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On the extension of Lipschitz-Hölder maps on Orlicz Spaces

by

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Abstract. Let φ be a Young's function, L^φ the corresponding Orlicz space over the measure space $(X, \mathscr{A}, \mu)$, $\varphi_0(x) = x^2$ and φ_s the inverse of the function $\varphi_s^{-1}(\varphi^{-1})^{1-s}(\varphi_0^{-1})^s$, $0 \leqslant s \leqslant 1$. A generalized interpolation theorem is used to find conditions so that for arbitrary $D \subset L^{\varphi_s}$, every Lipschitz map of order α, $0 < \alpha \leqslant 1$, from D into a Hilbert space H can be extended to a map of the same order on all of L^{φ_s}. Also, conditions are given so that the same is true if H is replaced by an intermediate Orlicz space.

1. Introduction. A map T from a subset D of a metric space (X, d_1) into a metric space (Y, d_2) satisfies a Lipschitz-Hölder continuity condition of order α and belongs to the class $\operatorname{Lip}(D, Y, \alpha)$ provided

$$d_2(Tx_1, Tx_2) \leqslant [d_1(x_1, x_2)]^\alpha, \quad x_1, x_2 \in D. \tag{1.1}$$

The problem of interest here is in extending maps in $\operatorname{Lip}(D, Y, \alpha)$ to maps in $\operatorname{Lip}(X, Y, \alpha)$ independent of D. Thus, the statement "extension holds for α" or "$e(X, Y, \alpha)$ holds" means that for arbitrary $D \subset X$, every map in $\operatorname{Lip}(D, Y, \alpha)$ can be extended to a map in $\operatorname{Lip}(X, Y, \alpha)$. Much of the literature has been concerned with the case $\alpha = 1$ (contraction maps) with X and/or Y Hilbert spaces. Early consideration of the problem includes results of MacShane [6] and Banach [1] in the case Y is the real line, and results of Kirszbraun [4] imply that $e(H, H, 1)$ holds for H a Hilbert space. Recently, it was shown by Hayden and Wells [3] that $e(X, H, \alpha)$ holds for any metric space X with $0 < \alpha \leqslant 1/2$ and, furthermore $e(L^p, H, \alpha)$ holds for $2\alpha \leqslant p \leqslant 2\alpha/2\alpha-1$ with $1/2 < \alpha < 1$. An extension of these results has been obtained in [10] with H replaced by an L^q space.

In this paper, the results in L^p spaces are extended to a class of Orlicz spaces which include intermediate spaces between a given Orlicz space and L^2. The main results are contained in Sections 4 and 5 while Section 3 is devoted to generalizing an interpolation theorem of Rao [9] which is the main tool used in attacking the problem.

2. Preliminaries. Let (φ, ψ) be a pair of complementary Young's functions ([5], pp. 12–13). Then $L^{\varphi}(X, \mathscr{A}, \mu)\,(=L^{\varphi})$ denotes the Orlicz space of measurable scalar valued functions on the measure space $(X, \mathscr{A}, \mu)$ such that $f \in L^{\varphi}$ iff $N_{\varphi}(f) < \infty$, where

$$N_{\varphi}(f) = \inf\Big\{k > 0:\ \int_X \varphi(|f|/k)\,d\mu \leqslant 1\Big\}. \tag{2.1}$$

Similarly we can define the space L^{ψ}. Another norm can be defined in L^{φ};

$$\|f\|_{\varphi} = \sup\Big\{\int_X |fg|\,d\mu:\ N_{\psi}(g) \leqslant 1\Big\}. \tag{2.2}$$

These norms are equivalent if every set of positive μ-measure contains a subset of positive finite μ-measure (cf [12]). In the case $\varphi(x) = |x|^p$, $p > 1$, it can be shown that $\|f\|_p = N_{\varphi}(f) = K\|f\|_{\varphi}$, where K is independent of f (cf [11]).

Definition 2.1. Let φ_1 and φ_2 be Young's functions and define φ_s to be the inverse of $\varphi_s^{-1} = (\varphi_1^{-1})^{1-s}(\varphi_2^{-1})^s$ for $0 \leqslant s \leqslant 1$, where φ^{-1} is the unique inverse of the Young's function φ.

The function φ_s is convex and has most of the common characteristics of φ_1 and φ_2 (cf [2] or [9]) including the property of satisfying a growth condition. That is, if the simple functions are dense in L^{φ_1} and L^{φ_2}, then the same is true in L^{φ_s}.

The complementary function of φ_s is not the same as the inverse of $\psi_s^{-1} = (\psi_1^{-1})^{1-s}(\psi_2^{-1})^s$ where ψ_1 and ψ_2 are the respective complements of φ_1 and φ_2. However, the complement of φ_s and ψ_s generate the same Orlicz space with equivalent norms (cf [9]). Since the complementary function is the only one of interest in this paper, ψ_s will denote the complement of φ_s unless otherwise stated. Also, it will be assumed that each Young's function is continuous with a continuous derivative.

3. Interpolation. In this section Theorem 1 in [9] is generalized similar to the way the Riesz–Thorin theorem was generalized in [3].

Let $(X_1, \mu_1), (X_2, \mu_2), \ldots, (X_n, \mu_n)$ be σ-finite measure spaces and let $\varphi = (\varphi_1, \ldots, \varphi_n)$ be an n-tuple of Young's functions such that φ_j satisfies a growth condition on (X_j, μ_j), $j = 1, 2, \ldots, n$. Define the direct sum $\oplus L^{\varphi_k}(\mu_k)$ by

$$\oplus L^{\varphi_k}(\mu_k) = \{f = (f_1, \ldots, f_n):\ f_k \in L^{\varphi_k}(\mu_k),\ k = 1, 2, \ldots, n\} \tag{3.1}$$

with usual addition and scalar multiplication. For each r, $1 \leqslant r \leqslant \infty$ and each n-tuple $\lambda = (\lambda_1, \ldots, \lambda_n)$ of positive weights, introduce the following norm on $\oplus L^{\varphi_k}(\mu_k)$,

$$\|f\|_{\varphi,r} = \begin{cases} \Big\{\sum_{k=1}^{n} \|f_k\|_{\varphi_k}^r \lambda_k\Big\}^{1/r} & 1 \leqslant r < \infty, \\ \max\limits_{1 \leqslant k \leqslant n} \|f_k\|_{\varphi_k} & r = \infty. \end{cases} \tag{3.2}$$

Denote by $L^{\varphi,r}(\lambda)$ the set of vectors f such that $\|f\|_{\varphi,r} < \infty$. The space $L^{\varphi,r}$ is a Banach space and if we let $\psi = (\psi_1, \ldots, \psi_n)$ with ψ_j the complementary function of φ_j, then it follows from lemma 1 in [7] that

$$\|f\|_{\varphi,r} = \sup\Big|\int fg\,d\lambda\Big| = \sup\Big|\sum_{k=1}^{n}\Big(\int_{X_k} f_k g_k\,d\mu_k\Big)\lambda_k\Big|, \tag{3.3}$$

where g varies over all simple functions in $L^{\psi,r'}$, $1/r + 1/r' = 1$, and

$$N_{\psi,r'}(g) = \Big\{\sum_{k=1}^{n}\big(N_{\psi_k}(g_k)\big)^{r'}\lambda_k\Big\}^{1/r'} \leqslant 1.$$

Definition 3.1. For two n-tuples $\varphi_1 = (\varphi_{11}, \varphi_{12}, \ldots, \varphi_{1n})$ and $\varphi_2 = (\varphi_{21}, \ldots, \varphi_{2n})$ define $\varphi_s = (\varphi_{s1}, \ldots, \varphi_{sn})$, $0 \leqslant s \leqslant 1$, where φ_{sk} is the inverse of the function $\varphi_{sk}^{-1} = (\varphi_{1k}^{-1})^{1-s}(\varphi_{2k}^{-1})^s$, $k = 1, 2, \ldots, n$.

Now, let $(Y_1, \nu_1), (Y_2, \nu_2), \ldots, (Y_m, \nu_m)$ be another sequence of σ-finite measure spaces, $\eta = (\eta_1, \ldots, \eta_m)$, and define the m-tuples Q_1 and Q_2 in the same manner as φ_1 and φ_2. Letting $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_m)$ we obtain the following interpolation theorem.

Theorem 3.1. *Let $1 \leqslant r_i, t_i \leqslant \infty$, $i = 1, 2$, $0 \leqslant s \leqslant 1$ with $1/r = (1-s)/r_1 + s/r_2$, $1/t = (1-s)/t_1 + s/t_2$ and suppose φ_i, Q_i, $i = 1, 2$ are defined on X and Y respectively with each component satisfying a growth condition. If T is a linear transformation from L^{φ_i,t_i} into L^{Q_i,r_i}, $i = 1, 2$ with bounds M_1 and M_2 respectively, then T takes $L^{\varphi_s,t}$ into $L^{Q_s,r}$ and*

$$\|Tf\|_{Q_s,r} \leqslant M_1^{1-s} M_2^s \|f\|_{\varphi_s,t} \tag{3.4}$$

Proof. The tuples φ_s and Q_s are those obtained from Definition 3.1 and from previous remarks, it follows that they satisfy a growth condition. The proof follows that of Theorem 1 in [9].

If R_i is the complementary function of Q_i, $i = 1, 2$, then the linearity of T and (3.3) imply that (3.4) will hold provided

$$\Big|\int Tfg\,d\eta\Big| \leqslant M_1^{1-s} M_2^s \tag{3.5}$$

holds for all simple vectors f on X and g on Y such that $\|f\|_{\varphi_s,t} = 1$, $N_{R_s,r'}(g) \leqslant 1$. Choose two such vectors $f = (f_1, \ldots, f_n)$, $g = (g_1, \ldots, g_m)$ and define $f_k = |f_k| e^{iu_k}$ and $g_k = |g_k| e^{iv_k}$ where u_k and v_k are real-valued simple measurable functions on X_k and Y_k respectively.

Let $\alpha_{sk} = \varphi_{sk}^{-1}$, $k = 1, 2, \ldots, n$, $\beta_{sj} = R_{sj}^{-1}$, $j = 1, 2, \ldots, m$ and extend these functions to the strip $0 \leqslant \operatorname{Re} z \leqslant 1$ by defining $\alpha_{zk} = (\varphi_{1k}^{-1})^{1-z}(\varphi_{2k}^{-1})^z$ and $\beta_{zj} = (R_{1j}^{-1})^{1-z}(R_{2j}^{-1})^z$ for each k and j.

For each $j = 1, 2, \ldots, n$, it follows (cf. [5], p. 92) that there exists a number $K_{sj} \geqslant 1$ such that

$$\int \varphi_{sj}\big(K_{sj}|f_j|/\|f_j\|_{\varphi_{sj}}\big)\,d\mu_j = K_{sj-1}. \tag{3.6}$$

Let $\gamma(z) = t[(1-z)/t_1+z/t_2]$, $\tau(z) = r'[(1-z)/r_1'+z/r_2']$ and define for each $j = 1, 2, \ldots, n$,

$$F_{zj} = (1/K_{sj})\|f\|_{\varphi_{sj}}^{\gamma(z)}\alpha_{zj}[\varphi_{sj}(K_{sj}|f_j|/\|f_j\|_{\varphi_{sj}})]e^{iu_j} \tag{3.7}$$

and $j = 1, 2, \ldots, m$,

$$G_{zj} = [N_{R_{sj}}(g_j)]^{\tau(z)}\beta_{zj}[R_{sj}(|g_j|/N_{R_{sj}}(g_j))]e^{iv_j}. \tag{3.8}$$

If we let $F_z = (F_{z1}, F_{z2}, \ldots, F_{zn})$ and $G_z = (G_{z1}, \ldots, G_{zm})$ then $F_s = f$ and $G_s = g$. Finally, define

$$H(z) = \int (TF_z)G_z d\eta = \sum_{k=1}^{m}\Big(\int_{Y_k}(TF_{zk})G_{zk}d\nu_k\Big)\eta_k.$$

Since f and g are simple, it can be shown that $H(z)$ is a finite linear combination of exponentials of the form d^z with $d>0$, which implies $H(z)$ is regular in the strip $0 \leqslant \operatorname{Re} z \leqslant 1$.

If $\operatorname{Re} z = 0$ then $TF_{zk} \in L^{Q_{1k}}$, $G_{zk} \in L^{R_{1k}}$ and by the Hölder inequality

$$\begin{aligned}|H(z)| &\leqslant \sum_{k=1}^{m}\|TF_{zk}\|_{Q_{1k}}N_{R_{1k}}(G_{zk})\eta_k \\ &\leqslant \|TF_z\|_{Q_1,r_1}N_{R_1,r_1'}(G_z) \leqslant M_1\|F_z\|_{\varphi_1,t_1}N_{R_1,r_1'}(G_z).\end{aligned}$$

Now

$$\begin{aligned}\|F_{iy}\|_{\varphi_1,t_1}^{t_1} &= \sum_{j=1}^{n}\|F_{(iy)j}\|_{\varphi_{1j}}^{t_1}\lambda_j \\ &= \sum_{j=1}^{n}\big\|\|f_j\|_{\varphi_{sj}}^{t_1\gamma(iy)}\big|\,\|(1/K_{sj})\alpha_{(iy)j}[\varphi_{sj}(K_{sj}|f_j|/\|f_j\|_{\varphi_{sj}})]\|_{\varphi_{1j}}^{t_1}\lambda_j.\end{aligned}$$

Denote by P_j the part inside the norms in the expression above, then according to ([5], pp. 92)

$$\begin{aligned}\|P_j\|_{\varphi_{1j}} &\leqslant (1/K_{sj})\Big(1+\int\varphi_{1j}(K_{sj}|P_j|)\Big) \\ &\leqslant (1/K_{sj})\Big(1+\int\varphi_{sj}(K_{sj}|f_j|/\|f_j\|_{\varphi_{sj}})\Big) = (1/K_{sj})(1+K_{sj}-1) = 1.\end{aligned}$$

Therefore

$$\|F_{iy}\|_{\varphi_1,t_1}^{t_1} \leqslant \sum_{j=1}^{n}\big|\|f_j\|_{\varphi_{sj}}^{t_1\gamma(iy)}\big|\lambda_j = \sum_{j=1}^{n}\|f_j\|_{\varphi_{sj}}^{t}\lambda_j = 1.$$

Since $\int R_{1j}\big(|\beta_{(iy)j}\;(R_{sj}|g_j|/N_{R_{sj}}(g_j))|\big) \leqslant 1$ it follows that $N_{R_{1j}}(|G_{(iy)j}|) \leqslant [N_{R_{sj}}(g_j)]^{r'/r_1'}$ and hence $N_{R_1,r_1'}(G_{iy}) \leqslant \{\sum_{j=1}^{m}[N_{R_{sj}}(g_j)]^{r'}\eta_j\}^{1/r_1'} \leqslant 1$. Thus $|H(iy)| \leqslant M_1$. Similarly, when $z = 1+iy$ we get $|H(1+iy)| \leqslant M_2$.

Hence by the "three-line theorem", we obtain (3.5).

4. Maps into Hilbert space. The results in this section are obtained by applying Theorem 3.1 and the following geometrical theorem proved by Hayden and Wells [3].

THEOREM 4.1. *Suppose X is a Banach space and $0 \leqslant \alpha \leqslant 1$. Then $e(X, H, \alpha)$ always holds provided that for any n points $x_1, x_2, \ldots, x_n$ of X, we have*

$$\sum_{i,j=1}^{n}C_iC_j\|x_i-x_j\|^{2\alpha} \leqslant 2\sum_{i=1}^{n}C_i\|x_i\|^{2\alpha} \tag{4.1}$$

whenever $C_i \geqslant 0$ and $\sum_{i=1}^{n}C_i = 1$.

The first application gives us another property that intermediate spaces inherit.

THEOREM 4.2. *Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space and φ_1, φ_2 Young's function satisfying a growth condition on X. If (4.1) holds for any n points in L^{φ_i}, $i = 1, 2$ and arbitrary collection $\{C_i\}_{i=1}^n$ of positive numbers such that $\sum_{i=1}^{n}C_i = 1$, then the same is true in L^{φ_s}, $0 \leqslant s \leqslant 1$, and hence $e(L^{\varphi_s}, H, \alpha)$ holds where $\varphi_s^{-1} = (\varphi_1^{-1})^{1-s}(\varphi_2^{-1})^s$.*

Proof. Let $C = \{C_i\}_{i=1}^n$ be a collection of positive numbers such that $\sum_{i=1}^{n}C_i = 1$, φ_i the constant n-tuple and Q_i the constant n^2-tuple with each component in both cases being φ_i, $i = 1, 2$. Define T from $L^{\varphi_i,2\alpha}(C)$ into $L^{Q_i,2\alpha}(C^2)$ by $Tf = (f_i-f_j)_{i,j=1}^n$ for each simple vector $f = (f_1, \ldots, f_n)$ with the norm defined in $L^{Q_i,2\alpha}$ so that $\|Tf\|_{Q_i,2\alpha}^{2\alpha} = \sum_{i,j=1}^{n}C_iC_j\|f_i-f_j\|_{\varphi_i}^{2\alpha}$, $i = 1, 2$. Thus by hypothesis $\|Tf\|_{Q_i,2\alpha} \leqslant 2^{1/2\alpha}\|f\|_{\varphi_i,2\alpha}$ and it follows from Theorem 3.1 with $r_1 = r_2 = t_1 = t_2 = 2\alpha$ and $M_1 = M_2 = 2^{1/2\alpha}$ that

$$\sum_{i,j=1}^{n}C_iC_j\|f_i-f_j\|_{\varphi_s}^{2\alpha} \leqslant 2\sum_{i=1}^{n}C_i\|f_i\|_{\varphi_s}^{2\alpha} \tag{4.2}$$

for all vectors $f = (f_1, \ldots, f_n)$ in L^{φ_s}.

THEOREM 4.3. *Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space, φ a Young's function satisfying a growth condition on X, and $0 < \alpha \leqslant 1$. If $\varphi_0(x) = x^2$ and φ_s, $0 \leqslant s \leqslant 1$ is the inverse of the function $\varphi_s^{-1} = (\varphi^{-1})^{1-s}(\varphi_0^{-1})^s$ then $e(L^{\varphi_s}, H, \alpha)$ holds for $2-1/\alpha \leqslant s \leqslant 1$.*

Proof. In the case $\alpha \leqslant 1/2$, the fact that the theorem is true for all s, $0 \leqslant s \leqslant 1$, follows from Theorem 4.1 (cf [3]). Thus we may assume $1/2 < \alpha \leqslant 1$.

Define the spaces as in the previous proof, let $t_1 = r_1 = 1$, $t_2 = r_2 = 2$ $M_1 = 2$, $M_2 = \sqrt{2}$, and apply Theorem 3.1 to the inequalities

$$\sum_{i,j=1}^{n}C_iC_j\|f_i-f_j\|_{\varphi} \leqslant 2\sum_{i=1}^{n}C_i\|f_i\|_{\varphi} \tag{4.3}$$

and

$$(4.4)\qquad \Big\{\sum_{i,j=1}^{n} C_iC_j\|f_i-f_j\|_{\varphi_0}^2\Big\}^{1/2} \leqslant \Big\{2\sum_{i=1}^{n} C_i\|f_i\|_{\varphi_0}^2\Big\}^{1/2}$$

to obtain

$$(4.5)\qquad \sum_{i,j=1}^{n} C_iC_j\|f_i-f_j\|_{\varphi_s}^{2/(2-s)} \leqslant 2\sum_{i=1}^{n} C_i\|f_i\|_{\varphi_s}^{2/(2-s)}.$$

The inequality (4.4) follows since it is true in L^2 and $\|f\|_{\varphi_0} = K\|f\|_2$. Setting $s = s' = 2-1/a$ in (4.5), we obtain (4.2) which implies that $e(L^{\varphi_{s'}}, H, a)$ holds in this case. For the spacial case $\varphi = \varphi_0$, it follows from (4.5) that

$$(4.6)\qquad \sum_{i,j=1}^{n} C_iC_j\|f_i-f_j\|_{\varphi_0}^{2a} \leqslant 2\sum_{i=1}^{n} C_i\|f_i\|_{\varphi_0}^{2a}, \qquad 1/2 < a < 1.$$

According to Theorem 4.2, inequalities (4.2) and (4.6) imply that $e(L^{\varphi_t}, H, a)$ holds for $0 \leqslant t \leqslant 1$ and $\varphi_t^{-1} = (\varphi_{s'}^{-1})^{1-t}(\varphi_0^{-1})^t$. If $s' \leqslant s \leqslant 1$, then setting $t = 1-[(1-s)/(1/a-1)]$ we see that $\varphi_t = \varphi_s$ and hence $e(L^{\varphi_s}, H, a)$ holds for $2-1/a \leqslant s \leqslant 1$.

COROLLARY 4.1. *Suppose $(X, \mathscr{A}, \mu)$ is a σ-finite measure space and $0 < a < 1$, then $e(L^p(\mu), H, a)$ holds for all $1 \leqslant p < \infty$ if $a \leqslant 1/2$ and for $2a \leqslant p \leqslant 2a/(2a-1)$ if $a > 1/2$.*

Proof. Let $\varphi(x) = |x|^k$, $1 < k < \infty$ and $\varphi_s^{-1} = (\varphi^{-1})^{1-s}(\varphi_0^{-1})^s$. Letting $s = (2/p)[(k-p)/(k-2)]$, then $\varphi_s = \varphi_p = |x|^p$ and from Theorem 4.3, $e(L^{\varphi_p}, H, a)$ holds provided $2-1/a \leqslant (2/p)[(k-p)/(k-2)] \leqslant 1$. Letting $k \to 1$ and $k \to \infty$, this inequality reduces to $2a \leqslant p \leqslant 2$ and $2 \leqslant p \leqslant 2a/(2a-1)$ respectively. Since $\|f\|_{\varphi_p} = K\|f\|_p$ for some constant K, we obtain the stated result.

Examples are given in [3] to show that these inequalities are sharp.

5. Maps into Orlicz spaces. Similar to the L^p case in [10], another application of Theorem 3.1 gives a new inequality which allows us to use a theorem of G. J. Minty [8] to obtain results in intermediate spaces.

Choose a collection $C = \{C_i\}_{i=1}^n$ of positive numbers such that $\sum_{i=1}^{n} C_i = 1$ and define $L^{\varphi,r}(C)$ as in Section 3. For a Young's function φ defined on the σ-finite measure space $(X, \mathscr{A}, \mu)$, let $L^{\varphi,r}(C^2)$ be the set of $(n-1)$-tuples $(f^1, f^2, \ldots, f^{n-1})$ in which f^j is an $(n-j)$-tuple of elements of L^φ. On this space introduce a new norm,

$$\|f\|_{\varphi,r} = \Big\{\sum_{i=1}^{n-1}\sum_{j>1} C_iC_j\|f_{j-1}^i\|_\varphi^r\Big\}^{1/r}, \qquad 1 \leqslant r < \infty$$

and

$$\|f\|_{\varphi,\infty} = \max_{\substack{1\leqslant i\leqslant n-1\\ j>1}} \|f_{j-1}^i\|_\varphi.$$

Define the operator T from the simple functions in $L^{\varphi,r}(C^2)$ into $L^{\varphi,r}(C)$ by $(Tf)_j = \sum_{i=1}^{n-j} C_{i+j}f_i^j - \sum_{k=1}^{j-1} C_kf_{j-k}^k$, $1 \leqslant j \leqslant n$, where the φ in defining $L^{\varphi,r}(C)$ denotes the constant n-tuple with each component φ. The following lemma is essentially established in [10].

LEMMA 5.1. *Suppose φ is a Young's function satisfying a growth condition on the σ-finite measure space $(X, \mathscr{A}, \mu)$ and $\varphi_0(x) = x^2$. Then for each simple vector f in $L^{\varphi,r}(C^2)$,*

$$(5.1)\qquad \|Tf\|_{\varphi_0,2} \leqslant \|f\|_{\varphi_0,2}$$

and

$$(5.2)\qquad \|Tf\|_{\varphi,\infty} \leqslant \|f\|_{\varphi,\infty}.$$

We can now apply Theorem 3.1 to obtain the following inequality.

LEMMA 5.2. *Suppose φ is a Young's function satisfying a growth condition on the σ-finite measure space $(X, \mathscr{A}, \mu)$ and $\varphi_0(x) = x^2$. Then for each $0 < s \leqslant 1$, with $\varphi_s^{-1} = (\varphi^{-1})^{1-s}(\varphi_0^{-1})^s$, we have,*

$$(5.3)\qquad \sum_{i,j=1}^{n} C_iC_j\|f_i-f_j\|_{\varphi_s}^{2/s} \geqslant 2\sum_{j=1}^{n} C_j\Big\|f_j - \sum_{i=1}^{n} C_if_i\Big\|_{\varphi_s}^{2/s}.$$

Proof. Apply Theorem 3.1 to the inequalities (5.1) and (5.2) with $t_1 = r_1 = \infty$, $t_2 = r_2 = 2$ and $M_1 = M_2 = 1$ we obtain

$$(5.4)\qquad \|Tf\|_{\varphi_s,2/s} \leqslant \|f\|_{\varphi_s,2/s}.$$

Let $f = (f_1, \ldots, f_n)$ be a simple measurable vector in $L^{\varphi,r}(C)$, then the simple vector $(f_1-f_2, f_1-f_3, \ldots, f_1-f_n, f_2-f_3, \ldots, f_{n-1}-f_n)$ is an element of $L^{\varphi,r}(C^2)$ where $f_i^j = f_j - f_{i+j}$.

$$\begin{aligned}\|Tf\|_{\varphi_s,2/s}^{2/s} &= \sum_{j=1}^{n} C_j\|(Tf)_j\|_{\varphi_s}^{2/s}\\ &= \sum_{j=1}^{n} C_j\Big\|\sum_{i=1}^{n-j} C_{i+j}(f_j-f_{i+j}) - \sum_{k=1}^{j-1} C_k(f_k-f_j)\Big\|_{\varphi_s}^{2/s}\\ &= \sum_{j=1}^{n} C_j\Big(\Big\|\sum_{i=j+1}^{n} C_i(f_j-f_i) - \sum_{k=1}^{j-1} C_k(f_k-f_j)\Big\|_{\varphi_s}^{2/s}\Big)\\ &= \sum_{j=1}^{n} C_j\Big(\Big\|\sum_{i=1}^{n} C_i(f_j-f_i)\Big\|_{\varphi_s}^{2/s}\Big)\\ &= \sum_{j=1}^{n} C_j\Big(\Big\|f_j - \sum_{i=1}^{n} C_if_i\Big\|_{\varphi_s}^{2/s}\Big)\end{aligned}$$

and

$$\|f\|_{\varphi_s,2/s}^{2/s} = \sum_{i=1}^{n-1}\sum_{j>i} C_iC_j\|f_i-f_j\|_{\varphi_s}^{2/s} = 1/2\sum_{i,j=1}^{n} C_iC_j\|f_i-f_j\|_{\varphi_s}^{2/s}.$$

Thus using (5.4),

$$2\sum_{j=1}^{n} C_j \Big\|f_j - \sum_{i=1}^{n} C_i f_i\Big\|_{\varphi_s}^{2/s} \leqslant \sum_{i,j=1}^{n} C_i C_j \|f_i - f_j\|_{\varphi_s}^{2/s}.$$

DEFINITION 5.1. Suppose Y is a vector space over the reals and X is a set. A map $K\colon Y\times X\times X \to R$ is called a *K-function* provided that

(i) for each $x_1, x_2 \in X$, K is finitely lower semicontinuous and convex on Y; and

(ii) for any sequence $(y_1, x_1), \dots (y_n, x_n)$ in $Y\times X$, any $x\in X$ and any sequences $C_1, \dots, C_n$ of nonnegative numbers with $\sum_{i=1}^{n} C_i = 1$ one has

$$\sum_{i,j=1}^{n} C_i C_j K(y_i - y_j; x_i, x_j) \geqslant 2\sum_{i=1}^{n} C_i K\Big(y_i - \sum_{j=1}^{n} C_j y_j; x_i, x\Big). \tag{5.5}$$

THEOREM 5.1 (MINTY). *Let Y be a linear space, X a space and K a K-function on $Y\times X\times X$. If $(y_1, x_1), \dots (y_n, x_n)$ is a finite sequence in $Y\times X$ such that $K(y_i - y_j; x_i, x_j) \leqslant 0$ for all i, j, $1\leqslant i, j\leqslant n$, and if $x\in X$, then there exists a vector $y\in Y$ such that $K(y_i - y; x_i, x) \leqslant 0$ for all i, $1\leqslant i\leqslant n$. Moreover y can be chosen in the convex hull of $\{y_1, y_2, \dots, y_n\}$.*

In order to extend Lipschitz–Hölder maps between Orlicz spaces, we combine Lemma 5.2 with the Minty criterion for extensions.

THEOREM 5.2 *Let φ_1 and φ_2 be Young's functions satisfying a growth condition on the measure space $(X, \mathscr{A}, \mu)$ and suppose L^{φ_1} is reflexive. If $2-1/a\leqslant t\leqslant 1$, $s = a(2-t)$ and $\varphi_s^{-1} = (\varphi_1^{-1})^{1-s}(\varphi_0^{-1})^s$, $\varphi_t^{-1} = (\varphi_2^{-1})^{1-t}(\varphi_0^{-1})^t$ then $e(L^{\varphi_t}, L^{\varphi_s}, a)$ holds.*

Proof. This proof follows closely the proof of Theorem 1 in [10].

Fix t and let $F\in \mathrm{Lip}(D, L^{\varphi_s}, a)$ where D is a subset of L^{φ_t}. Define $K\colon L^{\varphi_s}\times L^{\varphi_t}\times L^{\varphi_t}\to R$ by $K(g; f_i, f_j) = \|g\|_{\varphi_s}^{2/s} - \|f_i - f_j\|_{\varphi_t}^{2/(2-t)}$. Using (4.5) and (5.3), we see that

$$\sum_{i,j=1}^{n} C_i C_j K(g_i - g_j;\ f_i, f_j)$$

$$= \sum_{i,j=1}^{n} C_i C_j \|g_i - g_j\|_{\varphi_s}^{2/s} - \sum_{i,j=1}^{n} C_i C_j \|(f_i - f) + (f_j - f)\|_{\varphi_t}^{2/(2-t)}$$

$$\geqslant 2\sum_{j=1}^{n} C_j \Big\|g_j - \sum_{i=1}^{n} C_i g_i\Big\|_{\varphi_s}^{2/s} - 2\sum_{j=1}^{n} C_j \|f_j - f\|_{\varphi_t}^{2/(2-t)}$$

$$= 2\sum_{j=1}^{n} C_j \Big[\Big\|g_j - \sum_{i=1}^{n} C_i g_i\Big\|_{\varphi_s}^{2/s} - \|f_j - f\|_{\varphi_t}^{2/(2-t)}\Big]$$

$$= 2\sum_{j=1}^{n} K\Big(g_j - \sum_{i=1}^{n} C_i g_i; f_j, f\Big).$$

Since K is continuous and convex on L^{φ_s}, it is a K-function. In order to extend the domain of F to a point $h\in L^{\varphi_t}/D$, it is enough to show that $\bigcap_{f\in D} S_{F(f)} \neq \varphi$ where

$$S_{F(f)} = \{g\in L^{\varphi_s}\colon \|g - F(f)\|_{\varphi_s} \leqslant \|h - f\|_{\varphi_t}^{a}\}.$$

Fix $f_0\in D$ and define $S^0_{F(f)} = S_{F(f)} \cap S_{F(f_0)}$ for $f\in D$. Since L^{φ_1} is reflexive, it follows that L^{φ_s} is reflexive and hence $S_{F(f_0)}$ is weakly compact. Thus $\bigcap_{f\in D} S_{F(f)} = \bigcap_{f\in D} S^0_{F(f)} \neq \varphi$ if the finite intersection property holds. Suppose $f_1, f_2, \dots, f_n$ is a finite set in D and note that for $1\leqslant i, j\leqslant n$,

$$K\big(F(f_i) - F(f_j), f_i, f_j\big) = \|F(f_i) - F(f_j)\|_{\varphi_s}^{2/s} - \|f_i - f_j\|_{\varphi_t}^{2/(2-t)} \leqslant 0$$

since $\|F(f_i) - F(f_j)\|_{\varphi_s}^{2/s} \leqslant \|f_i - f_j\|_{\varphi_t}^{2a/s} = \|f_i - f_j\|_{\varphi}^{2/(2-t)}$.

By Theorem 5.1, there exists a g in L^{φ_s} such that $K\big(F(f_i) - g, f_i, h\big) \leqslant 0$ for $i = 1, 2, \dots, n$. Therefore

$$\|F(f_i) - g\|_{\varphi_s}^{2/s} - \|f_i - h\|_{\varphi_t}^{2/(2-t)} \leqslant 0$$

or

$$\|F(f_i) - g\|_{\varphi_s} \leqslant \|f_i - h\|_{\varphi_t}^{[2/(2-t)](s/2)} = \|f_i - h\|_{\varphi_t}^{a}.$$

COROLLARY 5.1 *Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space and $1 < p, q < \infty$. Then $e(L^p, L^q, a)$ holds for*

(i) *$2a < p < 2$ and $p/p - a \leqslant q$,*

(ii) *$2 < p < 2a/2a - 1$ and $aq \leqslant p/p - 1$.*

Proof. To prove (i), choose $k > 1$ sufficiently close to one so that $t = 2(p-k)/p(2-k) \geqslant 2 - 1/a$ and let $\varphi_2(x) = |x|^k$. With $s = a(2-t)$ and φ_s, φ_t as in Theorem 5.2, we can find $\varphi_1(x) = |x|^l$ so that $\varphi_s(x) = |x|^q$ provided that

$$1/q + a - 2(p-k)/p(2-k) \leqslant 1. \tag{5.6}$$

Letting $k\to 1$, we see that $1/q \leqslant (p-a)/p$ or $q\geqslant p/(p-a)$.

The proof of (ii) is similar if we choose k sufficiently large so that $t = 2(k-p)/p(k-2) \geqslant 2 - 1/a$.

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Drury's lemma and Helson sets

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Abstract. A generalization of a lemma of Drury [2] is used to obtain extrapolations of continuous functions on Helson sets by absolutely convergent Fourier transforms which are small on given closed sets of the complement. This simplifies the work of Varopoulos [3].

The idea of S. W. Drury [2] used in proving that the union of Sidon sets is a Sidon set can be generalized to give a result, Theorem 1 below, which is of considerable interest in its own right. The main point of this article is that the generalization allows one to obtain a simple proof of the fact that the union of Helson sets is a Helson set, a theorem obtained by Varopoulos [3] in an extremely complicated manner. Some of the broad lines of Varopoulos' argument remain, albeit in a simplified form. In particular, rather general locally compact commutative groups play an essential role so that even of one is only interested in Helson sets on the circle group the reasoning here perforce leaves the domain of classical harmonic analysis. In compensation, the methods used here give a sharper result than Varopoulos' note [4] on the classical situation, and genuinely less effort is required.

Our Theorem 2 below is new only in the sense that it is completely general. By contrast, Theorem 3 brings some precision to the estimates which is of interest even in the classical case, and it is a marked improvement over anything heretofore obtained.

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