that the family of all finite coverings of X by open sets with compact boundaries defines (in the sense of [1], Theorem 8.1.4) a uniformity $\mathfrak{U}$ compatible with the topology of X (see also [3], pp. 109–116). It is easy to check that this uniformity is totally bounded ([1], sec. 8.3).

Proposition 3. *Let X be a rim-compact* WPC *space and let Y be compact. Then every closed mapping $f\colon X\to Y$ is uniformly continuous with respect to the uniformity $\mathfrak{U}$.*

Proof. It is sufficient to prove that for any finite open covering $\mathfrak{U} = \{U_1, \ldots, U_n\}$ of Y there exists a finite open refinement $\mathfrak{V} = \{V_1, \ldots, V_l\}$ of the covering $f^{-1}(\mathfrak{U}) = \{f^{-1}(U_1), \ldots, f^{-1}(U_n)\}$ such that $\operatorname{Fr} V_i$ is compact for every $i\leqslant l$. Let $\widetilde{\mathfrak{V}} = \{\widetilde{V}_s\}_{s\in S}$ be an open refinement of $f^{-1}(\mathfrak{U})$ such that $\operatorname{Fr}(\widetilde{V}_s)$ is compact for every $s\in S$. Applying Theorem 1 to the covering $\widetilde{\mathfrak{V}}$, we obtain a finite subset $S' = \{s_1, \ldots, s_m\}$ of S such that $f(X\setminus\bigcup_{s\in S'}\widetilde{V}_s) = \{y_1, \ldots, y_k\}$. Let W_i, for $i = 1, \ldots, k$, be a neighbourhood of y_i contained in some U_p and let $\overline{W}_i\cap\overline{W}_j = \emptyset$ for $i\neq j$. One can easily prove that the set $A = \operatorname{Fr}(\bigcup_{s\in S'}\widetilde{V}_s)$ is compact and that $A\subset\bigcup_{i=1}^{m} f^{-1}(W_i)$. Now, let M_x, for every $x\in A$, be a neighbourhood of x with a compact boundary contained in $f^{-1}(W_i)$ for some i and let the family $\{M_{x_1}, \ldots, M_{x_p}\}$ be a finite covering of A. We shall denote M_{x_i} by M_i. We put $S_j = \{i\colon M_i\subset f^{-1}(W_j)\}$ for $j = 1, \ldots, k$ and we define $V_j = \big(f^{-1}(y_j)\setminus\bigcup_{s\in S'}\widetilde{V}_s\big)\cup \bigcup_{i\in S_j} M_i$. For every $j\leqslant k$ the set V_j is open as the union of the set $f^{-1}(W_j)\cap (X\setminus\bigcup_{s\in S'}\overline{V}_s)$ and some of the sets M_i. Moreover, the boundary of V_j is contained in

$$(X\setminus V_j)\subset\bigcup_{s\in S'}\widetilde{V}_s\cup\bigcup_{\substack{i=1,\ldots,j-1,\\ j+1,\ldots,k}}\overline{f^{-1}(W_i)}\,.$$

But, since $\overline{W}_i\cap\overline{W}_j = \emptyset$ for $i\neq j$ and $V_j\subset f^{-1}(W_i)$, we have $\operatorname{Fr}(V_j)\subset\bigcup_{s\in S'}\overline{\widetilde{V}}_s$. We put $A_j = \overline{V}_j\cap\bigcup_{s\in S'}\overline{\widetilde{V}}_s$ and we easily find that $A_j = \bigcup_{i\in S_j} M_i\cap\bigcup_{s\in S'}\overline{\widetilde{V}}_s$. Now, $\operatorname{Fr}(V_j)\subset\operatorname{Fr}(A_j)\cup\operatorname{Fr}(V_j\setminus A_j)$ and, since $(V_j\setminus A_j)\cap\bigcup_{s\in S'}\overline{\widetilde{V}}_s = \emptyset$, $\operatorname{Fr}(V_j\setminus A$ [illegible] A. On the other hand, applying the well-known formulas $\operatorname{Fr}(F\cup$ [illegible] $\subset\operatorname{Fr}(F)\cup\operatorname{Fr}(G)$ and $\operatorname{Fr}(F\cap G)\subset\operatorname{Fr}(F)\cup\operatorname{Fr}(G)$, we easily find th[illegible] $\operatorname{Fr}(A_j)\subset\bigcup_{i\in S_j}\operatorname{Fr}(M_i)\cup\operatorname{Fr}\big(\bigcup_{s\in S'}\overline{\widetilde{V}}_s\big)$. So $\operatorname{Fr}(V_j)$ is a closed subset of $A\cup\bigcup_{i\in S_j}\operatorname{Fr}(M_i)$ and hence $\operatorname{Fr}(V_j)$ is compact. Putting $V_j = \widetilde{V}_{s_{j-k}}$ for $j = k+1, \ldots, k+n$, we obtain the covering $\mathfrak{V} = \{V_1, \ldots, V_{n+k}\}$ satisfying the required conditions.

Theorem 2. *Let X be a locally compact* WPC *space and let Y be compact. The following conditions are equivalent:*

(i) *$f\colon X\to Y$ is closed,*

(ii) *there exists a compact subset Z of X such that the set $f(X\setminus Z)$ is finite.*

Proof. The implication (ii)→(i) follows from Proposition 1.

We now prove the implication (i)$\to$(ii). Let $\mathfrak{U} = \{U_t\}_{t\in T}$ be a covering of X by open sets with compact closures. Applying Theorem 1 to the covering $\mathfrak{U}$, we obtain a finite subset $T' \subset T$ such that $f(X\setminus \bigcup_{t\in T'} U_t)$ is finite. The set $Z = \bigcup_{t\in T'} \overline{U_t}$ satisfies the condition (ii).

We shall now give an example showing that the assumption of the weak paracompactness of X is essential in Proposition 3 and Theorem 2.

Example 1. Let X_1 be the space of all countable ordinals and let $X = X_1 \times I$, where I denotes the closed interval $[0,1]$. Take $Y = I$ and let $f\colon X\to Y$ be the projection. The space X is locally compact and, as X_1 is countably compact, the projection f is closed. The reader can easily prove that neither Proposition 3 nor Theorem 2 holds in this case.

The following example shows that the assumption of the local compactness of X is also necessary in Theorem 2.

Example 2. Let $X = \{0\}\times I \cup \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}\times (Q\cap I)$, where Q denotes the set of all rational numbers, and let $Y = \{0, 1, \frac{1}{2}, \ldots\}$. Let $f\colon X\to Y$ be the projection $f((x,y)) = x$.

The space X is metrizable and hence it is paracompact. It can easily be checked that the mapping f is closed and does not satisfy the condition (ii) of Theorem 2. As a corollary to Theorem 2 we prove

Proposition 4. *A mapping $f\colon X\to Y$ from a locally compact* WPC *space X into a compact space Y is closed iff there exist a compact space W and a quotient* [1] *mapping $g\colon X\to W$ satisfying the following conditions:*

(i) *there exists a compact subset Z of X such that $g|\operatorname{Int} Z$ is a homeomorphism and $g(X\setminus \operatorname{Int} Z)$ is finite,*

(ii) *there exists a mapping $h\colon W\to Y$ such that $f = h\circ g$.*

Proof. Let us suppose that f is closed. By Theorem 2 there exists a compact subset Z of X such that $f(X\setminus Z)$ is finite. The set $f(X\setminus \operatorname{Int} Z)$ is finite as the closure of $f(X\setminus Z)$. Let $f(X\setminus \operatorname{Int} Z) = \{y_1, \ldots, y_n\}$ and let $W = X/\{f^{-1}(y_i)\setminus \operatorname{Int} Z\}_{i=1,\ldots,n}$. The reader can easily prove that the space W and the mapping $g\colon X\to W$ satisfy the conditions (i) and (ii).

Now, if there exists a quotient mapping $g\colon X\to W$ satisfying the conditions (i) and (ii), then the closedness of f follows from Proposition 1 and the compactness of W.

Remark. It follows from Proposition 4 that any closed mapping from the Euclidean space E^n, where $n\geqslant 2$, into a compact space Y can be expressed as the composition of the quotient mapping g of E^n onto the sphere S^n obtained by matching to a point the complement of an open ball in E^n and of some mapping $h\colon S^n\to Y$.

[1] The mapping $f\colon X\to Y$ is quotient if and only if the set $U\subset Y$ is open iff $f^{-1}(U)$ is open in X. Let us notice that if R is an equivalence relation in X, then the mapping $f\colon X\to X/R$ is quotient, and that every closed mapping is quotient.

2. Extensions of closed mappings. In this section we study the relations between closed mappings of rim-compact WPC spaces and the Freudenthal compactification.

Definition 3. The *Freudenthal compactification* γX of a rim-compact space X is the least upper bound of all compactifications of X with a zero-dimensional (in the sense of ind) remainder.

Morita has proved ([5], Theorem 1) that X is the completion of the uniform space $(X, \mathfrak{U})$, where $\mathfrak{U}$ is the uniformity described above, and that $\operatorname{ind}(\gamma X\setminus X) = 0$. Let us notice that if X is locally compact, then $\gamma X\setminus X$ is compact, and so $\operatorname{Ind}(\gamma X\setminus X) = \operatorname{ind}(\gamma X\setminus X) = \dim(\gamma X\setminus X) = 0$.

We obtain from Proposition 3 the following

Theorem 4. *Let X be a rim-compact* WPC *space and let Y be compact. Then every closed mapping $f\colon X\to Y$ can be extended over γX.*

We can obtain further results concerned with the extension of closed mappings considering only locally compact spaces. We prove first the following

Lemma 1. *Let X be a locally compact space and let λX denote the least upper bound of all compactifications of X with a finite remainder. Then $\lambda X = \gamma X$.*

Proof. Obviously, $\lambda X \precsim \gamma X$. If $\gamma X\setminus X = \{a\}$ then the equality $\lambda X = \gamma X$ is evident. So we assume that $\overline{\overline{\gamma X\setminus X}} \geqslant 2$. Let us notice that if X is locally compact, then the remainder $rX\setminus X$ of any compactification rX is closed in rX. Now, since $\gamma X\setminus X$ is zero-dimensional, the family $\mathcal{F}$ of all mappings of $\gamma X\setminus X$ into the two-point discrete space separates points from closed sets in $\gamma X\setminus X$ and hence the diagonal mapping $\Delta\mathcal{F}$ is a homeomorphism. Since $\gamma X\setminus X$ is closed in γX, any mapping $f\in\mathcal{F}$ can be extended to a mapping $F\colon \gamma X\to r_f X$, where $r_f X$ is some two-point compactification of X and $F|X = id_X$. So the diagonal mapping $\underset{f\in\mathcal{F}}{\Delta} F\colon \gamma X\to \underset{f\in\mathcal{F}}{\mathbf{P}} r_f X$ is a homeomorphism and hence γX is the least upper bound of all two-point compactifications of X (see the proof of [1], Theorem 3.4.6). So we have $\gamma X \precsim \lambda X$, what finishes the proof.

Lemma 2. *Let X be a locally compact* WPC *space and let Y be compact. Then for any closed mapping $f\colon X\to Y$ there exists a compactification $a_f X$ with a finite remainder and the extension $a_f f\colon a_f X\to Y$ of the mapping f.*

Proof. Let Z be a compact subset of X such that $f(X\setminus Z)$ is finite (see Theorem 2) and let A be the closure of $X\setminus Z$ in βX. As $\beta f(\beta X\setminus X) \subset \beta f(A) \subset f(X\setminus Z)$, the set $\beta f(\beta X\setminus X)$ is finite. We define

$$a_f X = \beta X/\{\beta f^{-1}(y)\setminus X\}_{y\in f(X\setminus Z)}.$$

Since X is locally compact, $\beta X \setminus X$ is closed in βX, and so the quotient space $\alpha_f X$ is a compactification of X with a finite remainder. Putting $\alpha_f f([x]) = \beta f(x)$, we obtain the required extension of f.

THEOREM 3. *Let X be a locally compact* WPC *space. Every closed mapping $f\colon X \to Y$ into a compact space Y can be extended to $\gamma f\colon \gamma X \to Y$. If $\alpha X \not\geq \gamma X$ then there exist a compact space Y and a closed mapping $f\colon X \to Y$ which does not extend over αX.*

Proof. The first part of our theorem follows immediately from Lemmas 1 and 2 or, in a paralel manner from Theorem 2 and Proposition 3.

To prove the second part, let us notice that if $\alpha X \not\geq \gamma X$ then there exists a compactification rX with a finite remainder such that the embedding $r\colon X \to rX$ does not extend over αX. By [1], Theorem 3.4.5, this means that there exist two closed sets A and B in X such that $\overline{a(A)} \cap \overline{a(B)} \neq \emptyset$ and simultaneously $\overline{r(A)} \cap \overline{r(B)} = \emptyset$. Let $rX \setminus X = \{x_1, \ldots, x_n\}$ and let U_i be a neighbourhood of x_i in rX such that $\overline{U_i}$ intersects at most one of the sets A and B and $\overline{U_i} \cap \overline{U_j} = \emptyset$ for $i \neq j$. Let $Y = rX/\{\overline{U}_i\}_{i=1,\ldots,n}$. The space Y is compact and the quotient mapping $f\colon X \to Y$ is closed. The reader can easily prove that $\overline{f(A)} \cap \overline{f(B)} = \emptyset$ and hence f does not extend over αX.

Remark. The assumption of the weak paracompactness of X is essential. In fact, it is easy to check that the compactification βX of the space described in Example 1 is equal to $\beta X_1 \times I = \omega X_1 \times I$ and hence the remainder of any compactification of X is connected, and so $\gamma X = \omega X$. On the other hand, the projection $f\colon X \to I$ is closed and it cannot be extended over ωX.

To prove the functoriality of γX we introduce some notions:

DEFINITION 4. Let X be a locally compact WPC space. We denote by $CC'(X)$ the set all closed continuous real-valued bounded functions on X.

LEMMA 3. *The set $CC'(X)$ is a function ring containing all constant functions and separating points from closed sets in X.*

Proof. All constant functions belong to $CC'(X)$. Let $f, g \in CC'(X)$; both $f(X)$ and $g(X)$ are compact and, by Theorem 2, there exist two compact sets $Z_f, Z_g \subset X$ such that $f(X \setminus Z_f)$ and $g(X \setminus Z_g)$ are finite. The set $Z_f \cup Z_g$ is compact and the sets $f+g\big(X \setminus (Z_f \cup Z_g)\big)$ and $fg\big(X \setminus (Z_f \cup Z_g)\big)$ are finite. By Proposition 1 both $f+g$ and $f \cdot g$ are closed. Now, let $x \in X$ and let F be a closed subset of X not containing x. Since X is locally compact, there exists a neighbourhood V of x such that $\overline{V} \cap F = \emptyset$ and $\overline{V}$ is compact. By the Urysohn Lemma there exists a function f such that $f(x) = 1$ and $f(X \setminus V) \subset \{0\}$. It follows from Proposition 1 that $f \in CC'(X)$.

DEFINITION 5. The *ring* $CC(X)$ is the closure of $CC'(X)$ in the ring $C^*(X)$ of all continuous real-valued bounded functions on X with the topology of uniform convergence.

PROPOSITION 5. *If the mapping $f\colon X \to Y$ between two locally compact* WPC *spaces is closed, then $f^*\big(CC(Y)\big) \subset CC(X)$* (²).

Proof. It is clear that $f^*\big(CC'(Y)\big) \subset CC'(X)$. But f^* is continuous and hence $f^*\big(CC(Y)\big) \subset CC(X)$.

The reader can easily modify the well-known Theorem 71 in [7] so as to obtain the following

LEMMA 4. *Let X be a completely regular space and let R be a closed subring of $C^*(X)$ containing all constant functions and separating points from closed sets. Then the set $\mathfrak{M}$ of all proper maximal ideals of R with the topology generated by the basis $\mathfrak{B} = \{J \in \mathfrak{M}\colon f \notin J\}_{f \in R}$ is a compactification of X (to the point $x \in X$ corresponds the ideal of all functions from R vanishing in x), and R is exactly the set of those bounded functions which can be extended over $\mathfrak{M}$.*

DEFINITION 6. We denote by $\mathfrak{M}_X$ the set of all maximal ideals of $CC(X)$ with the topology described above.

LEMMA 5. *$\mathfrak{M}_X = \gamma X$ for every locally compact* WPC *space X.*

Proof. Let Y be a compact space and let $f\colon X \to Y$ be a closed mapping. We can regard Y as lying in a Tychonoff cube $\underset{s \in S}{\boldsymbol{P}} I_s$. Let f_s be the sth coordinate of f. Every mapping f_s can be extended to $F_s\colon \mathfrak{M}_X \to I$ and the diagonal mapping $\underset{s \in S}{\Delta} F_s\colon \mathfrak{M}_X \to \underset{s \in S}{\boldsymbol{P}} I_s$ is the required extension of f. So $\mathfrak{M}_X \geq X$. On the other hand, assuming that $\gamma X \not\geq \mathfrak{M}_X$ and denoting by $\widetilde{C}(X)$ the ring of all bounded functions on X extendable over γX, we find that $\widetilde{C}(X) \not\supset CC(X)$ and, by the density of $CC'(X)$ in $CC(X)$ and the closedness of $\widetilde{C}(X)$ in $C^*(X)$, that $CC'(X) \setminus \widetilde{C}(X) \neq \emptyset$. This means that there exists a closed bounded function on X which cannot be extended over γX, which is impossible. This contradiction establishes the equality $\gamma X = \mathfrak{M}_x$.

We can now prove the main theorem of this paper:

THEOREM 4. *Let both X and Y be locally compact* WPC *spaces. Then every closed mapping $f\colon X \to Y$ can be extended to $\gamma f\colon \gamma X \to \gamma Y$.*

Proof. By Proposition 5 the mapping $f^*\colon CC(Y) \to CC(X)$ is a continuous homomorphism. Let J be a proper maximal ideal of $CC(X)$. The set $f^{*-1}(J)$ is an ideal in $CC(X)$ and, as f^* preserves the unit element, $f^{*-1}(J)$ is proper. One can easily check that $f^{*-1}(J)$ is maximal and so

(²) $C^*(X)$ denotes the ring of all continuous real-valued bounded functions on X, and if $f\colon X \to Y$ is a continuous mapping, then $f^*\colon C^*(Y) \to C^*(X)$ is defined as follows: $f^*(\varphi) = \varphi \circ f$.

$f^{*-1}(J) \in \mathfrak{M}_Y$. We put $\gamma f(J) = f^{*-1}(J)$ and thus we obtain a mapping $\gamma f\colon \gamma X \to \gamma Y$. Notice that if J_x is the ideal of all the functions in $CC(X)$ vanishing at $x \in X$, then $\gamma f(J_x) = \{g \in CC(Y)\colon g(f(x)) = 0\}$, and so γf is an extension of f.

We shall now prove that γf is continuous. Let $U \in \mathfrak{B}_{\gamma Y}$, that is let $U = \{J \in \gamma Y\colon g \notin J\}$ for some fixed $g \in CC(Y)$. Then $\gamma f^{-1}(U) = \{J \in \gamma X\colon g \circ f \notin J\} \in \mathfrak{B}_X$. This finishes the proof of Theorem 4.

Remark 1. If the mapping f is onto Y, then the assumption of the weak paracompactness of Y is not essential since the closed image of the WPC space is also WPC ([9], Theorem 1).

Remark 2. Morita has proved in [3] that any closed mapping between locally compact paracompact spaces can be extended over the Freudenthal compactifications. Theorem 4, as is shown by the following example, is an essential generalization of this theorem.

Example 3. There exists a locally compact space weakly paracompact but not paracompact. Let $D_\mathfrak{n}$ be a discrete space of power $\mathfrak{n}$ and let $X = (\omega D_{\aleph_0} \times \omega D_\mathfrak{c}) \setminus \{(\omega_1, \omega_2)\}$ where $\{\omega_1\} = \omega D_{\aleph_0} \setminus D_{\aleph_0}$ and $\{\omega_2\} = \omega D_\mathfrak{c} \setminus D_\mathfrak{c}$. The space X is an open subset of the compact space $\omega D_{\aleph_0} \times \omega D_\mathfrak{c}$ and hence it is locally compact. The sets $A = (\omega D_{\aleph_0} \setminus \{\omega_1\}) \times \omega D_\mathfrak{c}$ and $B = \omega D_{\aleph_0} \times (\omega D_\mathfrak{c} \setminus \{\omega_2\})$ are open in X and are homeomorphic to the sums $\bigoplus_{n \in N} \omega D_\mathfrak{c}$ and $\bigoplus_{t \in I} \omega D_{\aleph_0}$, respectively. Hence, by [1], Theorem 5.1.9, A and B are paracompact. Now, let $\mathfrak{U} = \{U_s\}_{s \in S}$ be an open covering of X. We put $\mathfrak{U}_A = \{A \cap U_s\}_{s \in S}$ and $\mathfrak{U}_B = \{B \cap U_s\}_{s \in S}$. There exist, by the paracompactness of A and B, locally finite refinements $\mathfrak{V}_A$ and $\mathfrak{V}_B$ of $\mathfrak{U}_A$ and $\mathfrak{U}_B$, respectively. The union $\mathfrak{V} = \mathfrak{V}_A \cup \mathfrak{V}_B$ is the required open and point-finite refinement of U.

On the other hand, the sets $F = X \setminus A$ and $G = X \setminus B$ are closed in X and disjoint. It is easy to check that F and G cannot be separated by open sets. Hence X is not normal, and so it cannot be paracompact.

We can now give a characterization of closed mappings between locally compact WPC spaces.

Theorem 5. *Let both X and Y be locally compact* WPC *spaces. The mapping $f\colon X \to Y$ is closed iff it extends to $\gamma f\colon \gamma X \to \gamma Y$ and $\gamma f(x) \in Y$ for $x \in \gamma X \setminus X$ implies the existence of a neighbourhood U of x in γX such that $\gamma f(U) \subset \{\gamma f(x)\}$.*

Proof. The existence of such an extension γf follows from Theorem 4. Now, let $\gamma f(x) \in Y$ for some $x \in \gamma X \setminus X$ and let V be a neighbourhood of $\gamma f(x)$ with compact closure. The mapping $f | f^{-1}(\overline{V})$ satisfies the assumptions of Theorem 2 and hence there exists a compact set $Z \subset f^{-1}(\overline{V})$ such that the set $A = f(f^{-1}(\overline{V}) \setminus Z)$ is finite and $\gamma f(x) \in A$. Let W be a neighbourhood of $\gamma f(x)$ such that $W \cap A = \{\gamma f(x)\}$. The set $U = ((\gamma f)^{-1}(V) \setminus Z) \cap (\gamma f)^{-1}(W)$ is open and $x \in U$. Now, $\gamma f(U \cap X) = f(U \cap X) \subset W \cap f(f^{-1}(V) \setminus Z) \subset A \cap W = \{\gamma f(x)\}$, so $f(U) = \overline{f(U \cap X)} = \{\gamma f(x)\}$.

On the other hand, we assume that f satisfies the conditions of our theorem. Let $A = \overline{A} \subset X$ and let $\widetilde{A}$ be the closure of A in γX. Since $\widetilde{A}$ is compact, it is sufficient to prove that $Y \cap \gamma f(\widetilde{A}) = f(A)$. We prove that $\gamma f(\widetilde{A} \setminus A) \cap Y \subset f(A)$. In fact, let $y = \gamma f(x) \subset Y$ and let $x \in \widetilde{A} \setminus A \subset \gamma X \setminus X$. By assumption, there exists a neighbourhood U of x such that $\gamma f(U) = \{y\}$. But $x \in \widetilde{A}$ and hence $U \cap A \neq \emptyset$, and so $y \in f(U \cap A)$.

3. Applications.

Proposition 6. *If both X and Y are locally compact* WPC *spaces and Y has no isolated points, then every open-and-closed mapping $f\colon X \to Y$ is perfect.*

Proof. Let x be a point from $\gamma X \setminus X$. If $\gamma f(x) \in Y$, then there exists a neighbourhood U of x such that $\gamma f(U) = f(U \cap X) = \{\gamma f(x)\}$. Since f is open, $\{\gamma f(x)\}$ is open, which is impossible since Y has no isolated points. So $\gamma f(\gamma X \setminus X) \subset \gamma Y \setminus Y$ and f is perfect by ([1], Problem 3.X).

Proposition 7. *Let both X and Y be locally compact* WPC *but not compact spaces. Then every closed mapping from X onto Y is perfect if $\omega X = \gamma X$.*

Proof. We prove that $\gamma f(\gamma X \setminus X) \subset \gamma Y \setminus Y$. In fact, if $\gamma f(\gamma X \setminus X) \cap Y \neq \emptyset$ then, by assumption $\gamma f(\gamma X \setminus X) \subset Y$ and there exists a neighbourhood U of the unique point of $\gamma X \setminus X$ such that $\gamma f(U)$ is a single point. But $X \setminus U$ is compact and hence $Y = f(X) = f(U) \cup f(X \setminus U)$ is compact, which contradicts the assumption. So $\gamma f(\gamma X \setminus X) \subset \gamma Y \setminus Y$ and hence f is perfect.

It is interesting in the context of Proposition 7 to give a characterization of all locally compact WPC spaces such that $\gamma X = \omega X$.

Theorem 6. *If X is a non-compact locally compact* WPC *space, then the following conditions are equivalent:*

(i) $\gamma X = \omega X$,

(ii) *for any compact subset $Z \subset X$ and for every set A open-and-closed in $X \setminus \operatorname{Int} Z$ either A or $((X \setminus \operatorname{Int}(Z))) \setminus A$ is compact.*

Proof. We prove first that (i) $\Rightarrow$ (ii). We assume that there exists a compact subset $Z \subset X$ and two disjoint non-compact closed subsets A and B of $X \setminus \operatorname{Int} Z$ such that $X \setminus \operatorname{Int} Z = A \cup B$. Let us notice that $\operatorname{Fr}(A) \cup \operatorname{Fr}(B) \subset Z$. So, since Z is compact, we can find two sets $\widetilde{U}$ and $\widetilde{V}$ open in X and such that $\widetilde{U} \cap \widetilde{V} = \emptyset$, $\operatorname{Fr}(A) \subset \widetilde{U}$, $\operatorname{Fr}(B) \subset \widetilde{V}$. Taking $U = (\widetilde{U} \cup B) \setminus A$ and $V = (\widetilde{V} \cup A) \setminus B$, we obtain two disjoint open sets separating A and B. So the quotient space $X/\{A, B\}$ is T_2. One can easily check that the pace $Y = X/\{A, B\}$ is compact and the quotient mapping $f\colon X \to Y$ is closed (see Proposition 1). On the other hand, f cannot be extended over ωX and thus we obtain $\omega X \neq \gamma X$.

To prove that (ii) $\Rightarrow$ (i) we assume that $\gamma X\setminus X > 1$. Since $\operatorname{ind}(\gamma X\setminus X) = 0$ and X is locally compact, there exist two disjoint sets M and N such that both M and N are closed and non-void and $M \cup N = \gamma X\setminus X$. Let U and V be disjoint open subsets of γX separating M and N, and let $\overline{U} \cap \overline{V} = \emptyset$. The set $Z = X\setminus(U \cup V)$ is compact and the sets $\overline{U} \cap \cap (X\setminus \operatorname{Int} Z)$ and $\overline{V} \cap (X\setminus \operatorname{Int} Z)$ are non-compact disjoint open-and-closed in $X\setminus \operatorname{Int} Z$, which contradicts (ii). This contradiction establishes Theorem 6.

Corollary. *If for every compact subset Z of the locally compact non-compact* WPC *space X there exists a compact set $Z' \supset Z$ such that $X\setminus Z'$ is connected, then $\gamma X = \omega X$.*

Lemma 5. *Let X be compact and let Y be a* WPC *space. Then the product $X \times Y$ is weakly paracompact.*

The proof is analogous to the proof of ([1], Theorem 5.1.10).

Lemma 6. *Let X be a locally compact* WPC *space and let Y be* WPC. *Then the product $X \times Y$ is weakly paracompact.*

Proof. Let $\mathfrak{U} = \{U_t\}_{t\in T}$ be a point-finite covering of X by open sets with compact closures. Let $\mathfrak{V} = \{V_s\}_{s\in S}$ be an open covering of $X \times Y$. We denote by $\mathfrak{V}_t$ the restriction of $\mathfrak{V}$ to $\overline{U_t} \times Y$. By lemma 5 the space $\overline{U_t} \times Y$ is weakly paracompact and we can refine $\mathfrak{V}_t$ by a point-finite open covering of $\overline{U_t} \times Y$. We denote the restriction of this refinement to $U_t \times Y$ by $\mathfrak{W}_t$. One can verify that the covering $\mathfrak{W} = \bigcup_{t\in T} \mathfrak{W}_t$ is the required point-finite open refinement of $\mathfrak{V}$.

Proposition 8. *If both X and Y are connected non-compact locally compact* WPC *spaces, then $\gamma(X \times Y) = \omega(X \times Y)$.*

Proof. Let $Z \subset X \times Y$ be compact. The set $Z' = \pi_x Z \times \pi_y Z$, where π_x and π_y denote the projections of $X \times Y$ onto X and Y, respectively, satisfies the assumption of the Corollary to Theorem 6 and then $\gamma(X \times Y) = \omega(X \times Y)$.

Remark. Magill has proved in [4] the equality $\gamma(X \times Y) = \omega(X \times Y)$ for non-compact, connected, locally compact, metrizable X and Y.

Corollary. *If $n \geqslant 2$ then $\gamma E^n = \omega E^n = S^n$.*

Proof. $E^n = E \times E^{n-1}$ and it remains to apply Proposition 8.

Proposition 9. *Let X be a locally compact* WPC *space and let Y be a compact connected space. Then*

$$\gamma(X \times Y) = (\gamma X \times Y)/\{\{x\} \times Y\}_{x\in\gamma X\setminus X}.$$

Proof. Let Z be compact and let the mapping $f\colon X \times Y \to Z$ be closed. By Theorem 2 there exists a compact subset Z' of $X \times Y$ such that $f((X \times Y)\setminus Z)$ is finite. We put $Z' = \pi_x Z \times Y$. For every $x \in X$ the set $\{x\} \times Y$ is connected and hence if $(x_1, y_1), (x_2, y_2) \in (X \times Y)\setminus Z'$ and $x_1 = x_2$ then $f((x_1, y_1)) = f((x_2, y_2))$. Let $F\colon \gamma X \times Y \to Z$ denote the mapping defined by $F((x, y)) = \gamma f_y(x)$, where $f_y(x) = f((x, y))$. It is easy to check, using the density of $(X \times Y)\setminus Z'$ in $(\gamma X \times Y)\setminus Z'$, that the mapping F is continuous and if $x \in \gamma X\setminus X$ then $F((x, y_1)) = F((x, y_2))$ for any y_1, y_2. So, putting $\gamma f([x, y]) = F((x, y))$, we obtain the extension of f over $\gamma'(X \times Y) = (\gamma X \times Y)/\{\{x\} \times Y\}_{x\in\gamma X\setminus X}$. So $\gamma(X \times Y) \lesssim \gamma'(X \times Y)$.

On the other hand, the remainder $\gamma'(X \times Y)\setminus(X \times Y)$ is homeomorphic to $\gamma X\setminus X$; hence $\operatorname{ind}(\gamma'(X \times Y)\setminus(X \times Y)) = 0$ and so $\gamma'(X \times Y) \lesssim \gamma(X \times Y)$, which finishes the proof.

As an application of Proposition 9, we now define the notion of "c-homotopy" between two closed mappings.

Definition 7. The *c-homotopy* between two closed mappings $f_0, f_1\colon X \to Y$ is a closed mapping $F\colon X \times I \to Y$ such that $f_0 = F|X \times \{0\}$ and $f_1 = F|X \times \{1\}$. If there exists a c-homotopy between f_0 and f_1, then f_0 and f_1 are called *c-homotopic*.

Proposition 10. *If both X and Y are locally compact* WPC *spaces and $f_0, f_1\colon X \to Y$ are c-homotopic, then $\gamma f_0|\gamma X\setminus X = \gamma f_1|\gamma X\setminus X$.*

Proof. It easy to check that for any $t \in I$ the embedding $i_t\colon X \to X \times \times \{t\} \subset X \times I$ extends to the homeomorphic embedding $\gamma i_t\colon \gamma X \to \gamma(X \times I)$, where the space $\gamma(X \times I)$ is described in Proposition 9. Hence γi_t maps the remainder $\gamma X\setminus X$ homeomorphically onto $\gamma(X \times I)\setminus(X \times I)$. We can now easily check that $\gamma F([x \times I]) = \gamma f_t(x)$ for $x \in \gamma X\setminus X$ and any $t \in I$. So $\gamma f_0|\gamma X\setminus X = \gamma f_1|\gamma X\setminus X$.

Corollary. *If $f_0, f_1\colon X \to Y$ are c-homotopic, X and Y are locally compact* WPC *spaces and f_0 is perfect, then f_1 is also perfect.*

Proof. This is an obvious consequence of Proposition 10 and ([1], Exercise 11, p. 120).

Theorem 7. *Let X be a compact space and let Z be a closed subset of X such that $X\setminus Z$ is weakly paracompact. Then $X = \gamma(X\setminus Z)$ iff Z satisfies the following conditions*:

(i) $\operatorname{ind}(Z) = \dim(Z) = 0$,

(ii) *for every $x \in Z$ and every neighbourhood U of x there exists an open subset V of X such that*

(a) $x \in V \subset U$,

(b) $V\setminus Z \neq \emptyset$,

(c) *for every open-and-closed in $V\setminus Z$ subset A $x \notin \overline{A} \cap \overline{(V\setminus Z)\setminus A}$.*

Proof. We assume first that the conditions (i) and (ii) are satisfied. Let us notice that $X\setminus Z$ is dense in X and hence $X = \mu(X\setminus Z)$ is a compactification of $X\setminus Z$ with a zero-dimensional remainder; so $\mu(X\setminus Z) \lesssim \gamma(X\setminus Z)$. Now, let Y be a compact space and let $f\colon X\setminus Z \to Y$ be a closed mapping. By Theorem 2 there exists a compact subset Z_f of $X\setminus Z$ such that $f\big(X\setminus(Z\cup Z_f)\big)=\{y_1,\dots,y_n\}$. The set $W_i=\big(X\setminus(Z\cup Z_f)\big)\cap f^{-1}(y_i)$, for $i=1,\dots,n$, is open in $\big(X\setminus(Z\cup Z_f)\big)$ and hence open in X. Now, let V'_x, for every $x\in Z$, be an open subset of $X\setminus Z_f$ containing x and satisfying the conditions (b) and (c) of (ii). It is clear by (c) that $x\in\overline{(V'_x\setminus Z)\cap W_i}$ for only one i. Therefore, putting $V_x=\operatorname{Int}(V'_x\cup Z)\cap W_i$, we obtain a neighbourhood of x such that $V_x\setminus Z\subset W_i$ for some i. We denote by M_i the set of all such $x\in Z$ that $V_x\setminus Z\subset W_i$ $(i=1,\dots,n)$. It is clear that $Z=\bigcup_{i=1}^{n}M_i$ and the sets M_i are mutually disjoint. From (ii) it follows that $\operatorname{Int}Z=\emptyset$ and $V_x\subset W_i\cup M_i$ for every $x\in M_i$; so we have $M_i\subset\operatorname{Int}(W_i\cup M_i)$. This means in particular that the sets M_i are open-and-closed in Z. We put $F(x)=f(x)$ if $x\in X\setminus Z$ and $F(x)=y_i$ if $x\in M_i$. We shall now prove that F is continuous. Let U be an open subset of Y. We have $F^{-1}(U)=f^{-1}(U)\cup\bigcup_{i:\,y_i\in U}M_i=f^{-1}(U)\cup\bigcup_{i:\,y_i\in U}(M_i\cup W_i)$ and, since $M_i\subset\operatorname{Int}(W_i\cup M_i)$, $F^{-1}(U)$ is open in X. So F is the required extension of f and, by Theorem 3, $\mu(X\setminus Z)\succcurlyeq\gamma(X\setminus Z)$, which finishes the first part of the proof.

Let us assume now that $X=\gamma(X\setminus Z)$. Then the conditions (i) and (a), (b) of (ii) must be satisfied by the definition of γX. Assume now that for some $x\in Z$ the condition (ii.c) does not hold. This means that there exist two disjoint subsets A and B open in $X\setminus Z$ and such that $\operatorname{Int}(A\cup B\cup Z)$ is a neighbourhood of x in X and for every open in X set U containing x both $U\cap A$ and $U\cap B$ are non-empty. We define the space X' putting $X'=X\cup\{p\}$, where $p\notin X$, and we define the topology in X' as follows: the neighbourhoods of the points from $X\setminus\{x\}$ are unchanged. The basis of the neighbourhoods of x is the family $\{B\cap U\}$, where U is a neighbourhood of x in X, and the basis of the neighbourhoods of p is the family $\{A\cap U\}$, where U runs over all neighbourhoods of x in X. The reader can prove that the space X' is a compactification of $X\setminus Z$ with a zero-dimensional remainder and X' is essentially greater than X as a compactification of $X\setminus Z$. So $X\neq\gamma(X\setminus Z)$ and this contradiction establishes the last part of our proof.

Corollary 1. *Let X be a compact space and let Z be a closed subset of X such that*

(i) $\operatorname{ind}Z=\dim Z=0$,

(ii) *$X\setminus Z$ is weakly paracompact,*

(iii) *for every $x\in Z$ and every neighbourhood U of x there exists an open subset V of X such that*

(a) $x\in V\subset U$,

(b) *$V\setminus Z$ is non-empty and connected.*

Then $\gamma(X\setminus Z)=X$.

Corollary 2. *If M is a compact manifold of dimension greater than 1 and Z is a zero-dimensional closed subset of M, then $\gamma(M\setminus Z)=M$.*

Remark. The equality $\gamma E^n=\omega E^n=S^n$ if $n>1$ follows immediately from Corollary 2. It is sufficient to put $M=S^n$ and $Z=\{\omega\}\subset S^n$.

The result given in Corollary 2 has been obtained independently by *E. Ščepin*, regarding γX as the Wallman compactification of X.

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Added in proof. The generalization of Corollary 1 to Theorem 7 is given in J. R. McCartney, *Maximum zero-dimensional compactifications,* Proc. Cambridge Phil. Soc. 68 (1970), pp. 653–661.

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