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Some properties of l_p, $0<p<1$

by

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Abstract. This paper contains some results about l_p spaces for $0<p<1$. In particular, it contains a proof of the fact that every complemented infinite-dimensional subspace of l_p is isomorphic to l_p and a proof of the fact that every continuous linear map of a normed linear space into l_p is compact.

In [8] we investigated some of the properties of l_p spaces for $0<p<1$. We showed, for example, that each l_p contains a subspace no infinite-dimensional subspace of which is complemented in l_p, that each l_p contains an infinite-dimensional subspace not isomorphic to l_p, and that each p-normed separable F-space is a quotient of l_p. We add to these results here by showing that each complemented infinite-dimensional subspace of l_p is isomorphic to l_p and that each continuous linear mapping of a normed linear space into l_p is compact. We also give some other related results and state some problems concerning l_p spaces.

1. Terminology and Notation. The notation and terminology employed here seem to be fairly standard. We use the term *F-space* to denote a complete linear metric space and the symbol $[x_n]$ to denote the smallest subspace spanned by the sequence $\{x_n\}$, where the term subspace is used only for a closed linear manifold. A subspace X is *complemented* if and only if there is a continuous linear projection onto X, and two spaces are *isomorphic* if and only if they are linearly homeomorphic. For $0<p<1$, l_p will denote the F-space of all real sequences $x=\{x_n\}$ such that $\|x\|_p=\sum|x_n|^p<\infty$, and L_1 will denote the space of all Lebesgue summable functions on the interval $(0,1)$. Two basic sequences $\{a_n\}$ and $\{b_n\}$ in an F-space are said to be *equivalent* if and only if $\sum a_n a_n$ converges when and only when $\sum a_n b_n$ converges. Finally, we will say that a linear metric space X is a p-normed space if X admits a p-homogeneous norm.

2. Complements in l_p. It was shown by A. Pełczynski in [4] that each complemented infinite-dimensional subspace of l_p, $p\geqslant 1$, is isomorphic to l_p. His proof utilized the fact that each subspace of l_p, $p\geqslant 1$, contains a complemented subspace which is isomorphic to l_p. Since there are subspaces of l_p, $0<p<1$, which contain no complemented infinite-dimensional subspaces (cf. [8]), this proof does not apply when $0<p<1$; however, the theorem is true in this case as we will now show.

LEMMA 1. *Let* $a_n = (a_1^n, a_2^n, \ldots)$ *and* $b_n = (0, \ldots, a_{k_n}^n, a_{k_n+1}^n, a_{k_n+2}^n, \ldots a_{k(n+1)}^n, 0, \ldots)$ *be points in* l_p, $0<p<1$, *such that* $\|a_n\|_p = 1$, $\sum_{j=k_n}^{k(n+1)-1} |a_j^n| \geqslant r > 0$, *where* $\{k_n\}$ *is a strictly increasing sequence of positive integers,*

$$\sum_{j=k_n}^{k(n+1)-1} |a_j^n|^p \geqslant 1-1/2^{n+1}, \quad \textit{and} \quad \sum \|b_j - a_j\|_p < (1/2)r^p.$$

Then the following hold.

(1) $\{a_n\}$ *is a basic sequence equivalent to the unit vector basis in* l_p.

(2) *There is a continuous projection* P *of* l_p *onto the space* $[b_n]$ *and* P *is given by* $P(x) = \sum f_j(x) b_j$.

(3) *The mapping* A *given by* $A(x) = x - P(x) + \sum f_j(x) a_j$ *is an isomorphism of* l_p *onto* l_p.

Proof. By the proof of Theorem 2.1 of [8], $\{a_n\}$ is a basic sequence equivalent to the basic sequence $\{b_n\}$ which is equivalent to the unit vector basis in l_p.

Since $\sum_{j=k_n}^{k(n+1)-1} |a_j^n| \geqslant r > 0$ there exists a linear functional f_n and a corresponding sequence $(0, \ldots, f_{k_n}^n, \ldots, f_{k(n+1)-1}^n, 0, \ldots)$ such that $f_n(b_m) = \sum_{j=1}^{\infty} f_j^n a_j^m = \delta_{m,n}$ and $\sup_{j,n} |f_j^n| \leqslant 1/r$. It is then easy to see that $P(x) = \sum f_j(x) b_j$ is a continuous projection of l_p onto $[b_n]$.

The mapping $A(x) = x - P(x) + \sum f_j(x) a_j$ is a well defined continuous mapping on l_p since $\{a_j\}$ and $\{b_j\}$ are equivalent basic sequences. We will show that this mapping is one-to-one and onto l_p. Note that

$$\begin{aligned}\|I(x) - A(x)\|_p &= \|P(x) - \sum f_j(x) a_j\|_p = \|\sum f_j(x)(b_j - a_j)\|_p \\ &\leqslant \sum |f_j(x)|^p \|b_j - a_j\|_p \leqslant \sum \big((1/r)\|x\|_1\big)^p \|b_j - a_j\|_p \\ &\leqslant 1/r^p \|x\|_1^p \sum \|b_j - a_j\|_p < 1/2 \|x\|_1^p.\end{aligned}$$

Hence, $\|I(x) - A(x)\|_1 \leqslant \|I(x) - A(x)\|_p^{1/p} \leqslant (1/2)^{1/p} \|x\|_1$. A simple induction argument shows that $\|(I-A)^n x\|_p^{1/p} \leqslant (1/2)^{n/p} \|x\|_1$ for each positive integer n. Recall that

$$\begin{aligned}A\big(I + (I-A) + (I-A)^2 + \ldots + (I-A)^n\big) &\\ = \big((I + (I-A) + (I-A)^2 + \ldots + (I-A)^n\big) A &\\ = I - (I-A)^{n+1}.&\end{aligned}$$

Since

$$\|(I-A)^{n+1} x\|_p \leqslant (1/2)^{n+1} \|x\|_1^p, \ \lim \|(I-A)^{n+1} x\|_p = 0.$$

Moreover,

$$\begin{aligned}\|(I-A)^m x + \ldots + (I-A)^{m+k} x\|_p &\leqslant \|(I-A)^m x\|_p + \ldots + \|(I-A)^{m+k} x\|_p \\ &\leqslant \big((1/2)^m + \ldots + (1/2)^{m+k}\big) \|x\|_1^p.\end{aligned}$$

This implies that the series $\sum (I-A)^n x$ converges. Hence, it is easy to see that A is one-to-one and maps l_p onto l_p.

THEOREM 1. *If* X *is a subspace of* l_p, $0<p<1$, *then* X *contains no subspace which is both complemented in* l_p *and isomorphic to* l_p *if and only if* $S_X^p = \{x \in X \colon \|x\|_p \leqslant 1\}$ *is a precompact subset of* l_1, *i. e. if and only if given any* $\varepsilon > 0$ *there exists an integer* N *such that* $n \geqslant N$, $a = (a_1, a_2, \ldots) \in X$, *and* $\|a\|_p \leqslant 1$ *implies* $\sum_{j=n} |a_j| \leqslant \varepsilon$.

Proof. The equivalence of the last two conditions stated in the theorem follows from a theorem of Cohen and Dunford (cf. [7], p. 193). We will first show that if the second of these conditions is not satisfied, then X has a subspace Y which is both isomorphic to l_p and complemented in l_p.

Suppose that for some ε, $0 < \varepsilon \leqslant 1$, there is a sequence $\{a_n\}$ such that $a_n = (a_1^n, a_2^n, \ldots)$, $\|a_n\|_p \leqslant 1$ and given any integer N there exists an a_n such that $\sum_{j=N} |a_j^n| > \varepsilon$. We note that by using a diagonal process if necessary we can assume that $\lim_n a_k^n$ exists for each n. Let $a_k = \lim_n a_k^n$.

We select two strictly increasing sequences $\{h_n\}$ and $\{k_n\}$ of non-negative integers and a sequence $\{b_n\}$ of vectors inductively in the following manner. Let $b_0 = a_1$, $h_0 = 0$ and choose k_0 such that $\sum_{j=k_0}^{\infty} |a_j^1|^p < (1/8)(\varepsilon/2)^p$ and $\sum_{j=1}^{k_0} |a_j^1| > \varepsilon/2$. Having selected $h_1, \ldots, h_{2n}$, $k_1, \ldots, k_{2n}$, and b_n, select h_{2n+1} such that

$$\sum_{j=1}^{k_{2n}} |a_j^m - a_j|^p < 1/2^{n+5} (\varepsilon/2)^{2p} \tag{1}$$

for all $m \geqslant h_{2n+1}$. Select k_{2n+1} such that

$$\sum_{j=k_{2n+1}}^{\infty} |a_j^{h_{2n+1}}| < \varepsilon/2. \tag{2}$$

Then select h_{2n+2} such that

$$\sum_{j=k_{2n+1}}^{\infty} |a_j^{h_{2n+2}}| > \varepsilon. \tag{3}$$

Let $b_{n+1} = a_{h_{2n+2}} - a_{h_{2n+1}}$. Then $\|b_{n+1}\|_1 > \varepsilon/2$ and $\|b_{n+1}\|_p \leqslant 2$. Hence $(\varepsilon/2)^p \leqslant \|b_{n+1}\|_p \leqslant 2$. Finally, choose k_{2n+2} such that

$$(4)\qquad \sum_{j=k_{2n+2}}^{\infty} |b_j^{n+1}|^p < (1/2^{n+4})(\varepsilon/2)^{2p}$$

and

$$(5)\qquad \sum_{j=k_{2n+1}}^{k_{2n+2}} |b_j^{n+1}| > \varepsilon/2,$$

where $b_n = (b_1^n, b_2^n, \ldots)$. Let $c_n = b_n/\|b_n\|_p^{1/p}$; then $\|c_n\|_p = 1$. Now let $c_n = (c_1^n, c_2^n, \ldots)$, and $\tilde{c}_n = (0, \ldots, 0, c_{k_{2n+1}}^n, \ldots, c_{k_{2n+2}}^n, 0, \ldots)$. We will show that $\{c_n\}_1^\infty$ satisfies the conditions of Lemma 1 with $r = (1/2^{1/p})(\varepsilon/2)$. Let $K_{n+1} = \{j\colon j$ is an integer and $k_{2n}+1 \leqslant j \leqslant k_{2n+2}\}$. Then (1) implies that $\sum_{j=1}^{k_{2n}} |b_j^{n+1}|^p < 1/2^{n+4}(\varepsilon/2)^{2p}$ and (4) implies that $\sum_{j=k_{2n+2}}^{\infty} |b_j^{n+1}| < 1/2^{n+4}\times$ $\times(\varepsilon/2)^{2p}$. Hence

$$(6)\qquad \sum_{j\notin K_{n+1}} |b_j^{n+1}|^p < 1/2^{n+3}(\varepsilon/2)^{2p}.$$

Thus

$$\|c_n - \tilde{c}_n\|_p = \sum_{j\notin K_n} |c_j^n|^p = \sum_{j\notin K_n} |b_j^n|^p/\|b_n\|_p \leqslant \sum_{j\notin K_n} |b_j^n|^p/(\varepsilon/2)^p$$

$$< 1/2^{n+2}(\varepsilon/2)^{2p}/(\varepsilon/2)^p = 1/2^{n+2}(\varepsilon/2)^p.$$

This implies that

$$\sum_{n=1}^{\infty} \|c_n - \tilde{c}_n\|_p < \sum_{n=1}^{\infty} 1/2^{n+2}(\varepsilon/2)^p = 1/4(\varepsilon/2)^p = 1/2(1/2^{1/p}\varepsilon/2)^p.$$

Also

$$\sum_{j=k_{2n-2}+1}^{k_{2n}} |c_j^n| \geqslant \sum_{j=k_{2n-2}+1}^{k_{2n}} |c_j^n| = \sum_{j=k_{2n-2}+1}^{k_{2n}} |b_j^n|/\|b_n\|_p^{1/p}$$

$$\geqslant 1/2^{1/p} \sum_{j=k_{2n-2}+1}^{k_{2n}} |b_j^n| \geqslant 1/2^{1/p}\varepsilon/2$$

by (5). Finally

$$\sum_{j=k_{2n-2}+1}^{k_{2n}} |c_j^n|^p \geqslant 1 - \sum_{j\notin K_m} |c_j^n|^p \geqslant 1 - 1/2^{n+2}(\varepsilon/2)^{2p}$$

by (6). Thus, since $(\varepsilon/2)^{2p} < 1$,

$$\sum_{j=k_{2n-2}+1}^{k_{2n}} |c_j^n|^p \geqslant 1 - 1/2^{n+2}.$$

This shows that c_{n+1} satisfies the conditions of Lemma 1. Thus there is a projection P, $P(x) = \sum f_j(x)\tilde{c}_j$, of l_p onto $[\tilde{c}_j]$ and the mapping $A(x) = x - P(x) + \sum f_j(x)c_j$ is an isomorphism of l_p onto l_p. This implies that $Q = APA^{-1}$ is a continuous projection of l_p onto $[c_j]$. Since $[c_j]$ is a subspace of X which is isomorphic to l_p by Lemma 1, this completes the proof of the first implication.

Suppose that S_X^p is a precompact subset of l_1 and that Y is an infinite-dimensional subspace of X which is isomorphic to l_p. Since $S_Y^p \subset S_X^p$, S_Y^p is precompact in l_1 (cf. [6], p. 51), and so co S_Y^p is precompact in the l_1 topology of Y. Thus the Minkowski functional of co S_Y^p is a norm, $|\ |$, on Y which is strictly stronger than $\|\ \|_1$ on Y. This implies that there is a linear functional f which is continuous with respect to $|\ |$ which is not continuous with respect to $\|\ \|_1$. Thus f cannot be extended to a continuous linear functional on l_p. However, since Y is complemented in l_p, every continuous linear functional on Y can be extended to a continuous linear functional on l_p. This contradiction completes the proof of the theorem.

THEOREM 2. *Every complemented infinite-dimensional subspace of l_p, $0<p<1$, is isomorphic to l_p.*

Proof. Let X be an infinite-dimensional complemented subspace, of l_p. If X contains a subspace which is both complemented in l_p and isomorphic to l_p, then X is isomorphic to l_p by Proposition 4 of [4] (cf. [8]). Hence if X is not isomorphic to l_p, $S_X^p = \{x \in X\colon \|x\|_p \leqslant 1\}$ is a precompact subset of l_1 by Theorem 1. However, since X is complemented, the last part of the proof of Theorem 1 shows that S_X^p cannot be precompact. We conclude that X is isomorphic to l_p.

3. Extension Property. It seems natural to ask for a characterization of the infinite-dimensional subspaces of l_p for which every continuous linear functional can be extended continuously to all of l_p. It follows of course from the proof of Theorem 1 that no such subspace whose unit ball is l_1 precompact has this property. Although we cannot give a satisfactory characterization of these subspaces, we can show that the l_1 precompactness of their unit ball does not characterize them. Indeed, the referee has pointed out to us that a simple example of such a subspace can be obtained in the following manner. Select two isometric copies, X and Y, of l_p such that X and Y have "disjoint supports" and the unit ball in X is l_1 precompact while the unit ball of Y is not l_1 precompact (cf. [8]). Then if Z is the direct sum $Z = X \oplus Y$, Z is easily seen to have the desired properties. We will now give another example of such a subspace. This subspace X_p has some interesting properties which will be discussed in the following lemmas. In particular X_p (as a subset of l_1) is dense in U_0 where U_0 is a subspace of l_1 which is an $\mathscr{L}_1$ space without an unconditional basis which is not isomorphic to l_1 (cf. [1] and [3]). Furthermore, if $\hat{X}_p$ denotes the completion of X_p when X_p has the strongest locally convex topology weaker than the l_p topology, then $\hat{X}_p$ is isomorphic to l_1.

We begin by letting A be the continuous linear mapping of l_1 onto L_1 such that

$$A(b_{2^n+k}) = 2^n \chi_{[k2^{-n},(k+1)2^{-n}]},$$

$0 \leqslant k < 2^n$, $n = 0, 1, \ldots$, where $b_j = (0, \ldots, 0, 1_j, 0, \ldots)$. This mapping is essentially T_0, the mapping given by Lindenstrauss in [1]. It is stated without proof in [1] that the mapping T_0 is an isometry of l_1/U_0 onto L_1 where U_0 is the kernel of T_0. It is not difficult to see that A is also an isometry of l_1/X_1 onto L_1 where X_1 is the kernel of A in l_1. Let e_n be the element in l_1 defined by

$$e_n(i) = \begin{cases} 1 & \text{if } i = n, \\ -\frac{1}{2} & \text{if } i = 2n, 2n+1, \\ 0 & \text{otherwise} \end{cases}$$

for $n = 1, 2, \ldots$ Obviously $A(e_n) = 0$.

LEMMA 2. *For each p, $0 < p \leqslant 1$, the sequence $\{e_n\}$ is a monotone basis for a subspace of l_p.*

Proof. Let $\{a_j\}$ be any sequence of real numbers, and, for each n, let $a^n = a_1 e_1 + \ldots + a_n e_n$. If $[(n+1)/2]$ denotes the smallest integer greater than or equal to $(n+1)/2$, then

$$\begin{aligned} \|a^{n+1}\|_p - \|a^n\|_p &= |a_{n+1} - \tfrac{1}{2} a_{[(n+1)/2]}|^p + 2|\tfrac{1}{2} a_{n+1}|^p - |\tfrac{1}{2} a_{[(n+1)/2]}|^p \\ &\geqslant |\tfrac{1}{2} a_{[(n+1)/2]}|^p - |a_{n+1}|^p + 2/2^p\, |a_{n+1}|^p - |\tfrac{1}{2} a_{[(n+1)/2]}|^p \geqslant 0 \end{aligned}$$

because $0 < p \leqslant 1$. The theorem follows immediately when $p = 1$; the theorem follows for $0 < p < 1$ from the proof of a well-known theorem of M. M. Grynblum (see e. g. [9], p. 211).

LEMMA 3. *For $0 < p \leqslant 1$, let $X_p = \{x \in l_p \colon A(x) = 0\}$; then $\{e_j\}$ is a basis for X_p.*

Proof. Since $\{e_j\}$ is a basic sequence and $e_j \in X_p$, it is sufficient to show that the set of all finite linear combinations of the form $\sum a_k e_k$ is dense in X_p. To see that this is true, let $x = (x_1, x_2, \ldots)$ be any element in X_p, and let ε be any positive number. Choose a positive integer N such that

$$\sum_{j=2^N} |x_j|^p < \varepsilon/2,$$

and choose $\bar{x}_j$, $2^N \leqslant j \leqslant 2^{N+1}-1$, such that

$$A(x_1, \ldots, x_{2^N-1}, \bar{x}_{2^N}, \ldots, \bar{x}_{2^{N+1}-1}, 0, \ldots) = 0.$$

Since $A(0, \ldots, 0, \bar{x}_{2^N}, \ldots, \bar{x}_{2^{N+1}-1}, 0, \ldots) = A(0, \ldots, 0, x_{2^N}, x_{2^N+1}, \ldots)$, it follows that

$$\begin{aligned} A(0, \ldots, 0, \bar{x}_k, 0, \ldots) = A(0, \ldots, 0, x_k, 0, \ldots, 0, x_{2k}, x_{2k+1}, 0, \ldots \\ \ldots, 0, x_{4k}, \ldots, x_{4k+3}, 0, \ldots) \end{aligned}$$

for $2^N \leqslant k \leqslant 2^{N+1}-1$. Since $A\colon l_1/X_1 \to L_1$ is an isometry,

$$\begin{aligned} |\bar{x}_k| &= d\big((0, \ldots, 0, \bar{x}_k, 0, \ldots), X_1\big) \\ &= d\big((0, \ldots, 0, x_k, 0, \ldots, 0, x_{2k}, x_{2k+1}, 0, \ldots), X_1\big) \\ &\leqslant \|(0, \ldots, 0, x_k, 0, \ldots, 0, x_{2k}, x_{2k+1}, 0, \ldots)\| \end{aligned}$$

for $2^N \leqslant k \leqslant 2^{N+1}-1$, where $d(x, X_1)$ denotes the l_1 distance of x from X_1. Therefore,

$$\|(0, \ldots, 0, \bar{x}_{2^N}, \ldots, \bar{x}_{2^{N+1}-1}, 0, \ldots)\|_p \leqslant \|(0, \ldots, 0, x_{2^N}, x_{2^N+1}, \ldots)\|_p < \varepsilon/2$$

and this implies that

$$\|x - (x_1, \ldots, x_{2^N-1}, \bar{x}_{2^N}, \ldots, \bar{x}_{2^{N+1}-1}, 0, \ldots)\|_p < \varepsilon.$$

Since $(x_1, \ldots, x_{2^N-1}, \bar{x}_{2^N}, \ldots, \bar{x}_{2^{N+1}-1}, 0, \ldots) \in X_p$, this last inequality completes the proof.

LEMMA 4. *There exists a positive number K_p such that $K_p \overline{\mathrm{co}}_{X_p}(\{e_j\}) \supset \{x \in X_p \colon \|x\|_p \leqslant 1\}$ where $\overline{\mathrm{co}}_{X_p}(\{e_j\})$ is the l_p closure of the convex hull of the set $\{e_j\}$.*

Proof. If $x = a_1 e_1 + \ldots + a_n e_n$ and $\|x\|_p \leqslant 1$, then an application of the triangle inequality yields

$$|a_1|^p + 2^{1-p}\Big(\sum_1^n |a_j|^p\Big) - \sum_1^n |a_j|^p \leqslant 1.$$

Therefore,

$$\sum_1^n |a_j|^p \leqslant 1/(2^{1-p}-1) \quad \text{and} \quad \sum_1^n |a_j| \leqslant \big(1/(2^{1-p}-1)\big)^{1/p}.$$

By letting $K_p = \big(1/(2^{1-p}-1)\big)^{1/p}$ and noting that $\{e_j\}$ is a monotone basis for X_p, we see that the conclusion follows easily.

Since X_p is a locally bounded space in the topology it inherits from l_p, the strongest locally convex topology on X_p which is weaker than the topology of X_p is a norm topology. A neighborhood base at the origin for this topology can be obtained by taking the convex hull of l_p neighborhoods of the origin. This norm topology will be called the *Mackey topology* for X_p. The Mackey topology is obviously stronger than the topology induced on X_p by the l_1 norm. However, an even stronger statement is true as is shown by the following lemma.

LEMMA 5. *The Mackey topology on X_p is strictly finer than the topology induced on X_p by the l_1 norm.*

Proof. Since the Mackey topology is a norm topology, we can assume that its unit ball is the set $\mathrm{co}_{X_p}(\{x \in X_p: \|x\|_p \leqslant 1\})$. Since this set is, by Lemma 4, contained in $K_p \overline{\mathrm{co}}_{X_p}(\{e_j\})$, it suffices to show that $R\,\overline{\mathrm{co}}_{X_p}(\{e_j\})$ does not contain $\{x \in X_p: \|x\|_1 \leqslant 1\}$ for any real number R. Suppose to the contrary that $R\,\overline{\mathrm{co}}_{X_p}(\{e_j\})$ contains $\{x \in X_p: \|x\|_1 \leqslant 1\}$. Then, since $\{e_j\}$ is a monotone basis for X_1, it follows that $R\,\overline{\mathrm{co}}_{X_1}(\{e_j\}) \supset \{x \in X_1: \|x\|_1 \leqslant 1\}$. Since the l_1 topology on X_1 is obviously weaker than the norm topology whose unit ball is $\overline{\mathrm{co}}_{X_1}(\{e_j\})$, this implies that the two topologies on X_1 are equivalent. We will complete the proof by showing that this is impossible.

Consider the linear mapping T of l_1 into X_1 which maps $(0, \ldots, 0, 1_j, 0, \ldots)$ onto $\frac{1}{2}e_j$. This mapping is one-to-one and continuous. Since $\{e_j\}$ is a basis and the coefficient functionals are continuous, $\{x \in X_1: x = \sum a_i e_i \text{ and } \sum |a_i| \leqslant 1\}$ is an l_1 closed convex subset of X_1. This implies that the range of T contains $\overline{\mathrm{co}}_{X_1}(\{e_j\})$. Thus, since $R\,\overline{\mathrm{co}}_{X_1}(\{e_j\}) \supset \{x \in X_1: \|x\|_1 \leqslant 1\}$, T maps l_1 onto X_1. Hence X_1 is isomorphic to l_1; however, this gives a contradiction since Lindenstrauss has shown that X_1 is not isomorphic to l_1 (cf. [1]).

The above lemma clearly implies the following theorem.

THEOREM 3. *For each p, $0 < p < 1$, the set $\{x \in X_p: \|x\|_p \leqslant 1\}$ fails to be a precompact subset of l_1 and there is a continuous linear functional on X_p which cannot be extended continuously to all of l_p.*

Since each complemented subspace X has the *extension property*, i. e., each continuous linear functional on X can be extended to a continuous linear functional on l_p, one might suspect that the quotient space l_p/X determines those X which have the extension property. Thus we state the following.

PROBLEM 0. *If X is a subspace of l_p and l_p/X is locally convex, does X have the extension property?*

In connection with this problem, we recall (cf. [8]) that every separable Banach space B is the quotient of each l_p space, $0 < p < 1$. In fact if $\{z_n\}$ is any sequence dense in the unit ball of B and $\{e_n\}$ is the usual unit vector basis in l_p, then the linear extension of the mapping A such that $A(e_n) = z_n$ is a continuous linear mapping of l_p onto B for each p, $0 < p \leqslant 1$. A slightly stronger result is given in the following.

PROPOSITION. *Let B be a separable Banach space and $A: l_p \to B$ a mapping of the form given above. If $l_0 \bigcap_{p>0} l_p$ and $X_p = \{x \in l_p: A(x) = 0\}$, then B is a quotient of l_0 and every element y in l_p can be written as $y = u + v$ where $u \in l_0$ and v is in the p-closure of X_0.*

Proof. l_0 is an F-space when given the "sup" topology, and $A: l_0 \to B$ is clearly continuous in this case. By using the technique given in Theorem 4.1 of [8], one can show that A actually maps l_0 onto B. Hence B is a quotient of l_0. Since l_0 is dense in l_p and l_0/X_0 is isomorphic to B, X_0 is dense in X_p. If y is any element of l_p, there is an element u in l_0 such that $A(u) = A(y)$. Hence $y - u$ is in X_p.

4. Mappings.

In this section, we investigate properties of mappings into l_p spaces.

THEOREM 4. *Every continuous linear mapping of a normed linear space into l_p, $0 < p < 1$, is compact.*

Proof. Let X be a normed linear space and A a mapping of X into l_p. Then $A(x) = \sum f_j(x) e_j$ where $\{e_j\}$ is the unit vector basis in l_p and f_j is a continuous linear functional on X. Let S_X be the unit ball of X. If $A(S_X)$ is not precompact, then by Theorem B. IV. 2.1 of [7] there is an $\varepsilon > 0$ and a sequence $\{x_n\}$ of elements of S_X such that given any positive integer N there exists an integer n such that $\sum_{j=N}^{\infty} |f_j(x_n)|^p > \varepsilon$. Let $A(x_n) = y_n = (y_1^n, y_2^n, \ldots)$. By using a diagonalization process if necessary, we can assume that $\lim_n y_j^n$ exists for each n. Let $y_j = \lim_n y_j^n$. Select two strictly increasing sequences $\{h_n\}$ and $\{k_n\}$ of non-negative integers and a sequence $\{z_n\}$ of vectors in the following manner. Let $z_0 = y_1$, $h_0 = 0$, and choose k_0 such that

$$\sum_{j=k_0}^{\infty} |y_j^1|^p < \varepsilon/4 \quad \text{and} \quad \sum_{j=1}^{k_0} |y_j^1|^p > \varepsilon/2.$$

Having chosen $z_n, h_1, \ldots, h_{2n}$, and $k_1, \ldots, k_{2n}$, choose h_{2n+1} such that

$$\sum_{j=1}^{k_{2n}} |y_j^m - y_j|^p < \varepsilon/2^{n+2} \tag{1}$$

for every $m \geqslant h_{2n+1}$. Select k_{2n+1} such that

$$\sum_{j=k_{2n+1}}^{\infty} |y_j^{h_{2n+1}}|^p < \varepsilon/2; \tag{2}$$

then select h_{2n+2} such that

$$\sum_{j=k_{2n+1}}^{\infty} |y_j^{h_{2n+2}}|^p > \varepsilon, \tag{3}$$

and let $z_{n+1} = y^{h_{2n+2}} - y^{h_{2n+1}}$. Finally, use (2) and (3) to choose k_{2n+2} such that

$$\sum_{j=k_{2n+2}}^{\infty} |z_j^{n+1}|^p < \varepsilon/2^{n+3} \tag{4}$$

and

$$\sum_{j=k_{2n+1}}^{k_{2n+2}} |z_j^{n+1}|^p > \varepsilon/2, \tag{5}$$

where $z_{n+1} = (z_1^{n+1}, z_2^{n+1}, \ldots)$. Note that (1) implies

$$\sum_{j=1}^{k_{2n}} |z_j^{n+1}|^p < \varepsilon/2^{n+1}. \tag{6}$$

Also note that $A(x^{h_{2n+2}} - x^{h_{2n+1}}) = z_{n+1}$ and that for each positive integer m,

$$\left\|(1/m)\left((x^{h_2}-x^{h_1})+(x^{h_4}-x^{h_3})+\ldots+(x^{h_{2m}}-x^{h_{2m-1}})\right)\right\| \leqslant 2.$$

On the other hand

$$\|(1/m)(z_1+z_2+\ldots+z_m)\|_p$$
$$\geqslant (1/m^p)\Big(\Big(\sum_{j=1}^{k_2}|z_j^1|^p - \sum_{r\neq 1}\sum_{j=1}^{k_2}|z_j^r|^p\Big) + \Big(\sum_{j=k_2+1}^{k_4}|z_j^2|^p - \sum_{r\neq 2}\sum_{j=k_2+1}^{k_4}|z_j^r|^p\Big) +$$
$$+\ldots+\Big(\sum_{j=k_{2m-2}+1}^{k_{2m}}|z_j^m|^p - \sum_{r\neq m}\sum_{j=k_{2m-2}+1}^{k_{2m}}|z_j^r|^p\Big)\Big) \geqslant (1/m^p)\big((\varepsilon/2-\varepsilon/4)m\big)$$

by (4), (5), and (6). Since $\varepsilon/4\, m^{1-p} \to \infty$ as $m\to\infty$, $A(S_X)$ is not a bounded set in l_p. Thus $A: X\to l_p$ is not continuous.

COROLLARY. *If X is a reflexive Banach space and $T: X\to l_p$, $0<p<1$, is continuous, then $T(S_X)$ is compact where S_X is the unit ball of X.*

Proof. T is weakly continuous. Hence $T(S_X)$ is weakly compact and therefore closed.

We note that no continuous linear mapping of a Banach space into l_p, $0<p<1$, can have a closed infinite-dimensional range. We also note that Theorem 4 could easily be extended to show that a continuous linear mapping of a q-normed space into l_p is compact whenever $q>p$ and $0<p<1$.

5. Problems. It is known from [3] that every normalized unconditional basis of l_1 is equivalent to the unit vector basis $\{e_n\}$. This leads to the following.

PROBLEM 1. *Is every normalized unconditional basis of l_p, $0<p<1$, equivalent to $\{e_n\}$?*

It was shown in [5] that the identity map $I: l_1\to l_q$, $q\geqslant 1$ is absolutely summing if and only if $q\geqslant 2$. This leads to the following.

PROBLEM 2. *For precisely what values of q, $q>p$, is $I: l_p\to l_q$ absolutely summing?*

In [2], the authors show that if l_1/X is isomorphic to l_1/Y where X and Y are infinite-dimensional subspaces of l_1, then X is isomorphic to Y. Their proof is based on three properties of l_1 given in Lemma 2 of [2]. Since the first two of these properties hold in l_p, $0<p<1$, and it appears possible to replace the third property with a weaker condition, it seems natural to ask the following.

PROBLEM 3. *If X and Y are infinite-dimensional subspaces of l_p, $0<p<1$, and if l_p/X is isomorphic to l_p/Y, is X isomorphic to Y?*

PROBLEM 4. *Is every complemented subspace of L_p, $0<p<1$, isomorphic to L_q for some q, $0<q<1$?*

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