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Uniform algebras satisfying certain extension properties

by

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Abstract. In this paper we study a uniform algebra A on a compact metric space X. Here follows the main result. If for each closed subset F of X there is a closed neighborhood W of F and a constant k_F, such that for each $f\in A$ there is some $g\in A$ (resp. a sequence (g_n) in A) satisfying $g=f$ on F (resp. $\lim|g_n-f|_F=0$) while $|g|_W<k_F|f|_F$ (resp. $|g_n|_W<k_F|f|_F$), then $A=C(X)$ (resp. A is locally dense in $C(X)$).

Introduction. Let A be a uniform algebra on a compact space X, i. e. A is a closed separating subalgebra of $C(X)$ containing the constants. If F is a closed subset of X and if $f\in C(X)$ we put $|f|_F=\sup\{|f(x)|\colon x\in F\}$. The following two concepts will lead to the problems studied in this paper.

DEFINITION A. Let A be a uniform algebra on a compact space X. We say that A satisfies the local extension property on a closed set F in X if there is a closed neighborhood W of F and a constant C such that: $\forall f\in A$ there is a sequence (g_n) in A with $\lim|g_n-f|_F=0$ while $|g_n|_W\leqslant C|f|_F$ for all n.

DEFINITION B. Let A be a uniform algebra on a compact space X. We say that A satisfies the strong extension property on a closed set F in X if there is a closed neighborhood W of F and a constant C such that: $\forall f\in A$ there is some $g\in A$ satisfying $g=f$ on F while $|g|_W\leqslant C|f|_F$.

Now we can state the main results of this paper.

THEOREM 1. *Let A be a uniform algebra on a compact metric space X. If A satisfies the local extension property on each closed set in X, then A is locally dense in $C(X)$.*

THEOREM 2. *Let A be a uniform algebra on a compact metric space X. If A satisfies the strong extension property on each closed set in X, then $A=C(X)$.*

Finally we study a phenomena closely related to the local extension property. Let A be a uniform algebra with its maximal ideal space M_A and its Silov boundary S_A and let $f\in C(M_A)$. We say that f is boundedly approximable by A on a closed set F in M_A if there is a closed neighborhood

W of F and a constant C such that $\lim|g_n-f|_F = 0$ while $|g_n|_W \leqslant C$ for some sequence (g_n) in A.

THEOREM 3. *Let A be a uniform algebra and let $f \in C(M_A)$. Suppose that $M_A = F_1 \cup \ldots \cup F_n$, where F_i are closed subsets of M_A such that f is boundedly approximable by A on each F_i. If A_f is the uniform algebra on M_A which is generated by A and f, then $M_{A_f} = M_A$ and $S_{A_f} = S_A$.*

In Section 1–3 below we prove these theorems and make some additional remarks about them.

0. Preliminaries. Let X be a compact metric space with the metric d. If F is a closed subset of X we put $S(F, v) = \{x \in X\colon d(x, F) \leqslant v^{-1}\}$ for each positive integer v. By a measure on X we understand a complex-valued measure which is regular and defined on the Borel sets of X and with a finite total variation. These are denoted by $M(X)$, so here $M(X)$ is the dual space of $C(X)$. If $m \in M(X)$ and if W is a Borel set in X we put $|m|_W =$ total variation of m over W.

If A is a uniform algebra and if F is a closed subset of M_A we can introduce the restriction algebra $A|F = \{g \in C(F)\colon g = f$ on F for some f in $A\}$. In general $A|F$ is not a closed subalgebra of $C(F)$ so we denote its uniform closure by $A(F)$. Then $A(F)$ is a uniform algebra and it is well-known that $M_{A(F)}$ can be identified with the set $\mathrm{Hull}_A(F) = \{y \in M_A\colon |f(y)| \leqslant |f|_F$ for all f in $A\}$.

If A is a uniform algebra on a compact space X we may identify X with a closed subset of M_A. Under this identification X contains the Silov boundary S_A, i. e. $|f|_X = |f|_{M_A}$ holds for all f in A. If F is a closed subset of X such that $\mathrm{Hull}_A(F) \cap X = F$, then we say that F is an (A, X)-convex subset of X.

We shall freely use the following fundamental results about uniform algebras.

a) LOCAL MAXIMUM PRINCIPLE (abbr. to LMP) which asserts that if A is a uniform algebra and if W is a subset of $M_A \setminus S_A$, then $|f|_W = |f|_{\partial W}$ for all f in A, where ∂W is the topological boundary of W in M_A.

b) ŠILOV'S IDEMPOTENT THEOREM (SIT) which says that if F and G are two disjoint closed subsets of M_A such that $\mathrm{Hull}_A(F \cup G) = F \cup G$, then A contains a sequence (e_n) satisfying $\lim|e_n-1|_F = 0$ while $\lim|e_n|_G = 0$.

c) PRINCIPLE OF MINIMAL SUPPORTS (PMS). Here we start with a closed subset F of M_A and let $x \in \mathrm{Hull}_A(F)$. Then F contains a closed subset G (G is not necessarily unique) such that G is a minimal support of x. In particular G has the following property: If W is a relatively open subset of G and if (f_n) in A satisfy $\lim|f_n|_W = 0$ while $|f_n|_G \leqslant C$ for some constant C, then $\lim f_n(x) = 0$.

We refer to [4] for a proof and some applications of PMS.

1. Proof of Theorem 1. Firstly we collect some preliminary results. Let A be a uniform algebra on a compact space X. We say that A is boundedly normal on an open subset W of X if there is a constant C such that if F and G are two compact disjoint subsets of W, then A contains a sequence (e_n) satisfying $\lim|e_n-1|_F = 0$ and $\lim|e_n|_G = 0$ while $|e_n|_X \leqslant C$.

PROPOSITION 1.1. *Let A be a uniform algebra on a compact space X and suppose that A is boundedly normal on an open subset W of X for which the set $X \setminus W$ is (A, X)-convex. Then each $m \in M(X)$ which annihilates A is supported on the set $X \setminus W$.*

Proof. Let $K = X \setminus W$ and take a compact subset F in W. We firstly claim that F appears as an open and closed subset of $\mathrm{Hull}_A(K \cup F)$ in M_A. For the condition that K is (A, X)-convex firstly shows that the space $\mathrm{Hull}_A(K \cup F)$ contains an open set Ω with $F \subset \Omega$ while $\Omega \cap \mathrm{Hull}_A(K)$ is empty. Suppose next that $x \in \Omega \setminus F$ and choose a minimal support G for x with $G \subset K \cup F$. Here $G \cap F$ is a non-empty relatively open subset of G and then it is wellknown and easily verified that $G \cap F$ must be an infinite set (in fact even non-denumerable), so in particular $G \cap F$ contains two distinct points y_1 and y_2.

Now we choose disjoint closed neighborhoods W_i of each y_i so that W_i are compact subsets of W, i. e. here y_i are considered as points in W. By assumption we obtain a sequence (e_n) in A for which $\lim|e_n-1|_{W_1} = 0$ while $\lim|e_n|_{W_2} = 0$ while $|e_n|_X \leqslant C$. In particular $\lim|e_n-1| = 0$ on the set $W_1 \cap G$ while $|e_n|_G \leqslant C$ so by PMS we conclude that $\lim e_n(x) = 1$. In the same way we apply PMS to conclude that $\lim e_n(x) = 0$, a contradiction. Hence we have proved that $\Omega \setminus F$ is empty which means that F is an open subset of $\mathrm{Hull}_A(K \cup F)$.

Using the result above we can easily find closed neighborhoods K_1 of K, resp. F_1 of F in X so that $F_1 \subset\subset W$ while F_1 is an open and closed subset of $\mathrm{Hull}_A(K_1 \cup F_1)$. Using SIT we get a sequence (e_n) in A satisfying $\lim|e_n-1|_{F_1} = 0$ while $\lim|e_n|_{K_1} = 0$. Here $|e_n|_{F_1}$ and $|e_n|_{K_1} \leqslant 1$ may also be assumed.

If we put $G = (X \setminus \mathring{K}_1) \cap (X \setminus \mathring{F}_1)$, then G is a compact subset of W for which $G \cap F$ is empty. So if we put $C_n = |e_n|_G$ then we can choose f_n in A such that $|f_n-1|_F < n^{-1}$ and $|f_n|_G < C_n^{-1}$ while $|f_n|_X \leqslant C$. If we then put $h_n = e_n f_n$ we see that $\lim|h_n-1|_F = 0$ and $\lim|h_n|_{K_1} = 0$ while $|h_n|_X \leqslant C$.

Suppose now that $m \in M(X) \cap A^{\perp}$ while $|m|_W > 0$. Then W contains a compact subset F where $m_1 = m|F$ is a measure such that $\delta = m_1(F) > 0$ (if necessary we multiply m with a scalar to obtain this situation). Using the previous results it is easily seen that if G is a compact subset of $X \setminus F$, then A contains (e_n) satisfying $\lim|e_n-1|_F = 0$ and $\lim|e_n|_G = 0$

while $|e_n|_X \leqslant C$. In particular we can choose G so large that $|m-m_1|_{X\setminus G} < \delta(2C)^{-1}$ and then we see that $\liminf |\int e_n dm| > \delta/2$, a contradiction.

The following result is an easy consequence of SIT.

LEMMA 1.2. *Let A be a uniform algebra on a compact metric space and let F be a closed subset of X. If $x \in X\setminus \mathrm{Hull}_A(F)$ then there is an integer v such that for all closed subsets G of $S(x, v)$ we have $\mathrm{Hull}_A(F \cup G) = \mathrm{Hull}_A(F) \cup \mathrm{Hull}_A(G)$.*

From now on A and X are as in Theorem 1. We say that a closed set F in X is of type w_n if F satisfies the local extension property with the neighborhood $S(F, n)$ and the extension constant n. Here n is a positive integer and the assumption in Theorem 1 means that every closed set in X is of type w_n for some integer n. In Lemma 1.3–1.6 below we prove Theorem 1.

LEMMA 1.3. *Let $x \in X$. Then there is a closed neighborhood W of x and an integer N such that all closed (A, X)-convex subsets of $W\setminus\{x\}$ are of type w_N.*

Proof. Suppose this is false. Then we define a sequence (F_n) of disjoint (A, X)-convex sets which converge to the point x and finally derive a contradiction.

The sets F_n are defined inductivelly as follows. Suppose $F_1 \dots F_n$ have been obtained where F_i is not of type w_i and $R_n = F_1 \cup \dots \cup F_n$ is an (A, X)-convex subset of X not containing the point x. Using Lemma 1.2. we find an integer v such that the conclusion there is satisfied by R_n and the point x. Here we may assume that $v \geqslant n+1$ and by our assumption we can now choose an (A, X)-convex set F_{n+1} from $S(x, v)\setminus\{x\}$ where F_{n+1} is not of type w_{n+1}.

Having constructed the sets F_n as above we put $F = \left(\bigcup F_n\right) \cup \{x\}$. Then F is a closed set in X which is of type w_N say. Now the construction shows that $\mathrm{Hull}_A(F)$ is the disjoint union of the sets $\mathrm{Hull}_A(R_{N-1})$, $\mathrm{Hull}_A(F_N)$ and $\mathrm{Hull}_A(F\setminus R_N)$, so in particular $\mathrm{Hull}_A(F_N)$ is a closed and open subset of $\mathrm{Hull}_A(F)$. Hence we can apply SIT and obtain a sequence (e_n) from A satisfying $\lim|e_n-1|_F = 0$ while $\lim|e_n|_{F\setminus F_N} = 0$.

If we now let $f \in A$ satisfy $|f|_{F_N} < 1$ we see that $|fe_n|_F < 1$ for large n, so for large n we obtain g_n in A such that $|g_n - fe_n|_F < n^{-1}$ while $|g_n|_{S(F,N)} \leqslant N$. But then $\lim|g_n-f|_{F_N} = 0$ while $|g_n|_{S(F_N,N)} \leqslant N$, so that F_N is of type w_N, a contradiction.

LEMMA 1.4. *Let $x \in X$ and let W be a closed neighborhood of x such that all (A, X)-convex compact subsets of $W\setminus\{x\}$ are of type w_N. Then there is a closed neighborhood W_1 of x, with $W_1 \subset W$, such that if we put $B = A(W_1)$, then all (B, W_1)-convex subsets of $W_1\setminus\{x\}$ are of type w_N.*

Proof. Choose W_1 so small that $W_1 \subset S(x, 2N)$ and $\mathrm{Hull}_A(W_1) \cap X \subset W$. Let then F be a (B, W_1)-convex and compact subset of $W_1\setminus\{x\}$. If $G = \mathrm{Hull}_A(F) \cap X$, then $G \subset W\setminus\{x\}$ and hence G is of type w_N. Since $|f|_F = |f|_G$ for all f in A we can easily conclude that F is of type w_N with respect to B. Here $W_1 \subset S(x, 2N)$ also holds so we see that if $f \in B$ then there is a sequence (g_n) in B satisfying $\lim|g_n - f|_F = 0$ while $|g_n|_{W_1} \leqslant N|f|_F$.

LEMMA 1.5. *Let $B = A(W_1)$ be as in Lemma 1.4. Then $M_B = W_1$ and all compact subsets of $W_1\setminus\{x\}$ are (B, W_1)-convex.*

Proof. Suppose that $M_B\setminus W_1$ is not empty and let a be a point in this set. Choose a minimal support F for a with $F \subset W_1$. Then F is infinite so in particular we can choose two points y_1 and y_2 in $F\setminus\{x\}$. Next we choose small closed B-convex meighborhoods Ω_1 and Ω_2 of y_1 resp. y_2 in M_B, so that $\mathrm{Hull}_B(\Omega_1 \cup \Omega_2)$ is the disjoint union of the two sets $\mathrm{Hull}_B(\Omega_1)$ and $\mathrm{Hull}_B(\Omega_2)$. Then SIT gives a sequence (e_n) in B such that $\lim|e_n-1|_{\Omega_1} = 0$ while $\lim|e_n|_{\Omega_2} = 0$.

Because $\Omega = (\Omega_1 \cup \Omega_2) \cap W_1$ is (B, W_1)-convex we deduce from Lemma 1.4. that B contains a sequence (f_n) satisfying $|f_n|_{W_1} \leqslant N$ while $|f_n - e_n|_\Omega = 0$. But then PMS can be applied as in the proof of Proposition 1.1. which gives that $\lim f_n(a) = 1$ and $\lim f_n(a) = 0$ hold simultaneously, a contradiction.

Hence $M_B = W_1$ holds and in exactly the same way we can prove that all closed subsets of $W_1\setminus\{x\}$ are (B, W_1)-convex.

LEMMA 1.6. *Let $B = A(W_1)$ be as in Lemma 1.4. Then $B = C(W_1)$.*

Proof. If $W = W_1\setminus\{x\}$, then W is an open subset of M_B and $\{x\} = M_B\setminus W$ is B-convex. From Lemma 1.5. we know that all compact subsets of W satisfy w_N and then an application of SIT shows that B is boundedly normal on W with the constant N. Hence Proposition 1.1. shows that if $m \in M(W_1) \cap B^\perp$, then $\mathrm{supp}(m) \subset \{x\}$. But here B contains the constants so that $m = 0$ follows which means that $B = C(W_1)$.

Let us remark that Theorem 1 cannot be improved, i. e. we cannot expect that $A = C(X)$. Consider for example the disc algebra restricted to the unit circle. If we assume that $X = M_A$ in Theorem 1 then LMP shows that all closed subsets of M_A are A-convex. Whether we also have $A = C(M_A)$ in this case is an open question.

Finally we remark that Defintion A is given when the uniform algebra A is defined on a compact space X which need not be metric. We do not know if Theorem 1 remains true when X is not metric. Notice that we used the metric in Lemma 1.3.

2. Proof of Theorem 2. Let A be a uniform algebra with its maximal ideal space M_A. We introduce the so called zero-hull of a closed set F in M_A as follows. Put $Z(F) = \{x \in M_A\colon\ \forall f \in A$ with $f = 0$ on F we have $f(x) = 0\}$. Recall here that if $J_F = \{f \in A\colon\ f = 0$ on $F\}$, then J_F is a closed ideal in A and $Z(F)$ can be identified with the maximal ideal space of the Banach algebra A/J_F. In particular an application of SIT to A/J_F shows that each open and closed subset of $Z(F)$ must intersect F.

Suppose next that A is a uniform algebra on a compact space X and let F be a closed (A, X)-convex set in X satisfying the strong extension property. If we let W be the closed neighborhood of F which appears in Definition B, then the set $Z(F) \cap (W \setminus F)$ is empty. For if $x \in W \setminus F$ we can choose f in A so that $f(x) = 1$ while $|f|_F < (2C)^{-1}$. Now we can find $g \in A$ satisfying $g = f$ on F while $|g|_W \leqslant C(2C)^{-1} \leqslant 2^{-1}$ and then $h = g - f$ is zero on F while $h(x) \neq 0$.

Now we introduce some new concepts.

DEFINITION 2.1. Let A be a uniform algebra on a compact space X. We say that a closed subset F of X is *almost regular* if F is an open and closed subset of $Z(F) \cap X$.

DEFINITION 2.2. Let A be a uniform algebra on a compact space X. We say that A is *locally regular* at a point $x \in X$ if there is a closed neighborhood W of x in X such that each closed subset F in W satisfies $Z(F) \cap W = F$.

We shall now prove a result which together with Theorem 1 and the preceding discussion immediately gives Theorem 2.

THEOREM 2.3. *Let A be a uniform algebra on a compact metric space. Suppose that A is locally dense in X and that each closed (A, X)-convex subset of X is almost regular. Then $A = C(X)$.*

Before we prove Theorem 2.3. we need several preliminary results.

LEMMA 2.4. *Let A be a uniform algebra on a compact metric space. Suppose that each closed (A, X)-convex subset of X is almost regular. If now $x \in X$ is given, then there is an open neighborhood W of x in X such that A is locally regular at each point in $W \setminus \{x\}$.*

Proof. Let us say that a closed (A, X)-convex set F in X is of type ϱ_N if $d\big(F, (Z(F) \cap X) \setminus F\big) \geqslant N^{-1}$. Using the same argument as in Lemma 1.3. we can prove that there exists an open neighborhood W_1 of x in X such that every compact (A, X)-convex subset of $W_1 \setminus \{x\}$ is of type ϱ_N for a fixed integer N. By shrinking W_1 if necessary we may assume that $W_1 \subset S(x, 4/N)$. Then we see that if F is a compact (A, X)-convex subset of $W_1 \setminus \{x\}$, then $Z(F) \cap W_1 = F$ holds.

Suppose next that F is a compact subset of $W_1 \setminus \{x\}$ and let $y \in W_1 \setminus F$. If $z \in F$ we can choose a closed (A, X)-convex neighborhood V_z of z in X so that $V_z \subset W_1 \setminus \{x\}$. Then $Z(V_z) \cap W_1 = V_z$ so there is some f_z in A for which $f_z = 0$ on V_z while $f_z(y) \neq 0$. Finally F is covered by finitely many sets V_z and then we conclude that A contains some f for which $f(y) = 1$ while $f = 0$ on F. Hence $Z(F) \cap W_1 = F$ follows. This result shows of course that A is locally regular at every point in $W_1 \setminus \{x\}$.

Recall that if A is a uniform algebra on a compact space X and if X is identified with a subset of M_A, then X contains S_A. In addition S_A is the closure of the set C_A, where C_A consists of all generalized peak points for A, i. e. $x \in C_A$ if and only if $\{x\}$ is an intersection of peak sets. The set C_A is the so called Choquet boundary of A, i.e. $x \in C_A$ if and only if $m(\{x\}) = 0$ for all $m \in M(M_A) \cap A^{\perp}$.

PROPOSITION 2.5. *Let A be a uniform algebra on a compact space X. Let $x \in X$ be a generalized peak point for A and suppose in addition that $x \in \partial \Delta$, where $\Delta = M_A \setminus X$. If now U is a closed neighborhood of x in X, then $\mathrm{Hull}_A(U)$ is a closed neighborhood of x in M_A.*

Proof. Let W be a closed neighborhood of x in X as in Definition 2.2. If $\mathring{W}$ is its interior, then $F = X \setminus \mathring{W}$ is a closed subset of X and since $x \in C_A$ it is clear that x does not belong to $\mathrm{Hull}_A(F)$. Hence Lemma 1.2. shows that there is a closed neighborhood V of x in X such that $\mathrm{Hull}_A \times \times (V \cup F) = \mathrm{Hull}_A(V) \cup \mathrm{Hull}_A(F)$ holds. In addition we may assume that $V \subset \mathring{U}$ here.

Since A is locally regular at x we can choose f in A so that $f(x) = 1$ while $f = 0$ on the set $W \setminus \mathring{V}$. If we put $Z = \{y \in M_A\colon y \in M_A \setminus \mathrm{Hull}_A(F)$ and $f(y) \neq 0\}$, then Z is an open neighborhood of x in M_A and clearly $Z \cap X \subset V$.

Suppose now that $y \in Z \cap \Delta$. If G is a minimal support of y with $G \subset X$ it follows from PMS that $G \cap (W \setminus \mathring{V})$ is empty. Hence $y \in \mathrm{Hull}_A(F \cup V) = \mathrm{Hull}_A(V) \cup \mathrm{Hull}_A(F)$ and the definition of Z then implies that $y \in \mathrm{Hull}_A(V)$.

The result above shows that $Z \subset \mathrm{Hull}_A(V)$ and since $V \subset U$ it follows that $\mathrm{Hull}_A(U)$ is a neighborhood of x in M_A.

Proof of Theorem 2.3. Firstly we show that $M_A = X$ holds. For suppose that $\Delta = M_A \setminus X$ is not empty and let $F = \partial \Delta$ be its topological boundary. We claim that F is an (A, X)-convex subset of X. For if $x \in \mathrm{Hull}_A(F) \cap (X \setminus F) = U$ it follows easily form the LMP applied to the uniform algebra $A(F)$ that A cannot be locally regular at x. Notice namely that U is an open subset of $M_{A(F)} \setminus S_{A(E)}$ here.

Now Lemma 2.4. implies that A is locally regular at all points in X except for a finite set S. Hence we conclude that $U \subset S$, so that U is a finite open subset of $M_{A(F)} \setminus S_{A(F)}$. But LMP shows that $M_{A(F)} \setminus S_{A(F)}$ cannot contain isolated points so it follows that U must be empty.

The result above shows that F is (A, X)-convex so if we replace A by the uniform algebra $A(F)$ on F, then $A(F)$ will satisfy the same assumptions as A. So therefore we may assume that $X = \partial \Delta$ holds, since $M_A \setminus X = M_{A(F)} \setminus F$ holds by LMP.

If S is the finite set in X where A is not locally regular and if $x \in X \setminus S$ is in C_A, then Proposition 2.5. shows that if U is a closed neighborhood of x in X then $\mathrm{Hull}_A(U)$ is a neighborhood of x in M_A. Since A is locally dense in $C(X)$ we can choose U so small that $A(U) = C(U)$ and then $U = M_{A(U)} = \mathrm{Hull}_A(U)$ follows which means that x cannot belong to $\partial\Delta = X$.

We conclude that the set C_A is contained in the finite set S so that $S_A = S$ follows. But then $M_A = S_A \subset X$ follows, a contradiction.

Having proved that $M_A = X$ we can easily finish the proof. Since A is locally dense in $C(M_A)$ we see that if A is normal on M_A, then $A = C(M_A)$. In addition we know that A is normal on M_A if $F = Z(F)$ holds for all closed subsets in M_A.

So let F be a closed subset of M_A. Because A is locally dense in $C(M_A)$ the LMP shows that F is A-convex so by assumption F is an open and closed subset of $Z(F)$. But we recall that $Z(F) = M_B$, where B is the Banach algebra A/J_F and then SIT shows that $Z(F) = F$ holds. This proves that A is normal on M_A so that $A = C(M_A)$.

3. Proof of Theorem 3. The proof which follows is using the methods developed by *C. E.* Rickart in [5]. We recall that in [1] it is proved that if $f \in C(M_A)$ is such that to each point x in M_A there is an open neighborhood W of x and a sequence (f_n) in A satisfying $\lim |f_n - f|_W = 0$, then $M_A = M_{A_f}$ and $S_A = S_{A_f}$ holds. We also remark that it is essential to assume that f is boundedly approximable on the closed sets F_i in Theorem 3. For in [2] there is an example with $M_A = F_1 \cup F_2 \cup F_3$ and some $f \in C(M_A)$ which is approximable by A on each closed set F_i and yet $M_{A_f} \neq M_A$.

Proof of Theorem 3. Suppose that the set $\Delta = M_{A_f} \setminus M_A$ is not empty. If $X = \partial\Delta$ is its topological boundary and if we put $B = A_f(X)$, then $M_B = \mathrm{Hull}_{A_f}(X)$ and the LMP shows that M_B contains the set Δ. Since $A|X$ can be identified with a subalgebra of B we get a canonical map π from M_B into M_A which satisfies $g(y) = g(\pi(y))$ for all $y \in M_B$ and all $g \in A|X$. Recall that π is continuous here and in addition π maps M_B into the set $\mathrm{Hull}_A(X)$ in M_A.

Next we make the following induction hypothesis. If $x \in X$ is a generalized peak point for B, i. e. if $x \in C_B$, then x belongs to at least k different F_i. Trivially this holds when $k = 1$.

Suppose next that $x \in C_B$ only belongs to k sets F_i, say $F_1 \ldots F_k$. By assumption A contains k sequences $(g_n^{(i)})$ such that $\lim |g_n^{(i)} - f|_{F_i} = 0$ while $|g_n^{(i)}|_{W_i} \leqslant C$ for all i and n and certain closed neighborhoods W_i of each F_i and a constant C. If we put $W = W_1 \cap \ldots \cap W_k$, then W is a closed neighborhood of x in M_A.

We can choose a closed neighborhood V of x in M_B so that $\pi(V) \subset W$ while $\pi(V) \cap F_i$ are empty for all $i > k$. If we then take a point y in $V \cap \Delta$, then the LMP applied to B shows that the set $\partial V \cap (V \cap S_B)$ contains a minimal support G for y. Using the fact that $x \in C_B$ we can choose g in B so that $g(x) = 1$ while $|g|_{M_B} \leqslant 1$ and $|g|_V < 1/2$. If we now let $y \in V \cap \Delta$ satisfy $|g(y)| > 3/4$ and if G is chosen as above then the set $U = V \cap S_B \cap G$ contains a relatively open subset of G.

Now $\mathring{V} \cap S_B$ is a relatively open subset of S_B which is contained in X and we know that $F_i \cap \mathring{V} \cap S_B$ are empty for all $i > k$. So by the induction hypothesis every point in the set $\mathring{V} \cap C_B$ must belong to $F_1 \cap \ldots \cap F_k$ and since S_B is the closure of C_B we conclude that $\mathring{V} \cap S_B \subset F_1 \cap \ldots \cap F_k$.

Now $g_n^{(i)}$ are considered as elements of B and since $\pi(V) \subset W$ we see that $|g_n^{(i)}|_V \leqslant C$ for all i and n. Hence PMS can be applied to the point y and its minimal support to give $\lim |g_n^{(i)}(y) f - f(y)| = 0$ for $i = 1 \ldots k$. We also know that $\pi(y) \in F_1 \cup \ldots \cup F_k$ so let $\pi(y) = z$ with $z \in F_1$ say. Then $g_n^{(1)}(y) = g_n^{(1)}(z)$ for all n and since $\lim g_n^{(1)}(z) = f(z)$ we conclude that $f(y) = f(z)$. Here $z \in M_B \cap M_A$ and since $f(z) = f(y)$ while $z = \pi(y)$ we conclude that $y = z$ in M_B. But then y cannot belong to Δ from which it was chosen, a contradiction.

Hence we have proved that each point in C_B is contained in at least $k+1$ different sets F_i, so by induction we get $C_B \subset F_1 \cap \ldots \cap F_n$ and hence S_B is also contained in this intersection. But then we easily get a contradiction again. For let $y \in \Delta$ and put $\pi(y) = z$ where $z \in E_1$ say. Since $S_B \subset E_1$ we see that $f(y) = \lim g_n^{(1)}(y) = \lim g_n^{(1)}(z) = f(z)$ so that $y = z$ in M_B which means that y cannot belong to Δ, from which it was chosen, a contradiction. Hence Δ must be empty which proves that $M_{A_f} = M_A$. In exactly the same way we can prove that $S_{A_f} = S_A$.

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