Liftings of compact sets of mappings through a light proper mapping are compact

by

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For any map $p\colon T\to B$ and space Z another map $\overline{p}\colon T^Z\to B^Z$ exists defined by $\overline{p}(f)=p\cdot f$ (T^Z and B^Z are the spaces of maps from Z to T and B respectively in the compact open topology). In order that a map $f\colon Z\to B$ or more generally a homotopy $h\colon Z\times I\to B$ might be lifted to T, it is necessary for $\overline{p}$ to behave well.

This note considers the behaviour of $\overline{p}$ in the case that p is light. The following is shown. Let $p\colon (T, d)\to B$ be a light proper onto mapping and let Z be a locally compact, locally connected, and separable metric space. Then $\overline{p}\colon T^Z\to B^Z$ is a light proper mapping.

A theorem due to Whyburn (1934) and Floyd (1950) states: Let $p\colon T\to B$ be a light proper onto mapping on the metric spaces T and B. If, furthermore, p is an open mapping then for every path $\alpha\colon I\to B$ and $x\in T$ with $p(x)=\alpha(0)$, there exists a path $\beta\colon I\to T$ such that $p\cdot\beta=\alpha$ and $\beta(0)=x$. The converse holds if B is locally path connected. Hereafter, this result will be referred to as theorem W–F.

As consequences of these two theorems conditions are given for light proper mappings to possess covering homotopy and isotopy properties (this is a generalization of the Whyburn–Floyd theorem), and to be Hurewicz fibrations. Theorems of McAuley and Tulley on the lifting of cells also follow.

1. Definitions. A metric space will be denoted as a pair (T, d) with d the metric on the set T. $S(x,\varepsilon)$ will denote $\{y\in T\mid d(x,y)<\varepsilon\}$. A mapping $p\colon T\to B$ is light iff every point inverse is totally disconnected. The map p is open if the image of every open set is an open set and finally p is proper if the preimage of every compact set in B is compact in T. As a notational convenience a space Z will be called *acceptable* iff Z is a locally compact, locally connected, and separable metric space.

Further if S^k and D^k are the standard k dimensional sphere and cell respectively, then a space is LC^n iff for any point x and neighborhood U of x, there exists a neighborhood V of x so that each map $m\colon S^k \to V$ has an extension $m' = D^{k+1} \to U$ for $0 \leqslant k \leqslant n$.

2. Sections. A special case of the main theorem will be proved here. Specifically if $p\colon T \to B$ is a map, let $S(p) = \{s \in T^B \mid p \cdot s = I_B\}$ be a topological space with the compact open topology. In the case that (T, d) is a metric space and B is a locally compact, second countable space, $S(p)$ is metrizable as a countable sum of pseudometrics of the form

$$d_i(s_1, s_2) = \min\{1/2^i, \sup\{d(s_1(x), s_2(x)) \mid x \in K_i\}\}$$

where $\{K_1, K_2, \ldots\}$ is a sequence of compact sets in B whose interiors cover B.

Completeness of the fibers of p is enough to insure that $S(p)$ is complete. Compactness in $S(p)$ is hard to attain in general, but reasonable conditions are given for the case of light mappings.

(2.1). Theorem. *If $p\colon (T, d) \to B$ is a light proper mapping onto the acceptable space B, then $S(p)$ is a compact metrizable space.*

Proof. Ascoli's theorem yields the compactness of $S(p)$ if it can be shown that $S(p)$ is equicontinuous (see [7], page 155). To this end let $b \in B$ and suppose that $S(p)$ is not equicontinuous at $b \in B$. Then there exists $\varepsilon > 0$, a sequence $\{s_n\}$ in $S(p)$, and two sequences $\{y_n\}$, $\{z_n\}$ in B satisfying 1. $d(s_n(y_n), s_n(z_n)) > \varepsilon$ for $n \in N$ and 2. $\{s_n(b)\}$ is convergent in $p^{-1}(b)$. Furthermore there is a sequence of compact connected sets $\{C_1, C_2, \ldots\}$ with 3. $C_1 \supset C_2 \supset \ldots$ and 4. $C_n \subset S(b, 1/n)$. Without loss it may be assumed that 5. $\{y_n, z_n\} \subset C_n$ for $n \in N$. Observe that $\{s_n(C_n)\}$ is a sequence of connected sets each of diameter at least $\varepsilon > 0$. Thus, since p is proper, $C = \limsup s_n(C_n)$ is a connected set of diameter at least $\varepsilon > 0$; and consequently, since p is light, cannot lie $p^{-1}(b)$. However if $x \in C - p^{-1}(b)$, there exists a sequence $\{x_k\} \to x$ with $x_k \in s_{n_k}(C_{n_k})$. Clearly $p(x_k)$ converges to $b \in B$ and hence $x \in p^{-1}(b)$. This is a contradiction which concludes the proof.

3. Pullbacks extended. To make effective use of this theorem the usual notion of the pullback of a mapping will be extended.

(3.0) Definition. If N denotes the positive integers, let $1/N = \{x \in R \mid x = 0 \text{ or } x = 1/n \text{ for } n \in N\}$.

Note that $1/N$ is a compact metric space.

(3.1). Definition. If $p\colon T \to B$ is an onto mapping and $\{g_n\} \subset B^Z$ is a sequence converging in the compact open topology to $g_0 \in B^Z$, let $[p, \{g_n\}] = \{(z, x, y) \in Z \times T \times 1/N \mid g_0(z) = p(x) \text{ if } y = 0 \text{ or } g_{1/y}(z) = p(x)$ if $y \neq 0\}$. Define $\pi\colon [p, \{g_n\}] \to Z$ by $\pi(z, x, y) = z$. The mapping π is the *pullback of the sequence* $\{g_n\}$.

(3.2). Lemma. *If T and Z are metric spaces then so is $[p, \{g_n\}]$. Also if $p\colon T \to B$ is a proper (or light) mapping then $\pi\colon [p, \{g_n\}] \to Z$ is a proper (light) mapping.*

Proof. Only the proof that π is a proper mapping, given that p is proper, will be demonstrated. To this end let $K \subset Z$ be a compact set. Define $\sigma(K) = \bigcup_{n=0}^{\infty} g_n(K)$ and note that $\pi^{-1}(K) \subset K \times p^{-1}(\sigma(K)) \times 1/N \subset [p, \{g_n\}]$. The compactness of $\pi^{-1}(K)$ follows if $p^{-1}(\sigma(K))$ is compact and this is true iff $\sigma(K)$ is compact in B. Thus let $\{O_m\}_{m \in M}$ be an open cover of $\sigma(K)$ in B and extract a finite subcollection $O_{m_1}, O_{m_2}, \ldots, O_{m_k}$ which cover $g_0(K)$. Let $O = \bigcup_{i=1}^{k} O_{m_i}$ and note that since $g_n \to g_0$ in the compact open topology, for all except finitely many subscripts we have $g_n(K) \subset O$. It is now clear that a finite subcover can be found for $\sigma(K)$ which then concludes the proof.

4. The basic theorem. As mentioned before, for each map $p\colon T \to B$ and space Z a map $\bar{p}\colon T^Z \to B^Z$ is defined by $\bar{p}(f) = p \cdot f$. If $F \subset B^Z$, define $LF(p) = \{g \in T^Z \mid p \cdot g \in F\}$. Consequently $\bar{p} \mid LF(p)\colon LF(p) \to F$ is a mapping. We will record this mapping more briefly as $\bar{p}\colon LF \to F$ as long as no confusion arises.

(4.1). Theorem. *Let $p\colon (T, d) \to B$ be a light proper onto mapping and let Z be an acceptable space. Then if $F \subset B^Z$ the mapping $\bar{p}\colon LF \to F$ is light and proper.*

Proof. To see that $\bar{p}$ is a light mapping suppose that $f_1, f_2 \in \bar{p}^{-1}(g)$ with $g \in F \subset B^Z$. If $f_1 \neq f_2$ there is some $z \in Z$ for which $f_1(z) \neq f_2(z)$. Define $e_z\colon \bar{p}^{-1}(g) \to p^{-1}(g(z))$ by $e_z(f) = f(z)$ and note that e_z is continuous. Now if f_1 and f_2 were in a connected subset of $\bar{p}^{-1}(g)$ it would follow that $e_z(f_1) = e_z(f_2)$. This is not so, and hence $\bar{p}$ is light.

To show that $\bar{p}$ is a proper map it is sufficient to consider a sequence $\{f_n\} \subset LF$ so that $p \cdot f_n = g_n$ converges to $g_0 \in F$. If it can be shown that a subsequence of $\{f_n\}$ converges to a map f_0 covering g_0, then $\bar{p}$ is proper. To accomplish this, construct the pullback $\pi\colon [p, \{g_n\}] \to Z$ and consider $S(\pi) =$ space of this sections for π. Define a sequence $\{s_n\} \subset S(\pi)$ by $s_n(z) = (z, f_n(z), 1/n)$. Theorem 2.1 applies so that a section $s\colon Z \to [p, \{g_n\}]$ and a subsequence $\{s_{n_k}\}$ converging to s are obtained. Letting $\pi_2\colon [p, \{g_n\}] \to T$ be the natural projection on the second coordinate, define $f_0\colon Z \to T$ as $f_0 = \pi_2 \cdot s$. It follows that $f_0 = \pi_2(\lim s_{n_k}) = \lim \pi_2 \cdot s_{n_k} = \lim f_{n_k}$; and also that f_0 is continuous with $p \cdot f_0 = g_0$. This concludes the proof.

There are immediate corollaries.

(4.2). Corollary. (James Hill, see [2]). *If $p\colon (T, d) \to B$ is a light proper onto mapping with the property that each homeomorphism $h\colon I^n \to B$, $n \geqslant 2$,* *can be lifted to T, then each homeomorphism of S^n into B can be lifted to T.*

Proof. It is sufficient to note that there is a sequence of homeomorphisms of I^n into S^n whose limit is a mapping onto S^n.

(4.3). COROLLARY. *Let $p\colon (T, d)\to B$ be a light proper onto mapping of metric spaces with B LC^0. Then p is an open mapping iff for each path $\alpha\colon I\to B$ there exists a commuting diagram of onto maps*

$$\begin{tikzcd} L\alpha\times I \arrow[r, "\bar{\alpha}"] \arrow[d, "\pi_2"'] & p^{-1}(\alpha I) \arrow[d, "p"] \\ I \arrow[r, "\alpha"'] & \alpha(I) \end{tikzcd}$$

with $L\alpha$ a totally disconnected compact metric space and $\bar{\alpha}(f, t) = f(t)$.

Proof. This follows immediately from theorem F and use of (4.1) with $F = \{\alpha\}$.

A similar diagram exists for α with domains other than $[0, 1]$ provided, of course, that they are acceptable spaces. Furthermore, it is clear that in all cases the factor $L\alpha$ can be replaced by the Cantor set if the definition of $\bar{\alpha}$ is suitably modified.

5. Light mappings and the CHP. A homotopy $h\colon Z\times I\to B$ induces maps $h_t\colon Z\to B$ defined $h_t(z) = h(z, t)$. If $F\subset B^Z$ then $h\colon Z\times I\to B$ is said to be a *homotopy through* F if $h_t\in F$ for $0\leqslant t\leqslant 1$.

A mapping $p\colon T\to B$ is said to have the *Z-CHP through F* (the covering homotopy property with respect to Z through F) iff for each homotopy $h\colon Z\times I\to B$ through F and map $g\colon Z\to T$ with $h(z, 0) = p\cdot g(z)\forall z\in Z$, there exists a homotopy $H\colon Z\times I\to T$ with $p\cdot H = h$ and $H(z, 0) = g(z)\ \forall z\in Z$. The map p is said to have the *Z-CHP* if it has the Z-CHP through B^Z.

(5.1). DEFINITION. Let $p\colon T\to B$ be a map and let Z be a topological space. If $F\subset B^Z$, define p to be *full over* F if given $f\in LF(p)$ and a (compact open) neighborhood U of f, there exists a neighborhood V of $p\cdot f\in F$ so that if $g\in V\cap F$ there exists $f'\in U$ with $p\cdot f' = g$.

(5.1). THEOREM. *Suppose $p\colon (T, d)\to B$ is a light proper onto mapping, Z is an acceptable space, and $F\subset B^Z$. Then if p is full over F, p has the Z-CHP through F. Furthermore if F is locally path connected then the converse is true.*

Proof. Let $h\colon Z\times I\to B$ be a homotopy through F and let $g\colon Z\to T$ be a map with $p\cdot g(z) = h(z, 0)\ \forall z\in Z$. Define $\alpha\colon I\to F$ by $\alpha(t) = h_t$.

Let G be the path component of F containing the range of α. Consider the map $\bar{p}\colon LF\to F$ and note that since p is full over F, $\bar{p}(LF)$ is an open subset of F. But applying (4.1), $\bar{p}(LF)$ is a closed subset of F. Consequently since $\bar{p}(g)\in G$, $\bar{p}\colon LG\to G$ is a light proper open onto mapping. Applying theorem W-F there exists a path $\beta\colon I\to LG$ with $\bar{p}\cdot\beta = \alpha$ and $\beta(0) = g$. Finally define $H\colon Z\times I\to T$ by $H(z, t) = \beta(z)(t)$. The map so defined is the required covering homotopy. If F is LC^0, theorem W-F provides the converse.

Remarks. Theorem 5.1 is a generalization of the Whyburn–Floyd theorem since in the simple case that Z is a singleton set, their theorem is immediately recovered.

Note also that whenever B is a compact metric ANR the Z-CHP for $p\colon T\to B$ is equivalent to the fullness of p over B^Z.

Let $H(Z, B)$ be the space of homeomorphisms of Z into B with the compact open topology.

(5.2). COROLLARY. *Suppose $p\colon (T, d)\to B$ is a proper light onto mapping and Z is an acceptable space. Then if p is full over $H(Z, B)$, p has the covering isotopy property with respect to the space Z. The converse is true if $H(Z, B)$ is locally arcwise connected.*

(5.3). COROLLARY. *Suppose $p\colon (T, d)\to B$ is a proper light onto mapping. If p is full over B^I, then p has the path lifting property, that is, p is a Hurewicz fibration. The converse holds if B is LC^1 space.*

Proof. Use of (5.1) insures that p has the I-CHP. It follows easily that since p is light, liftings of paths are unique given the initial point (see [8]). A theorem of Ungar yields the conclusion. His proof will be given here since it is immediate from (4.1).

Define $X = \{(t, \beta)\in T\times B^I \mid p(t) = \beta(0)\}$ and consider the following commuting diagram

$$\begin{tikzcd} T^I \arrow[r, "\pi"] \arrow[d, "\bar{p}"'] & X \\ B^I \arrow[ur, dashed, "\pi_2"'] & \end{tikzcd}$$

with $\pi(\alpha) = \big(\alpha(0), p\cdot\alpha\big)$ and $\pi_2(t, \beta) = \beta$. Use of (4.1) and remarks above show that π is a proper, onto injection and hence is a homeomorphism, thus π^{-1} is a path lifting function. The converse follows since B^I is LC^0 whenever B is LC^1.

Remarks. The results in section five have all been proved assuming that p is full over a large family of functions. Interesting results can be obtained if more specialized choices of F are made. For example the following result is easily shown.

(5.4). Corollary (McAuley–Tulley). *Let* $p\colon (T, d)\to I^2$ *be a light proper onto mapping. Defining* $F = \{\alpha\colon I\to I^2 \mid (\exists x \in I)\, \alpha(t) = (x, t)\forall t \in I\}$, *$p$ is full over F iff for each* $\beta\colon I\to T$ *with* $p\cdot\beta(t) = (0, t)$ *there is a section* $s\colon I^2\to T$ *for p extending* β.

Analogues of this theorem can be stated for cells of higher dimension (see [5] and [6]).

As another example, McAuley (in [5]) attempted to eliminate some of the pathology of light open mappings by defining a twist free mapping. A light open onto mapping $p\colon T\to B$ is twist free if for each homeomorphism $h\colon S^1\to B$ and $x \in p^{-1}(h(1, 0))$, there exists a homeomorphism $H\colon S^1\to T$ with $p\cdot H = h$ and $H(1, 0) = x$.

A conjecture of McAuley is partially answered by the following.

(5.5). Corollary. *If* $p\colon (T, d)\to B$ *is a proper twist free onto mapping and p is full over* $H(S^1, B)$ *then any 2 cell in B can be lifted to T.*

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Homeotopy groups of orientable 2-manifolds

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1. Introduction. Let X be a topological space, and let $H(X)$ denote the group of homeomorphisms of X onto itself topologized by the compact open topology. The arc-component of the identity $H_0(X)$ is a normal subgroup of $H(X)$ and $\mathcal{H}(X) = H(X)/H_0(X)$ is the group of the arc-components of $H(X)$, which is called the *homeotopy group of X*. The equivalence relation defined by $H_0(X)$ is called *isotopy*. We can also define the isotopy relation in a subgroup $H'(X)$ of $H(X)$ and the group generated by the isotopy classes will be called the *isotopy group of $H'(X)$*, which is denoted by $\pi_0[H'(X)]$. J will denote the group of integers and J_2 the integers mod 2. In 1914, Tietze [10] showed that the homeotopy group of the 2-sphere is J_2. This was proven again by Kneser in 1926 [7], Baer in 1928 [2], Schreier and Ulam in 1934 [9], and most recently by Fisher in 1960 [4]. In [7] Kneser also obtained a result that the homeotopy group of a disk is J_2. In 1923, Alexander [1] proved that the isotopy group of homeomorphisms of an n-cell onto itself leaving the boundary pointwise fixed is trivial. This result has been a most important tool for further development in this area of study. In 1962, in terms of the winding number of a homeomorphism of an annulus, Gluck [5] proved that the isotopy group of homeomorphisms of a closed annulus onto itself leaving the boundary pointwise fixed is J. He also showed that the homeotopy group of an annulus is $J_2\times J_2$.

In this paper we compute the homeotopy group and isotopy groups of various subgroups of the homeomorphism group of the manifold obtained from the 2-sphere by removing the interiors of three disjoint subdisks. Further we deal with the orientable 2-manifold with n boundary curves.

2. Preliminaries. In this section we give preliminary results which will be used in the next section.

Basic notations

M_n will denote an orientable 2-manifold with n boundary curves,