References

[1] Dunham Jackson, *The theory of approximation*, vol. XI, A. M. S. Pub., New York.
[2] P. P. Korovkin, *Linear operators and approximation theory*, India.
[3] S. M. Lozinski, *On convergence and summability of Fourier series and interpolation processes*, Math. Sbornik 14 (56) (1944), pp. 175–268.
[4] I. P. Natanson, *Constructive function theory*, New York.
[5] A. Sharma and A. K. Varma, *Trigonometric interpolation* Duke Math. J., vol. 32, No. 2 (1965), pp. 341–350.
[6] O. Shisha and B. Mond, *The degree of approximation to periodic functions by linear positive operators*, Jour. of Approximation Theory 1 (1968), pp. 335–339.
[7] A. K. Varma. *Simultaneous approximation of periodic continuous functions and their derivative*, Israel J. Math. 6 (1968), pp. 67–79.
[8] A. Zygmund, *Trigonometric series*, vol. II, New York 1959.

Received December 18, 1970 (280)

STUDIA MATHEMATICA T. XLI. (1972)

Fractional powers of operators and Bessel potentials on Hilbert space

by

MICHAEL J. FISHER* (Missoula, Montana)

Abstract. Two candidates for the title "the Bessel potential" over a real separable Hilbert space are studied with the theory of fractional powers of operators and shown to define equivalent Sobolev spaces $L_p^a(H)$. $L_p^a(H)$ is shown to be equivalent to $D(T^a)$ when $(-T)$ is the infinitesmal generator of the Poisson integral and when $D(T^a)$ is equipped with the graph norm. The Bessel potentials of purely imaginary order are shown to be bounded on the reflexive $L_p(H)$ and to form a strongly continuous boundary value group for the Bessel potentials J^a with $\mathrm{Re}(a) > 0$.

Introduction. In [3] we defined the Bessel potential over a real separable Hilbert space, H, and studied the family of singular integral operators G^a: $L_p^a(H) \to L_p(H)$, where $L_p^a(H)$ is the image of $L_p(H)$ under the Bessel potential J^a. $J^a(f) = \Gamma(a)^{-1}\int_0^\infty P_t(f)t^{a-1}e^{-t}dt$, where $P_t(f)$ is the Poisson integral of f; [2]. The norm in $L_p^a(H)$ is $\|g\|_{a,p} = \|f\|_p$ when $g = J^a(f)$. The purpose of this paper is to examine the Bessel potential operators more closely than they were studied in [3]. Specifically, we shall examine two prominent candidates for the designation of "the Bessel potential" over an infinite dimensional Hilbert space and show that the spaces $L_p^a(H)$ defined using these operators are equivalent to the domain of a certain closed densely defined operator when this domain is equipped with the graph norm. Secondly, we shall examine the semigroup J^a in $\mathrm{Re}(a) \geqslant 0$ and show that the boundary values, $J^{i\gamma}$, form a strongly continuous group of bounded operators on $L_p(H)$ if $1 < p < \infty$. The paper closes with a discussion of the infinitesmal generators of J^β, $\beta > 0$, and $J^{i\gamma}$.

Throughout this paper K, $K(a)$, $K(p, a)$ etc. (M, $M(a)$, $M(p, a)$, etc.) denote positive (complex) constants which depend only on the parameters shown. If T is a linear operator on a Banach space X, $D(T)$ denotes the domain of T and $R(T)$ denotes the range of T.

* Research supported in part by the National Science Foundation grants NSF-GP-8839 and NSF-GP-24574.

1. Definitions and Preliminaries. Let H be a real separable Hilbert space, let n_t denote the weak normal distribution on H which has variance parameter $t/2$ and is centered at the origin in H, and let B be a Hilbert–Schmidt operator on H. Let $n = n_2$, $L_p(H) = L_p(H, n)$ be the Banach space of p-power integrable complex valued functions on H, and let $y \to T_y$ denote the regular representation of the additive group of H acting on $L_p(H)$. See Section 1 of [2] for the measure theoretic preliminaries. Define

$$H_t(f) = \int\limits_H T_y f \, dn_t \circ B^{-1}(y) \quad \text{and} \quad P_z(f) = \int\limits_0^\infty H_t(f) N_t(z)\, dt/t,$$

where $N_t(z) = (\pi t)^{-1/2} z \exp(-t^{-1}z^2)$. $P_z(f)$ is the Poisson integral of f. The strongly continuous contraction semi-groups H_t and P_z were studied in [2]; P_z extends as an analytic semi-group to $|\arg(z)| < \pi/4$.

Set

$$J_1^\alpha(f) = \Gamma(\alpha)^{-1} \int\limits_0^\infty P_t(f) t^{\alpha-1} e^{-t} dt$$

and

$$J_2^\alpha(f) = \Gamma(\alpha/2)^{-1} \int\limits_0^\infty H_t(f) t^{(\alpha-2)/2} e^{-t} dt.$$

In the nomenclature of Komatsu's theory of fractional powers of operators [4], $J_1^\alpha(f) = (1+T)^{-\alpha} f$ and $J_2^\alpha(f) = (1+T^2)^{-\alpha/2}$, where $P_z(f) = \exp(-zT)f$. We shall show in Section 3 that when $\alpha > 0$, the $(J_i^\alpha)^{-1} J_k^\alpha$, $i, k = 1, 2$, are bounded operators on $L_p(H)$ and that $L_p^\alpha(H)$, whether defined by using J_1^α or J_2^α, is equivalent to $D(T^\alpha)$ with the graph norm. We shall use Komatsu's definition and theory [4] of the powers T^α.

Our comparative study of the Bessel potentials J_k^α will require no information about the operator T beyond the facts that $(-T)$ and $(-T^2)$ generate bounded strongly continuous semi-groups P_z and H_t on a reflexive Banach space X. In Section 4, where we study $J^{i\gamma}$, we need only the basic definitions and introduction of [2]; Section 5 has the same prerequisites. For these reasons we shall not review the formal definitions and basic measure theory of the spaces $L_p(H)$ and $L_p^\alpha(H)$, see [2, 3]. We shall concentrate instead on listing some of the results from the Balakrishnan–Komatsu theory of fractional powers of operators. In what follows T denotes a closed, densely defined operator on a reflexive Banach space X such that $P_z = \exp(-zT)$ is a bounded, strongly continuous semi-group on X.

Early work on the theory of fractional powers of operators is surveyed in [7]. Balakrishnan [1], defined fractional powers T^α, $0 < \alpha < 1$, for an operator $(-T)$ which generates a bounded semi-group. In [1] the semi-group generated by $(-T^\alpha)$ is studied, formulas for the resolvent in terms of $\exp(-zT)$ are given, and properties of T which are inherited by T^α are listed. Komatsu [4–I, II, III, IV] has developed an extensive theory of fractional powers of operators. In [4–I, II] it is assumed that A is a linear operator (not necessarily densely defined) such that the negative half line is in the resolvent set of A and $\|t(t+A)^{-1}\| \leq M$ for all $t > 0$. A^α is defined for all complex α in Section 4 of [4–I]. For our purposes it will be sufficient to recall some of Komatsu's results for the case when $(-A)$ generates a bounded, strongly continuous semi-group on a reflexive Banach space X.

K-1. If $0 < \mathrm{Re}(\alpha) < 1$, $A^\alpha x = \dfrac{\sin \pi\alpha}{\pi} \displaystyle\int\limits_0^\infty t^{\alpha-1} A(t+A)^{-1} x\, dt$ when $x \in D(A)$, the domain of A; [4–I, p. 299].

K-2. If $0 < \mathrm{Re}(\alpha) < \sigma < n$, n a positive integer, then

$$A^\alpha x = \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int\limits_0^\infty t^{\alpha-1}\big(A(t+A)^{-1}\big)^m x\, dt$$

for $x \in D(A^N)$ when $N > m > n$; [4-II, P. 92].

K-3. If $(-A)$ generates a bounded strongly continuous semi-group T_t on X, then if $x \in D(A)$ and $0 < \mathrm{Re}(\alpha) < \sigma < 1$,

$$A^\alpha x = \Gamma(-\alpha)^{-1} \int\limits_{0+}^\infty (T_t x - x) t^{-\alpha-1} dt; \qquad [4\text{–I, p. } 325].$$

More formally, K-1 and K-3 define an operator A_σ^α on a subspace D^σ of X; D^σ is defined in [4-I]. If A_+^α denotes the smallest closed extension of A_σ^α, whose existence is proved in [4-I, Prop. 4.1], then $A^\alpha = A_+^\alpha$. Similarly K-2 defines an operator on a natural subspace of X and its smallest closed extension is $A_+^\alpha = A^\alpha$ as is shown in [4-II]. When $\mathrm{Re}(\alpha) < 0$, $A_{-\sigma}^\alpha$ is defined by equation 4.10 of [4-I, p. 304] and $A_{-\sigma}^\alpha$ is shown to have a smallest closed extension A_-^α which is independent of σ. When $\mathrm{Re}(\alpha) = 0$, $A^\alpha x$ is defined by equation 4.11 of [4-I, p. 305] for $x \in D^\sigma \cap R^\tau$. There is the important

K-4. For every complex α, $A_{\sigma\tau}^\alpha$ has the smallest closed extension A_0^α which is independent of σ and τ when $-\tau < \mathrm{Re}(\alpha) < \sigma$. If $\mathrm{Re}(\alpha) > 0$, $A_0^\alpha = A_+^\alpha$ on $D(A_+^\alpha) \cap \overline{R(A)}$ and if $\mathrm{Re}(\alpha) < 0$, $A_0^\alpha = A_-^\alpha$.

If A has a bounded inverse, $R^\sigma = X$ and A_-^α is everywhere defined and analytic if $\mathrm{Re}(\alpha) < 0$. If $x \in D^\sigma$, $A^\alpha x$ is analytic in $\mathrm{Re}(\alpha) < \sigma$. If $-(n+1) < \mathrm{Re}(\alpha) < 0$

$$A_-^\alpha = \frac{-\sin \pi\alpha}{\pi} \frac{n!}{(\alpha+1)\dots(\alpha+n)} \int\limits_0^\infty t^{\alpha+n}(t+A)^{-n-1} dt$$

and

K-5. If $\mathrm{Re}(\alpha) > 0$, then $A_+^\alpha = A_0^\alpha$ is the inverse of $A_0^{-\alpha} = A_-^{-\alpha}$; the $D(A_+^\alpha)$ is contained in the $R(A_-^{-\alpha})$. See Section 5 of [4-I].

K-6. (i) If $\mathrm{Re}(\alpha)\cdot\mathrm{Re}(\beta) > 0$, then $A_\mp^\alpha A_\mp^\beta = A_0^\alpha A_\mp^\beta = A_\mp^{\alpha+\beta}$ in the sense of the product of operators.

(ii) If α and β are any complex numbers, then $[A_0^\alpha A_0^\beta]_C = A_0^{\alpha+\beta}$, where $[T]_C$ denotes the smallest closed extension of T.

(iii) If A has a bounded inverse and if $\mathrm{Re}(\alpha) > 0$, then $A_0^\alpha A_0^\beta = A_0^{\alpha+\beta}$. See Section 7 of [4-I].

From the assumption that $\|t(t+A)^{-1}\| \leqslant M$ for $t > 0$ and the resolvent equation it follows that $(t+A)^{-1}$ exists for t in the sector $|\arg(t)| < \mathrm{Arc}\sin(M^{-1})$ and that $t(t+A)^{-1}$ is bounded on each ray of this sector. Let $M(\theta) = \sup\{\|t(t+A)^{-1}\|\colon |\arg(t)| = \theta\}$, $\theta > 0$; $M(\theta)$ is an increasing function of θ. An operator A is said to be of type $(\omega, M(\theta))$, $0 \leqslant \omega < \pi$, if A is closed, densely defined, the resolvent set of $(-A)$ contains the sector $|\arg(t)| < \pi-\omega$, and $\sup\{\|t(t+A)^{-1}\|\colon |\arg(t)| = \theta\} \leqslant M(\theta) < \infty$ holds for all $0 \leqslant \theta < \pi-\omega$. An operator A is of type $(\omega, M(\theta))$ for an $\omega < \pi/2$ if and only if $(-A)$ generates a semi-group T_t which has an analytic extension to the sector $|\arg(t)| < \pi/2-\omega$ such that the extension is uniformly bounded on each sector $|\arg(t)| \leqslant \pi/2-\omega-\varepsilon$, for $\varepsilon > 0$.

K-7. If A is an operator of type $(\omega, M(\theta))$ and $0 < \alpha\omega < \pi/2$, then $(-A_+^\alpha)$ is the generator of a strongly continuous semi-group $\exp(-tA_+^\alpha)$ which is analytic in the sector $|\arg(t)| < \pi/2-\alpha\omega$ and uniformly bounded on each smaller sector $|\arg(t)| < \pi/2-\alpha\omega-\varepsilon$, for $\varepsilon > 0$. See Section 10 of [4-I].

K-8. Let A be of type $(\omega, M(\theta))$. Then $(A_+^\alpha)^\beta = A_+^{\alpha\beta}$ if $0 < \alpha < \pi/\omega$ and $\mathrm{Re}(\beta) > 0$.

K-9. If $0<\alpha<1$ and if $T_t = \exp(-tA)$, $T_t^\alpha x = \exp(-tA_+^\alpha)x = \int_0^\infty T_s x N(\alpha, t, s)\,ds$, where $N(\alpha, t, s) = (2\pi i)^{-1}\int_{\sigma-i\infty}^{\sigma+i\infty} \exp(us-tu^\alpha)\,du$; [7].

Let $P_z = \exp(-zT)$ be the Poisson integral defined in the introduction; P_z and T have the following properties:

P-1. P_z is a strongly continuous, contraction semi-group on $L_p(H)$; [2].

P-2. P_z admits an extension as an analytic semi-group the sector $|\arg(z)| < \pi/4$; P_z is uniformly bounded in each smaller sector.

This follows from the fact that $N_t(z)$ is analytic in $\mathrm{Re}(z) > 0$ and the fact that the integral $P_z(f) = \int_0^\infty H_t(f)N_t(z)\,dt/t$ converges uniformly in compacts of $|\arg(z)| < \pi/4-\varepsilon$, for $\varepsilon > 0$.

P-3. If $H_t f = \int_H T_y f\, dn_t \circ B^{-1}(y) = \exp(-tA)f$, then $T = (A)^{1/2}$; [7, p. 264].

P-4. If B is a one-one Hilbert–Schmidt operator, T is one-one on $L_p(H)$ and the range of T, $R(T)$, is dense in $L_p(H)$.

If T is one-one on $L_p(H)$, the $R(T)$ is dense in $L_p(H)$ by Theorem 3.1 of [4-I]. To show that T is one-one, it suffices to show that T^2 is one-one. If $T^2 f = 0$, $H_t f = f$ for all $t > 0$. If A_h denotes the infinitesimal generator of T_{tBh}, $t > 0$, then $A_h H_{t^2}(f) = A_h f$ for all h in H. From the formula for $A_h H_{t^2}$ given in [2], it can be seen that $Z_{\delta\varrho}(f) = \int_\delta^\varrho A_h H_{t^2}(f)\,dt = A_h f(\varrho-\delta)$ is a bounded operator on $L_p(H)$ with norm at most $K\log(\varrho/\delta)$. After dividing by $(\varrho-\delta)$, one shows that this inequality implies that $A_h f = 0$ for all h in H by letting $\varrho\to\infty$. $A_h f = 0$ for all h in H implies that $T_{tBh} f = f$ for all $t > 0$. A well-known result due to Hormander implies that for tame functions g, $\|T_{tBh}g+g\|_p \to 2^{1/p}\|g\|_p$ as t tends to $+\infty$. Since the tame functions in $L_p(H)$ are dense in $L_p(H)$, this limit holds for all f in $L_p(H)$. For the f with $T^2 f = 0$, $2\|f\|_p = \|T_{tBh}f+f\|_p \to 2^{1/p}\|f\|_p$. This implies that $f = 0$ and T is one-one.

2. Basic properties of Bessel potentials. In Section 3 of [3] we studied the Bessel potential J^α which is mentioned in the introduction of the present paper. We showed that for $\mathrm{Re}(\alpha) > 0$, J^α is bounded on $L_p(H)$, J^α is strongly analytic, $\lim\{J^\alpha f\colon |\arg(\alpha)| \leqslant \theta < \pi/2,\ \alpha\to 0\} = f$, $J^\alpha J^\beta = J^{\alpha+\beta}$, J^α is the α-th Komatsu power of J', J^α is one-one on $L_p(H)$, $R(J^\alpha)$ is dense in $L_p(H)$, $T^\alpha J^\alpha$ is bounded on $L_p(H)$, and $(-A_h)^\alpha J^\alpha$ is bounded on $L_p(H)$ when A_h is the infinitesimal generator of the translation semi-group T_{tBh}, $t > 0$, and B is the one-one Hilbert–Schmidt operator of Section 1. In Sections 2 and 3 of the present paper we shall study abstract Bessel potentials by using the theory of fractional powers of operators.

Let T be a one-one, closed, densely defined operator on a reflexive Banach space X such that $(-T)$ and $(-T^2)$ generate strongly continuous, contraction semi-groups $P_z = \exp(-zT)$ and $H_t = \exp(-tT^2)$. Then P_z can be written as an integral of H_t as in Section 1 and P_z is analytic in $|\arg(z)| < \pi/4$. Let J_1^α and J_2^α be as in Section 1 when $\mathrm{Re}(\alpha) > 0$. Recall that since X is reflective, it $T_t = \exp(-tA)$ is a bounded strongly continuous semi-group, then $X = N(A)\otimes\overline{R(A)}$ by Theorem 3.1 of [4-I]. Since T is one-one, $R(T)$ and $R(T^2)$ are dense in X. We begin by listing some of the basic properties of the J_k^α.

THEOREM 1. *Let α and β be positive real numbers. Then for $k = 1, 2$:*

(1) $\|J_k^\alpha\| \leqslant 1$.

(2) $\lim\{J_k^\alpha f\colon \alpha\to 0\} = f$.

(3) $J_k^\beta = (J_k^1)^\beta$, *the β-th power of J_k^1.*

(4) $J_k^\alpha = (1+T^k)^{-\alpha/k}$.

(5) $R(J_k^\alpha)$ *is dense in* $L_p(H)$.

(6) $J_k^\alpha J_k^\beta = J_k^{\alpha+\beta}$.

(7) J_k^α *is one-one on* $L_p(H)$.

Proof. Let T_t denote a contraction semi-group, $T_t = \exp(-tA)$, which stands for H_t or P_y in the proof. Set $J^\beta f = \Gamma(\beta)^{-1}\int_0^\infty T_t f t^{\beta-1}e^{-t}dt$. (1) follows from Minkowski's integral inequality. For (2) let $f \in X$, $\varepsilon > 0$, and $\delta > 0$ such that if $0 < t < \delta$, $\|T_t f - f\| < \varepsilon$. Choose $\eta > 0$ such that if $0 < \beta < \eta$, $\Gamma(\beta)^{-1}\int_\delta^\infty t^{\beta-1}e^{-t}dt < \varepsilon$. Then if $\beta < \eta$, $\|J^\beta f - f\|$ $\leqslant \|\Gamma(\beta)^{-1}\int_0^\delta (T_t f - f)t^{\beta-1}e^{-t}dt\| + \|\Gamma(\beta)^{-1}\int_\delta^\infty (T_t f - f)t^{\beta-1}e^{-t}dt\| \leqslant \varepsilon + 2\varepsilon\|f\|$. This proves (2). To prove (3) and (4) let $\beta > 0$ and set $Jf = J^1 f = \int_0^\infty T_t f e^{-t}dt$ $= (1+A)^{-1}f$; we will show that $(1+A)^{-\beta}f = J^\beta f$. Let $0 < \beta < 1$; then for x in X

$$\begin{aligned} J^\beta x &= \Gamma(\beta)^{-1}\int_0^\infty u^{\beta-1}e^{-u}T_u x\,du \\ &= \Gamma(\beta)^{-1}\Gamma(1-\beta)^{-1}\int_0^\infty\Big(\int_0^\infty t^{-\beta}e^{-u}e^{-tu}dt\Big)T_u x\,du \\ &= \Gamma(\beta)^{-1}\Gamma(1-\beta)^{-1}\int_0^\infty\Big(\int_1^\infty (t-1)^{-\beta}e^{-tu}dt\Big)T_u x\,du \\ &= \Gamma(\beta)^{-1}\Gamma(1-\beta)^{-1}\int_1^\infty (t-1)^{-\beta}\int_0^\infty e^{-tu}T_u x\,du\,dt \\ &= \Gamma(\beta)^{-1}\Gamma(1-\beta)^{-1}\int_0^\infty (t-1)^{-\beta}(t+A)^{-1}x\,dt \\ &= \Gamma(\beta)^{-1}\Gamma(1-\beta)^{-1}\int_0^\infty v^{-\beta}(v+1+A)^{-1}x\,dv. \end{aligned}$$

Since $(v+1+A)^{-1} = J(vJ+1)^{-1}$, set $w = v^{-1}$ to get

$$J^\beta x = \frac{\sin\pi\beta}{\pi}\int_0^\infty w^{\beta-1}J(w+J)^{-1}x\,dw$$

by K-1. This completes the proof of (3) and (4) for $0 < \beta < 1$; the general result now follows from K-6. (5) follows from K-5 since $J_k^\alpha = (1+T^k)^{-\alpha/k}$ $= [(1+T^k)^{\alpha/k}]^{-1}$ and $D\big((1+T^k)^{\alpha/k}\big) \subset R(J_k^\alpha)$. (6) is a consequence of (4) and K-6; an elementary direct computation also verifies the desired identity. If $J_k^\alpha(f) = 0$, $J_k^{\alpha+r}(f) = 0$ for $r > 0$ by (6). Since $J_k^\alpha(f)$ extends to an analytic function in $\mathrm{Re}(\alpha) > 0$, the uniqueness principle for analytic functions implies that $J_k^\alpha(f) = 0$ for all $\mathrm{Re}(\alpha) > 0$. By (1), $f = 0$. So (7) holds and J_k^α is one-one.

Let T^α, $\alpha > 0$, be as in K-1, K-2, or K-3; we shall examine $T^\alpha J_k^\alpha$.

THEOREM 2. *$T^\alpha J_k^\alpha$ is a bounded operator on X for $k = 1, 2$ if $\alpha > 0$. These operators are given by:*

$$T^\alpha J_1^\alpha x = x - \frac{\sin\pi\alpha}{\pi}\int_0^1 J_u x u^\alpha(1-u)^{-\alpha}du \quad \textit{for} \quad 0 < \alpha < 1,$$

and

$$T^\alpha J_2^\alpha x = x - \frac{\sin(\pi\alpha/2)}{\pi}\int_0^1 K_u x u^{\alpha/2}(1-u)^{-\alpha/2}du$$

for $0 < \alpha < 2$, *where* $J_u x = \int_0^\infty e^{-ut}P_t x\,dt$ *and* $K_u x = \int_0^\infty e^{-ut}H_t x\,dt$.

Proof. To prove the theorem for J_k^α, $k = 1, 2$ note first that $T(1+T)^{-1}$ $= 1-(1+T)^{-1}$ and $T^2(1+T^2)^{-1} = 1-(1+T^2)^{-1}$ are bounded operators, so that we need only prove the theorem for $0 < \alpha < k$ for J_k^α; K-6 can be used to complete the proof of the boundedness of $T^\alpha J_k^\alpha$. Let $T_t = \exp(-tA)$ be a bounded, strongly continuous semi-group on X. We shall show that

$$A^\beta(1+A)^{-\beta}x = x - B(\beta, 1-\beta)^{-1}\int_0^1 (u+A)^{-1}xu^\beta(1-u)^{-\beta}du$$

for $0 < \beta < 1$. Here $B(x, y)$ is the β-function.

For $0 < \beta < 1$, $J^\beta x = \Gamma(\beta)^{-1}\int_0^\infty T_t x t^{\beta-1}e^{-t}dt = (1+A)^{-\beta}x$ and $A^\beta x$ $= \Gamma(-\beta)^{-1}\int_0^\infty (T_y x - x)y^{-1-\beta}dy$. If $x \in D(A)$,

$$A^\beta x = -\Gamma(1-\beta)^{-1}\int_0^\infty T'_y x y^{-\beta}dy,$$

where $T'_y x = \dfrac{\partial}{\partial y}T_y x$. Let $Lg(u)$ denote the Laplace transform of g at u. Then

$$A^\beta J^\beta x = -\frac{\sin\pi\beta}{\pi}\int_1^\infty L(t^\beta A_0^\beta T_t x)(u)\,du,$$

where $A_0^\beta = \Gamma(-\beta)A^\beta$. Then

$$t^\beta A_0^\beta(T_t x) = t^\beta\int_{0+}^\infty \frac{\partial}{\partial y}T_{y+t}xy^{-\beta}dy = \int_{0+}^\infty \frac{\partial}{\partial t}T_{y+t}xt^\beta y^{-\beta}dy = \int_{0+}^\infty tT'_tT_{ty}xy^{-\beta}dy.$$

But $T_t' T_{ty} x = (y+1)^{-1} \frac{\partial}{\partial t} T_{t(y+1)} x$. Then

$$L(t^\beta A_0^\beta T_t x)(u) = \int_{0+}^{\infty} (y+1)^{-1}\Big(-\frac{\partial}{\partial u} u\Big) L(T_{t(y+1)} x)(u) y^{-\beta} dy$$

$$= \int_{0+}^{\infty} (y+1)^{-2}\Big(-\frac{\partial}{\partial u} u\Big) J_{u(y+1)^{-1}}(x) y^{-\beta} dy,$$

where $J_v x = \int_{0+}^{\infty} e^{-vt} T_t x\, dt$. Since $\Big\|\Big(-\frac{\partial}{\partial u} u\Big) J_{u(y+1)^{-1}}(x)\Big\| \leqslant K(y+1)u^{-1}\|x\|$, we consider an interchange of integrals in $\int_1^R L\big(t^\beta A_0^\beta T_t(x)\big)(u)\, du$.

$$\int_1^R (1+y)^{-2}\Big(-\frac{\partial}{\partial u} u\Big) J_{u(1+y)^{-1}}(x)\, du = (y+1)^{-2}[J_{(1+y)}-1 - RJ_{R(1+y)}-1](x).$$

Since $\|RJ_{R(y+1)^{-1}}(x)\| \leqslant K(y+1)\|x\|$, the dominated convergence theorem implies that

$$A^\beta J^\beta x = -\frac{\sin\pi\beta}{\pi}\int_{0+}^{\infty} (J_{(1+y)^{-1}}(x) - (y+1)x) y^{-\beta}(1+y)^{-2} dy$$

$$= -\frac{\sin\pi\beta}{\pi}\int_{0+}^{1} (J_u x - u^{-1}x) u^\beta (1-u)^{-\beta} du$$

$$= x - \frac{\sin\pi\beta}{\pi}\int_0^1 J_u x u^\beta (1-u)^{-\beta} du.$$

Since $\|J_u x\| \leqslant Ku^{-1}\|x\|$, $\|A^\beta J^\beta x\| \leqslant (1+K)\|x\|$ by the triangle inequality for integrals. This completes the proof of Theorem 2.

Remark 2.1. When P_t and H_t are semi-groups on $L_p(H)$ of the type given in the introduction, $T^\alpha J_k^\alpha$ is given by convolution with a measure on H for all $\alpha > 0$; see [3]. If $X = L_p$ of an Abelian group and if P_t and H_t are given by convolution with measures, it is easy to see from Theorem 2 above or from Theorem 4 of [3], that $T^\alpha J_k^\alpha$ is given by convolution with a measure.

Remark 2.2. Many of the properties of the J_k^α in Theorem 1 also hold for complex α when $\mathrm{Re}(\alpha) > 0$. Theorem 2 holds for these complex α with no change in proof or notation.

Remark 2.3. For any k in $0 < k \leqslant 2$, $J_k^\alpha = (1+T^k)^{-\alpha/k}$ defines a strongly analytic semi-group in α when our present assumptions on $P_z = \exp(-zT)$ are in force. It will not be hard to see in what follows that any of the J_k^α, $k = k_1, k_2$ are equivalent for $0 < k_1, k_2 \leqslant 2$. The basis for this assertion is the set of results presented in Section 10 of [4-I] regarding the semi-groups generated by fractional powers of operators.

3. Equivalence of J_1^α and J_2^α. By statement (7) of Theorem 1, J_k^α is one-one on X for $k = 1, 2$. Define $X_k^\alpha = R(J_k^\alpha)$ with the norm $\|y\|_{\alpha,k} = \|x\|$ if $J_k^\alpha x = y$. In this section we shall complete the proof of the equivalence of the norms $\|_{\alpha,k}$ by proving that $(J_k^\alpha)^{-1} J_i^\alpha$ is a bounded operator on X for $k, i = 1, 2$ and $\alpha > 0$; this leads to the conclusion that X_k^α is equivalent to $D(T^\alpha)$ when this domain is equipped with the graph norm. The following lemma will be useful.

LEMMA 3.1. *Let $(-A)$ be the infinitesmal generator of a bounded, strongly continuous semi-group on X and let $0 < \alpha < 1$. If $x \in D(A)$, $(1+A)^\alpha x = A^\alpha x + Bx$, where B is a bounded operator on X.*

Proof. By K-1

$$(1+A)^\alpha x = \frac{\sin\pi\alpha}{\pi}\int_0^\infty t^{\alpha-1}(A+1)(t+1+A)^{-1} x\, dt$$

$$= \frac{\sin\pi\alpha}{\pi}\int_0^\infty t^{\alpha-1}(t+1+A)^{-1} x\, dt + \frac{\sin\pi\alpha}{\pi}\int_0^\infty t^{\alpha-1} A(t+1+A)^{-1} x\, dt.$$

Since $\|(t+1+A)^{-1}\| \leqslant K(t+1)^{-1}$, the first integral on the right represents a bounded operator on X. By the resolvent equation, $(t+1+A)^{-1} - (t+A)^{-1} = -(t+1+A)^{-1}(t+A)^{-1}$. Then

$$\frac{\sin\pi\alpha}{\pi}\int_0^\infty t^{\alpha-1} A(t+1+A)^{-1} x\, dt$$

$$= A^\alpha x - \frac{\sin\pi\alpha}{\pi}\int_0^\infty t^{\alpha-1} A(t+1+A)^{-1}(t+A)^{-1} x\, dt.$$

Since $\|A(t+A)^{-1}\| \leqslant K$ and $\|(t+1+A)^{-1}\| \leqslant K(t+1)^{-1}$, the last integral on the right represents a bounded operator on X. Thus $(1+A)^\alpha$ has the desired form.

THEOREM 3. *If $\alpha > 0$, $(J_k^\alpha)^{-1} J_i^\alpha$ are bounded operators on X if $i, k = 1, 2$.*

Proof. Consider $(1+T^2)^{\alpha/2}(1+T)^{-\alpha}$ and write $\alpha = 2n+\beta$, where $0 \leqslant \beta < 2$. Then by Lemma 3.1, $(1+T^2)^{\beta/2} = (T^2)^{\beta/2} + B$, where B is a bounded operator. By K-8, $(T^2)^{\beta/2} = T^\beta$. Since $(1+T^2)^{\alpha/2} = (1+T^2)^n (T^\beta + B) = \sum_{k=0}^{n} A_k T^{2k} + \sum_{k=0}^{n} B_k T^{2k+\beta}$, where A_k and B_k are bounded operators on X, and since $T^\gamma(1+T)^{-\alpha} = T^\gamma(1+T)^{-\gamma}(1+T)^{-(\alpha-\gamma)}$ for $\gamma \leqslant \alpha$, Theorem 2 implies that $(1+T^2)^{\alpha/2}(1+T)^{-\alpha}$ is a bounded operator on X.

Similarly, write $\alpha = N+\gamma$, where N is a non-negative integer and $0 \leqslant \gamma < 1$. Then $(1+T)^\alpha = (1+T)^N(T^\gamma + B) = \sum_{k=0}^{N} A_k T^k + \sum_{k=0}^{N} B_k T^{k+\gamma}$, where A_k and B_k are bounded operators on X. Since if $\delta \leqslant \alpha$, $T^\delta(1+T^2)^{-\alpha/2} = (T^2)^{\delta/2}(1+T^2)^{-\delta/2}(1+T^2)^{-(\alpha-\delta)/2}$ by K-8 and Theorem 2 implies that $(1+T)^\alpha(1+T^2)^{-\alpha/2}$ is a bounded operator on X. This completes the proof of Theorem 3.

Our methods give the following

COROLLARY 3.1. *X_k^α, $k = 1, 2$, is equivalent to $D(T^\alpha)$ when $D(T^\alpha)$ is equipped with the graph norm.*

Proof. By Theorem 3, it suffices to prove that $X_1^\alpha = D(T^\alpha)$. If $y = J_1^\alpha x$, then $\|y\| + \|T^\alpha J_1^\alpha x\| \leqslant K(\alpha)\|x\| = K(\alpha)\|y\|_{\alpha,1}$ by Theorem 2. Write $\|y\|_{\alpha,1} = \|x\| = \|(1+T)^\alpha y\|$, and expand $(1+T)^\alpha = \sum_{k=0}^{N} A_k T^k + \sum_{k=0}^{N} B_k T^{k+\beta}$ by Lemma 3.1. By Theorem 6.5 of [4-I], $D(T^\alpha) \subset D(T^\gamma)$ continuously if $\gamma < \alpha$. Thus $\|(1+T)^\alpha y\| \leqslant K(\alpha)(\|y\| + \|T^\alpha y\|)$ and the proof is complete.

COROLLARY 3.2. *X_k^α consists precisely of those elements x of X for which $T^\alpha x$ is also in X.*

Stein [6] has studied the questions dealt with in this section over finite dimensional Euclidean spaces. Fourier multiplier techniques are used in [6] to prove the above results.

4. BOUNDEDNESS OF $J^{i\gamma}$. In this section we shall prove that $J^{i\gamma}$, γ real, is a group of bounded operators on $L_p(H)$ if $1 < p < \infty$, and we will study the relationship between $J^{i\gamma}$ and the closed operator $(1+A)^{-i\gamma}$. If $\gamma = 0$, $J^{i\gamma}$ is the identity on $L_p(H)$ and if $\gamma \neq 0$, set

$$J^{i\gamma}f = \Big[\lim_{\varepsilon\to 0} \Gamma(i\gamma)^{-1}\int_\varepsilon^\infty P_t f t^{i\gamma-1}e^{-t}dt + \Gamma(i\gamma+1)^{-1}\varepsilon^{i\gamma}f\Big].$$

Here $P_t f$ is the Poisson integral of f as defined and studied in [2], and which is briefly described in Section 1 of this paper. In [2] it was shown that there is a unique Borel probability measure $p(E)$ on H such that if $p_t(E) = p(E/t)$ for $t > 0$, $P_t f = \int_H T_y f\, dp_t(y) = \int_H T_{ty} f\, dp(y)$ for all f in $L_p(H)$; $y \to T_y$ is the regular representation of the additive group of H acting on $L_p(H)$.

For $\alpha = \beta + i\gamma$, $\beta \geqslant 0$ and γ real, define $J_\varepsilon^\alpha f = f$ if $\alpha = 0$ and if $\operatorname{Re}\alpha \neq 0$ define

$$J_\varepsilon^\alpha f = \Gamma(\alpha)^{-1}\int_\varepsilon^\infty P_t f t^{\alpha-1}e^{-t}dt + \varepsilon^\alpha\Gamma(\alpha+1)^{-1}f \quad \text{for } \varepsilon > 0.$$

Then for $\operatorname{Re}\alpha > 0$, J_ε^α converges strongly to $J^\alpha f$ as ε tends to zero. Define $J^\alpha f = \lim\{J_\varepsilon^\alpha f\colon \varepsilon \to 0^+\}$ for $\operatorname{Re}\alpha \geqslant 0$ if this limit exists. Let $1 < p < \infty$ and $1/p + 1/q = 1$.

THEOREM 4. *$J^{i\gamma}$ is a bounded operator on $L_p(H)$ for all real γ and $\|J^{i\gamma}\| \leqslant Kpq(|\gamma|+1)^2|\Gamma(i\gamma+1)|^{-1}$.*

Proof. First consider $(T_\varepsilon^a f)(x) = \int_\varepsilon^\infty f(x-t)\exp(-t/a)t^{i\gamma-1}dt$ on $L_p\big((-\infty, \infty), dx\big)$. Let $g(t) = t^{i\gamma-1}$ if $t > 0$ and $g(t) = 0$ if $t < 0$. Then for $a > 0$,

$$(T_\varepsilon^a f)(x) = \int_{|t|>\varepsilon} f(x-t)\exp(-|t|/a)g(t)\,dt.$$

Since

$$\exp(-|t|/a) = (\pi)^{-1}\int_{-\infty}^{\infty} e^{-ity}\frac{a\,dy}{1+y^2a^2},$$

set $h(a, y) = a(\pi)^{-1}(1+y^2a^2)^{-1}$ and write

$$(T_\varepsilon^a f)(x) = \int_{-\infty}^{\infty} e^{-ixy}\int_{|t|>\varepsilon} f(x-t)e^{i(x-t)y}g(t)\,dt\,h(a, y)\,dy.$$

By Minkowski's integral inequality and by Theorem 1 of [5],

$$\|T_\varepsilon^a\|_p \leqslant Kpq(|\gamma|+1)^2|\gamma|^{-1} \quad \text{for all} \quad \varepsilon > 0.$$

We may write

$$J_\varepsilon^{i\gamma}f = \Gamma(i\gamma)^{-1}\int_H\Big[\int_\varepsilon^\infty T_{ty}f t^{i\gamma-1}e^{-t}dt + \frac{\varepsilon^{i\gamma}}{i\gamma}f\Big]dp(y).$$

If f is a bounded tame function on H, then the rotational invariance of the normal distribution can be used as in the proof of Proposition 3 of [2] to show that as a consequence of the above inequality for $T_\varepsilon^{\|y\|}$,

$$\Big\|\int_\varepsilon^\infty T_{ty}f t^{i\gamma-1}e^{-t}dt\Big\|_p \leqslant K(\gamma, p)\|f\|_p$$

for all $\varepsilon > 0$. The bounded tame functions are dense in $L_p(H)$. Minkowski's integral inequality can be used to complete the proof that $\|J_\varepsilon^{i\gamma}f\|_p \leqslant Kpq(|\gamma|+1)^2|\Gamma(i\gamma+1)|^{-1}$ for all $\varepsilon > 0$.

To prove convergence as $\varepsilon \to 0^+$, write

$$J_\varepsilon^{i\gamma}f = \Gamma(i\gamma)^{-1}\int_H\Big[\int_1^\infty T_{ty}f t^{i\gamma-1}e^{-t}dt + \int_\varepsilon^1 T_{ty}f t^{i\gamma-1}(e^{-t}-1)\,dt + \Big(\int_\varepsilon^1 T_{ty}f t^{i\gamma-1}dt + \frac{\varepsilon^{i\gamma}f}{i\gamma}\Big)\Big]dp(y).$$

The first and second integrals on the right converge absolutely. If f is a bounded tame function with bounded derivatives, write the third quantity as

$$\int\limits_{\varepsilon}^{1} T_{ty}f\,t^{i\gamma-1}dt+\frac{\varepsilon^{i\gamma}}{i\gamma}f=\int\limits_{\varepsilon}^{1}(T_{ty}f-f)\,t^{i\gamma-1}dt+\frac{f}{i\gamma}.$$

Since $\|T_{ty}f-f\|_p\leqslant Kt\|y\|$ and since $\int\limits_H \|y\|\,dp(y)<\infty$, the third quantity on the right above converges as $\varepsilon\to 0^+$. Thus $J_{\varepsilon}^{i\gamma}$ converges strongly to a bounded operator $J^{i\gamma}$ and $\|J^{i\gamma}\|\leqslant Kpq(|\gamma|+1)^2|\Gamma(i\gamma+1)|^{-1}$.

The next theorem shows that $\int\limits_{\varepsilon}^{\infty} P_tft^{i\gamma-1}dt$ converges with respect to a certain summability method.

THEOREM 5. *$J^{i\gamma}f=\lim\{J^{\beta+i\gamma}f\colon\beta\to 0^+\}$ for all f in $L_p(H)$.*

Proof. The integral $\Gamma(\alpha)^{-1}\int\limits_1^{\infty}P_tft^{\alpha-1}e^{-t}dt$, $\alpha=\beta+i\gamma$, converges uniformly to $\Gamma(i\gamma)^{-1}\int\limits_1^{\infty}P_tft^{i\gamma-1}e^{-t}dt$ as $\beta\to 0^+$. It is sufficient to consider the limit of $\int\limits_0^1 P_tft^{\alpha-1}e^{-t}dt$. This last integral is

$$\int\limits_0^1 \beta x^{\beta-1}\int\limits_x^1 P_tft^{i\gamma-1}e^{-t}dt\,dx.$$

The function $\beta x^{\beta-1}$ gives a regular summability method on $0\leqslant x\leqslant 1$.

Since the integral $\int\limits_{0+}^{1}P_tft^{\alpha-1}(e^{-t}-1)\,dt$ converges strongly to $\int\limits_{0+}^{1}P_tft^{i\gamma-1}(e^{-t}-1)\,dt$ as $\beta\to 0^+$, we consider

$$\lim_{\beta\to\infty}\int\limits_0^1\beta x^{\beta-1}\int\limits_x^1 P_tft^{i\gamma-1}dt\,dx.$$

From Section 3 of [5], we have that this last limit exists if

$$\lim_{\varepsilon\to\infty}\left[\int\limits_{\varepsilon}^{1}P_tft^{i\gamma-1}\,dt+\frac{\varepsilon^{i\gamma}}{i\gamma}f\right]$$

exists; when the last limit exists these limits are equal. Theorem 4 shows that the last limit exists, so that

$$J^{i\gamma}f=\lim\{^{\beta+i\gamma}f\colon\beta\to 0^+\}.$$

COROLLARY 4.1. *If α is a real positive number, $J^{\alpha}J^{i\gamma}=J^{\alpha+i\gamma}$.*

Proof. For $0<\varepsilon<\alpha$, $\|J^{\alpha}J^{i\gamma}f-J^{\alpha+i\gamma}f\|_p\leqslant\|J^{\varepsilon}J^{i\gamma}f-J^{\varepsilon+i\gamma}f\|_p$ by statements 6 and 1 in Theorem 1. By the triangle inequality $\|J^{\varepsilon}J^{i\gamma}f-J^{\varepsilon+i\gamma}f\|_p\leqslant\|J^{\varepsilon}J^{i\gamma}f-J^{i\gamma}f\|_p+\|J^{i\gamma}f-J^{\varepsilon+i\gamma}f\|_p$. By statement 2 of Theorem 1 and by Theorem 5, the terms on the right tend to 0 as $\varepsilon\to 0^+$. This verifies the required identity.

COROLLARY 4.2. *$T^{\alpha}=J^{\alpha+i\gamma}$, $\alpha\geqslant 0$, forms a bounded, strongly continuous family of operators on $L_p(H)$ with $\|T^{\alpha}\|\leqslant Kpq(|\gamma|+1)^2|\Gamma(i\gamma+1)|^{-1}$.*

Proof. Since $T^{\alpha}=J^{\alpha}J^{i\gamma}$ and J^{α} is strongly continuous for $\alpha>0$ and $J^{i\gamma}$ is bounded, T^{α} is strongly continuous. The estimate for $\|T^{\alpha}\|$ follows from the estimate for J^{α}, $\|J^{\alpha}\|\leqslant 1$. and the estimate for $\|J^{i\gamma}\|$ which is given in Theorem 4.

COROLLARY 4.3. *$J^{i\gamma}J^{i\beta}=J^{i(\gamma+\beta)}$ for all real γ and β.*

Proof. If $\varepsilon>0$, $J^{2\varepsilon+i\gamma}J^{i\beta}=J^{i\gamma}J^{2\varepsilon+i\beta}=J^{2\varepsilon+i(\gamma+\beta)}$ by part 6 of Theorem 1 and Corollary 4.1. By Theorem 5, if we take the limit as $\varepsilon\to 0^+$ in this equation, we get the desired result.

COROLLARY 4.4. *$J^{i\gamma}$, γ real, is a strongly continuous group of operators on $L_p(H)$ with $J^{i0}=$ the identity and $(J^{i\gamma})^{-1}=J^{-i\gamma}$.*

Proof. Because of Corollary 4.3 we need only show that $\lim\{J^{i\gamma}f\colon\gamma\to 0\}=f$ for each f in $L_p(H)$. The bound on $\|J^{i\gamma}\|$ is

$$Kpq(|\gamma|+1)^2|\Gamma(i\gamma+1)|^{-1}=Kpq(\pi|\gamma|)^{-1/2}(\sinh\pi|\gamma|)^{+1/2}(|\gamma|+1)^2,$$

since $|\Gamma(i\gamma)|=(\pi)^{1/2}(\gamma\sinh\pi\gamma)^{-1/2}$ as follows from the well-known identity for $\Gamma(z)\Gamma(1-z)$. Thus $\|J^{i\gamma}\|$ is bounded on any compact neighborhood of $\gamma=0$ and $\lim\{J^{\varepsilon+i\gamma}f\colon\varepsilon\to 0^+\}=J^{i\gamma}f$ uniformly on $-1\leqslant\gamma\leqslant 1$. Because of the strong continuity of J^{α} in $\operatorname{Re}\alpha>0$, the following equality completes the proof:

$$\lim_{\gamma\to 0}J^{i\gamma}f=\lim_{\gamma\to 0}\lim_{\varepsilon\to 0}J^{\varepsilon+i\gamma}f=\lim_{\varepsilon\to 0}\lim_{\gamma\to 0}J^{\varepsilon+i\gamma}f=\lim_{\varepsilon\to 0}J^{\varepsilon}f=f.$$

COROLLARY 4.5. *$J^{i\gamma}f=(1+T)^{-i\gamma}f=[(1+T)^{i\gamma}]^{-1}f$, for all f in $L_p(H)$, and $(1+T)^{-i\gamma}$ is a bounded operator on $L_p(H)$ for all real γ.*

Proof. By Corollary 5.3 of [4-I], $J^{\alpha+i\gamma}$ is the inverse of $(1+T)^{\alpha+i\gamma}$ if $\alpha>0$. Because the fractional powers are strongly continuous on a dense set of $L_p(H)$ in a strip $-\tau<\alpha<\sigma$, $-\infty<\gamma<\infty$, by Theorem 8.2 of [4-I], we have that $J^{i\gamma}f=(1+T)^{-i\gamma}f=[(1+T)^{i\gamma}]^{-1}f$ for a dense set of f's in $L_p(H)$. Since the $J^{\alpha+i\gamma}=((1+T)^{-1})^{\alpha+i\gamma}=(1+T)^{-\alpha-i\gamma}$ are uniformly bounded in $\alpha>0$, a corollary of the Uniform Boundedness Principle implies that $(1+T)^{-i\gamma}=[(1+T)^{i\gamma}]^{-1}$ is a bounded operator on $L_p(H)$ and the desired equality holds.

COROLLARY 4.6. *$J^{i\gamma}$ is the $i\gamma$-th Komatsu power of $J=J^1$ for all real γ.*

Proof. Denote the $i\gamma$-th Komatsu power of J by $(J)^{i\gamma}$. By Theorem 8.2 of [4-I], for a dense set of f in $L_p(H)$, $(J)^{i\gamma}f=\lim\limits_{\alpha\to 0^+}(J)^{\alpha+i\gamma}f=\lim\limits_{\alpha\to 0^+}J^{\alpha+i\gamma}f$ $=J^{i\gamma}f$; the second equality follows from part 4 of Theorem 1 and the strong analyticity of $(J)^{\alpha}$ and J^{α}. Since $J^{\alpha+i\gamma}=(J)^{\alpha+i\gamma}$ are uniformly bounded in $\alpha\geqslant 0$, $(J)^{i\gamma}$ is bounded by the Uniform Boundedness Principle, so that $J^{i\gamma}f=(J)^{i\gamma}f$ for all f in $L_p(H)$.

COROLLARY 4.7. *For any complex number α in $\operatorname{Re}\alpha>0$, $L_p^{\alpha}(H)=L_p^{\operatorname{Re}\alpha}(H)$, with equivalent norms.*

Proof. Let $\alpha = \beta + i\gamma$. Then because of the boundedness and invertibility of $J^{i\gamma}$, $\|J^\beta f\|_p \leqslant K(\gamma)\|J^\alpha f\|_p \leqslant K_1(\gamma)\|J^\beta f\|_p$.

Remark 4.1. The method used in Section 4 can be used to show that $J_2^{i\gamma}$ is a bounded operator on $L_p(H)$. The basic fact used above was that if

$$_\varepsilon U_y^\alpha f = \Gamma(\alpha)^{-1}\int_0^\infty T_{ty} f t^{\alpha-1} e^{-t}\,dt + \frac{\varepsilon^\alpha}{\Gamma(\alpha+1)} f,$$

and if

$$U_y^\alpha f = \Gamma(\alpha)^{-1}\int_0^\infty T_{ty} f t^{\alpha-1} e^{-t}\,dt \qquad \text{for } \operatorname{Re}\alpha > 0,$$

then $_\varepsilon U_y^{i\gamma}$ converges strongly to a bounded operator $U_y^{i\gamma}$ as $\varepsilon \to 0^+$. Furthermore, $U_y^{i\gamma}$ is the strong limit of $U_y^{\beta+i\gamma}$. Thus only minor modifications in the argument are needed to prove that $J_2^{i\gamma}$ is bounded. Write

$$J_2^\alpha(f) = 2\Gamma(\alpha/2)^{-1}\int_0^\infty H_{t^2}(f) t^{\alpha-1}\exp(-t^2)\,dt$$

$$= 2\Gamma(\alpha/2)^{-1}\int_H\int_0^\infty T_{ty} f t^{\alpha-1} e^{-t^2}\,dt\,dn\circ B^{-1}(y).$$

$\exp(-t^2)$ is the Fourier transform of a bounded measure on the real line and $t^{-1}(\exp(-t^2)-1)$ is bounded near zero, so that

$$_\varepsilon J_2^\alpha f = 2\Gamma(\alpha/2)^{-1}\left[\int_\varepsilon^\infty H_{t^2}(f) t^{\alpha-1} e^{-t^2}\,dt + \frac{\varepsilon^\alpha}{\alpha} f\right]$$

is uniformly bounded and converges if $\operatorname{Re}\alpha \geqslant 0$. As above $J_2^{i\gamma} f = \lim\{_\varepsilon J_2^{i\gamma} f\colon \varepsilon \to 0^+\}$ and one shows that $J_2^{i\gamma} = \lim_{\beta\to 0^+} J_2^{\beta+i\gamma} f$. Again $J_2^{i\gamma} = (1+T^2)^{i\gamma/2}$ and the $J_2^{i\gamma}$ form a strongly continuous group of bounded operators on $L_p(H)$ whose norm depends only on γ and p. An estimate for $\|J_2^{i\gamma}\|$ can be written easily from the estimate, given above, for $J_1^{i\gamma}$.

Remark 4.2. If $2 \geqslant k > 0$ is a real number, one can reason as above to show that $J_k^\alpha = (1+T^k)^{-\alpha/k}$ has boundary values $J_k^{i\gamma}$ bounded on $L_p(H)$. Here one has to use the special function $f_k(t, u)$ given in [7] to represent $\exp(-tT^k)$ as a semi-group given by convolution with a Borel probability measure.

5. The semi-groups J^β and $J^{i\gamma}$. We shall study the spectrums and infinitesmal generators of J^β, $\beta > 0$, and $J^{i\gamma}$, γ real. Since J^α is an analytic semi-group with bounded boundary values, $J^{i\gamma}$, a well-known theorem in semi-group theory states that if A is the infinitesmal generator of J^β, $\beta > 0$, then $J^{i\gamma} = \exp(i\gamma A)$. In what follows A denotes the infinitesmal generator of J^β, $\beta > 0$.

THEOREM 6. *A function f in $L_p(H)$ is in $D(A)$ if and only if*

$$f_* = \int_0^\infty P_t f e^{-t}\log t\,dt$$

is in $D(T)$ when $P_t = \exp(-tT)$. When this is the case $Af = Cf + (1+T)f_*$, where C is Euler's constant.

Proof. Let $J = J^1$. If f is in $D(A)$, $AJ(f) = JA(f) = \frac{\partial}{\partial\beta} J^\beta(f)|_{\beta=1}$ $= -\Gamma'(1)J(f) + \int_0^\infty P_t(f) e^{-t}\log t\,dt$. Since JAf and Jf are in $D(T)$, f_* is in $D(T)$ and $Af = Cf + (1+T)f_*$; $C = -\Gamma'(1)$ is Euler's constant $= \lim\left(\sum_{k=1}^n k^{-1} - \log n\right)$.

Conversely if f_* is in $D(T)$, then

$$\int_a^b J^t\big(Cf + (1+T)f_*\big)\,dt = \int_a^b J^t(1+T)(CJf + f_*)\,dt$$

$$= \int_a^b J^t(1+T)\frac{\partial}{\partial u} J^u(f)|_{u=1}\,dt = (1+T)\int_a^b \frac{\partial}{\partial t} J^{t+1}(f)\,dt$$

$$= (1+T)[J^{b+1}(f) - J^{a+1}(f)].$$

Thus $\lim_{a\to 0^+}\int_a^b J^t\big(Cf + (1+T)f_*\big)\,dt = J^b(f) - f$, and f is in $D(A)$ since

$$\lim_{b\to 0^+} b^{-1}\int_0^b J^t\big(Cf + (1+T)f_*\big)\,dt = Cf + (1+T)f_*.$$

COROLLARY 5.1. *The infinitesmal generator of the group $J^{i\gamma}$ acting on $L_p(H)$ is iA, where $Af = Cf + (1+T)f_*$ when $f_* = \int_0^\infty P_t(f)e^{-t}\log t\,dt$ is in the domain of T; C is Euler's constant.*

Proof. By a well-known theorem in semi-group theory, $J^{i\gamma} = \exp(i\gamma A)$ when $J^\beta = \exp(\beta A)$ for $\beta > 0$. The properties of A are given in Theorem 5.

If $\beta > 0$, the spectrum of J^β is contained in the unit disk. For $J^{i\gamma}$ there is:

THEOREM 7. *The spectrum of $J^{i\gamma}$ lies in the annulus* $\exp(-\pi|\gamma|/2) \leqslant |\lambda| \leqslant \exp(\pi|\gamma|/2)$.

Proof. By Theorem 4, $\|J^{i\gamma}\| \leqslant Kpq(|\gamma|+1)^2|\Gamma(i\gamma+1)|^{-1}$. Since $|\Gamma(i\gamma)| = \pi^{1/2}(\gamma\sinh\pi\gamma)^{-1/2}$, Corollary 4.3 implies that the spectral radius of $J^{i\gamma}$ is $r(J^{i\gamma}) = \lim_{n\to\infty}\|(J^{i\gamma})^n\|^{1/n} = \lim_{n\to\infty}\|J^{i\gamma n}\|^{1/n} \leqslant \exp(\pi|\gamma|/2)$. Since $J^{i\gamma}$ is invertible and since $\sigma(J^{i\gamma})$ is a compact set in the complex plane, there is a real number A such that $\sigma(J^{i\gamma})$ lies in the annulus $A \leqslant |\lambda| \leqslant \exp(\pi|\gamma|/2)$. Since $\sum_{n=0}^{\infty}\lambda^{-n-1}J^{in\gamma}$ converges to $(\lambda - J^{i\gamma})^{-1}$ on $|\lambda| > \exp(\pi|\gamma|/2)$ and since $-\sum_{n=0}^{\infty}\lambda^{n}J^{-i(n+1)\gamma}$ converges to $(\lambda - J^{i\gamma})^{-1}$ in $|\lambda| < \exp(-\pi|\gamma|/2)$, we have the desired results.

References

[1] V. Balakrishnan *Fractional powers of closed operators and the semi-groups generated by them*, Pacific J. Math. 10 (1960), pp. 419–437.

[2] M. J. Fisher, *Harmonic functions and their conjugates on Hilbert space*, Trans. Amer. Math. Soc. 137 (1969), pp. 387–405.

[3] — *Singular integrals and fractional powers of operators*, ibidem (November, 1971).

[4] H. Komatsu, *Fractional powers of operators I*, Pacific J. Math. 19 (1966), pp. 285–346; II. Pacific J. Math. 21 (1967), pp. 89–111; III. J. Math. Soc. Japan 21 (1969), pp. 205–220; IV. J. Math. Soc. Japan 21 (1969), 221–228.

[5] B. Muckenhoupt, *On certain singular integrals*, Pacific J. Math. 10 (1960), pp. 239–261.

[6] E. M. Stein, *The characterization of functions arising as potentials*, Bull. Amer. Math. Soc. 67 (1961).

[7] K. Yosida, *Functional Analysis*, Berlin 1965, pp. 259–268.

THE UNIVERSITY OF MONTANA
MISSOULA, MONTANA 59801

Received December 27, 1970 (283)

STUDIA MATHEMATICA T. XLI. (1972)

Some remarks on the Gurarij space

by

P. WOJTASZCZYK (Warszawa)

Abstract. Complementably universal properties of the Gurarij space of universal disposition are proved. Some linearly isomorphic equivalences between Banach spaces whose duals are L_1 spaces are stated.

A predual of L_1 is a Banach space X such that X^* is linearly isometric to $L_1(\mu)$ for some measure μ.

DEFINITION. A separable space X is a *space of universal disposition* iff for every finite dimensional Banach spaces $F \supset E$ and every isomorphism $T: E \to X$ and every $\varepsilon > 0$ there is an isomorphism $\tilde{T}: F \to X$ such that $\tilde{T}|E = T$ and $\|\tilde{T}\|\cdot\|\tilde{T}^{-1}\| \leqslant (1+\varepsilon)\|T\|\|T^{-1}\|$.

Such a space was first constructed by Gurarij [1] and next by Lazar and Lindenstrauss [3].

In this note we prove the following

THEOREM. *Let X be a separable predual of L_1. Then there exists a Banach space of universal disposition Γ_X, $\Gamma_X \supset X$ and there is a projection of norm one from Γ_X onto X.*

The proof of this Theorem is a slight modification of Gurarij's proof [1].

By [5], Theorem 4.2 there exists a Banach space Y such that:

(*) Y is a separable predual of L_1 and for any separable predual of L_1, say X, and any $\varepsilon > 0$ there exist an embedding $T: X \to Y$, $\|T\|\,\|T^{-1}\| \leqslant 1+\varepsilon$ and a projection of norm one from Y onto $T(X)$.

By [4] Remark c after Theorem 4 there exists a separable predual of L_1, say W, such that any separable predual of L_1 is a quotient space of W.

If we apply the above Theorem for $X = Y$ or $X = W$ we obtain

COROLLARY 1. *The spaces Y and W can be choosen to be of universal disposition.*

COROLLARY 2. *Every space which satisfies (*) is isomorphic to every space of universal disposition.*