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Drury's lemma and Helson sets

by

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Abstract. A generalization of a lemma of Drury [2] is used to obtain extrapolations of continuous functions on Helson sets by absolutely convergent Fourier transforms which are small on given closed sets of the complement. This simplifies the work of Varopoulos [3].

The idea of S. W. Drury [2] used in proving that the union of Sidon sets is a Sidon set can be generalized to give a result, Theorem 1 below, which is of considerable interest in its own right. The main point of this article is that the generalization allows one to obtain a simple proof of the fact that the union of Helson sets is a Helson set, a theorem obtained by Varopoulos [3] in an extremely complicated manner. Some of the broad lines of Varopoulos' argument remain, albeit in a simplified form. In particular, rather general locally compact commutative groups play an essential role so that even of one is only interested in Helson sets on the circle group the reasoning here perforce leaves the domain of classical harmonic analysis. In compensation, the methods used here give a sharper result than Varopoulos' note [4] on the classical situation, and genuinely less effort is required.

Our Theorem 2 below is new only in the sense that it is completely general. By contrast, Theorem 3 brings some precision to the estimates which is of interest even in the classical case, and it is a marked improvement over anything heretofore obtained.

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0. Introduction. Let G be a locally compact commutative group. Write $A(G)$ for the Fourier algebra of G, i.e. the space of continuous functions $f\colon G \to C$ which have a representation $f(x) = \int_{G^\wedge} \langle x, \xi\rangle f^\wedge(\xi)\,d\xi$ where $G^\wedge$ is the character group and $f^\wedge \in L_1(G^\wedge)$. Then $A(G)$ is a Banach algebra for the orfinary addition and multiplication of functions and the norm $\|f\|_A = \|f^\wedge\|_1$. The dual Banach space to $A(G)$ is denoted by $PM(G)$; the elements of this space are called *pseudomeasures*. Let E be a closed subset of G; we write $A(E, G)$ for the quotient algebra of $A(G)$ obtained by restricting functions to E.

Consider the situation in which E is a compact subset of G, $H \xrightarrow{\pi} G$ is a continuous homomorphism of locally compact commutative groups, and $\theta\colon E \to H$ is a continuous section, i.e. a continuous map such that $\pi\circ\theta$ is the identity on E. Then $g \mapsto g\circ\pi$ gives a morphism of Banach algebras (algebraic homomorphism of norm $\leqslant 1$) $A(E, G) \xrightarrow{\pi_*} A(\theta E, H)$. The question arises whether $h \mapsto h\circ\theta$ gives a continuous map $A(\theta E, H) \to A(E, G)$. The simplest result is this.

THEOREM 0. *Suppose that $H \xrightarrow{\pi} G$ is a monomorphism and that E is a compact subset of G with a continuous section $\theta\colon E \to H$. Then $A(E, G) \xrightarrow{\pi_*} A(\theta E, H)$ is an (isometric) isomorphism. In particular, given $h \in A(H)$ and $\beta > 1$ there exists $g \in A(G)$ such that*

(i) $\|g\|_A \leqslant \beta\|h\|_A$,

(ii) $g = h\circ\theta$ *on* E.

The requirement that π be a monomorphism is quite severe. A slight relaxation is permissible ([1] Theorem 1).

COMPLEMENT TO THEOREM 0. *If it is only assumed that π has a discrete kernel then π_* still has a continuous inverse, i.e. there is a constant α such that the last sentence of Theorem 0 holds with "$\beta > 1$" replaced by "$\beta > \alpha$".*

This is as far as one can go without special assumptions. In fact if one assumes that there exists a constant α such that the above is valid with no restrictions on π and θ then (E, G) must be Helson α.

DEFINITION. The pair (E, G) where E is a closed subset of the locally compact commutative group G, is *Helson* α if given $\varphi \in C_0(E)$ and $\beta > \alpha$ there exists $f \in A(G)$ with $\|f\|_A \leqslant \beta\|\varphi\|_\infty$ such that $f = \varphi$ on E.

The generalization of Theorem 0 as it stands with (E, G) assumed to be a Helson set is trivial. The essential point of the next is condition (iii) which gives a control outside E.

THEOREM 1. *Suppose that $H \xrightarrow{\pi} G$ is a homomorphism and that E is a compact subset of G with a continuous section $\theta\colon E \to H$. Suppose further that (E, G) is Helson α. Then given $h \in A(H)$, $\beta > \alpha$, $\delta > 0$, and V a neighborhood of the identity in G, there exists $g \in A(G)$ such that*

(i) $\|g\|_A \leqslant \beta^2\|h\|_A$,

(ii) $|h\circ\theta - g| < \delta$ *on* E,

(iii) $|g(x)| \leqslant \beta^2 \sup_{\pi y \in Vx} |h(y)|$ *for all* $x \in G$.

The key step in the proof of Theorem 1 is the special case we call "*Drury's Lemma*" in which $H = G\times H_1$ and π is projection on the first coordinate. The idea of Drury [2] applies to G discrete and H_1 finite. It was extended by Varopoulos [3] to G compact and H_1 profinite. It is essential in what follows that we have Theorem 1 in full generality. (We could, with only slight annoyance, restrict our attention to G and H compact which would allow for less "abstraction" in the proof of Theorem 1.)

Our main application of Theorem 1 is to prove

THEOREM 2. *There exists a continuous function $\omega\colon (0, 1] \to [1, \infty)$ with the following property. If (E, G) is Helson α and $\varphi \in C_0(E)$, $\beta > \alpha$ and F, a closed subset of G disjoint from E, are given then for each ε with $0 < \varepsilon \leqslant 1$ one can find $f \in A(G)$ such that*

(i) $\|f\|_A \leqslant \beta^2\|\varphi\|_\infty\,\omega(\varepsilon)$,

(ii) $f = \varphi$ *on* E,

(iii) $|f| \leqslant \beta^2\|\varphi\|_\infty\varepsilon$ *on* F.

The purely existential Theorem 2 suffices to prove that the union of two Helson sets is Helson. We do, however, have something new to contribute about the nature of the function ω.

THEOREM 3. *One can take the ω of Theorem 2 so that $\omega(\varepsilon) \leqslant \varepsilon^{-1/2}$ for all $\varepsilon \leqslant 1$. For small ε one has $\log\omega(\varepsilon) < \frac{1}{4}(\log\log 1/\varepsilon)^2$.*

From Theorems 2 and 3 we easily deduce

COROLLARY. *The union of a Helson α with a Helson β is Helson $H(\alpha, \beta) \leqslant \frac{1}{2}3^{3/2}(\alpha^3 + \beta^3)$.*

REMARKS. After the first draft of this paper was submitted I learned of the work of Stegeman [5] where the estimate $\omega(\varepsilon) \leqslant \varepsilon^{-1/2}$ for $\varepsilon \leqslant \frac{1}{4}$ was obtained in the context of Theorem 2$'V$ below. The particular case $H(1, 1) \leqslant 3^{3/2}$ of the above Corollary is a slight improvement over Stegeman's Theorem 3 resulting from the removal of the limitation $\varepsilon \leqslant \frac{1}{4}$. In fact, for $\varepsilon < \frac{1}{4}$, we have $\omega(\varepsilon) < \varepsilon^{-1/2}$, and the above estimate of $H(\alpha, \beta)$ is very crude unless α and β are both near 1 so that values of $\varepsilon > \frac{1}{4}$ play a role. For fixed α we have $H(\alpha, \beta) = o\big(\alpha^2\beta\exp\{(\frac{1}{2}\log\log\beta)^2\}\big)$ as $\beta \to \infty$. If $\alpha \geqslant \beta$ then $H(\alpha, \beta) = o\big(\alpha^2\exp\{(\frac{1}{2}\log\log\alpha)^2\}\big)$ as $\alpha \to \infty$. The factor α^2 comes from the inequalities of Theorem 2.

A special case of Theorem 2 is

THEOREM 2$'$. *Let (E, G) be Helson 1 with E compact. Then given $\gamma > 1$, F a closed subset of G disjoint from E, and $0 < \varepsilon \leqslant 1$ there exists $f \in A(G)$ such that*

(i) $\|f\|_A < \gamma\omega(\varepsilon)$,

(ii) $f = 1$ *on* E,

(iii) $|f| \leqslant \varepsilon$ *on* F.

Theorem 1 allows us to pass from Theorem 2′ to Theorem 2 almost immediately. We therefore concentrate on Theorem 2′. The simplest special case is this.

PROPOSITION 1. *Let* E_n *be the canonical generators of* $\boldsymbol{Z}^n$. *Given* $0 < \varepsilon < 1$ *there exists* $f \in A(\boldsymbol{Z}^n)$ *such that*

(i) $\|f\|_A \leqslant \varepsilon^{-1/2}$,

(ii) $f = 1$ on E_n,

(iii) *if* $x \notin E_n$ *then* $f(x) = 0$ *or* $f(x) = \varepsilon^{k(x)}$, *where* $k(x) > 0$ *is an integer.*

The proof of Proposition 1 is a simple construction. We shall now fix the function ω. We put $\omega_n(\varepsilon) = \inf\|f\|_A$ taken over $f \in A(\boldsymbol{Z}^n)$ with $f = 1$ on E_n and $|f| \leqslant \varepsilon$ elsewhere. Then ω is defined by $\omega(\varepsilon) = \sup_n \omega_n(\varepsilon)$. The existence and properties of this function are consequences of Proposition 1. Only the existence of a continuous ω such that Theorem 2′ holds for $(E_n, \boldsymbol{Z}^n)$ is used in the proof of Theorem 2.

We now sketch the ideas for the proof of Theorem 2. A useful technical device is

LEMMA 1. *In order that the conclusion of Theorem 2 hold for* (E, G) *it is necessary and sufficient that for each complex Radon measure of bounded variation* $\mu \in M(G)$ *and each* ε, $0 < \varepsilon \leqslant 1$ *we have*

$$\int_E |d\mu| \leqslant \alpha^2(1+\alpha^2\varepsilon)^{-1}[\omega(\varepsilon)\|\mu\|_{PM} + \varepsilon\|\mu\|_M]$$

where $\|\mu\|_{PM} = \|\mu^\wedge\|_\infty$ *and* $\|\mu\|_M =$ *total variation of* μ.

It follows immediately that it suffices to prove Theorem 2 for E compact. The crucial role of Theorem 1 is in proving the transition step

LEMMA 2. *A sufficient condition that Theorem 2 hold for* (E, G) *is that there exist* $(E^\sim, H)$ *Helson 1 satisfying the conclusion of Theorem 2′, a continuous map* $\theta\colon E \to H$ *with* $\theta E = E^\sim$, *and a continuous homomorphism* $H \xrightarrow{\pi} G$ *such that* $\pi\circ\theta = \mathrm{id}_E$.

The case of Sidon sets is now settled. If (E, G) is Helson α with E discrete, Lemma 1 allows to reduce to the case of E finite. If E has n elements then have $\theta\colon E \to E_n$ and $\boldsymbol{Z}^n \xrightarrow{\pi} G$ so that Lemma 2 can be applied using $(E_n, \boldsymbol{Z}^n)$ and the known result, Proposition 1. We state the result as

THEOREM 2D. *Theorem 2 holds for* E *discrete. If* G *is also discrete then given* $\varphi \in C(E)$ *there exists a Fourier-Stieltjes transform* $f \in B(G)$ *such that*

(i) $\|f\|_B \leqslant \alpha^2\omega(\varepsilon)\|\varphi\|_\infty$,

(ii) $f = \varphi$ *on* E,

(iii) $|f| \leqslant \alpha^2\varepsilon\,\|\varphi\|_\infty$ *outside* E.

The second sentence of Theorem 2D is given by Drury [2] for the function $\varphi = 1$ and with $\omega(\varepsilon)$ replaced by $4\varepsilon^{-1}$. The proof indicated above is a minor variation of his proof with the improvement being gained by using Proposition 1 rather than Riesz products.

An analysis of the proof reveals the fact that Lemma 2 could be applied because Proposition 1 is available for the *free* discrete commutative group generated by the finite set E. What is needed for the general case is the *free compact commutative group* $\Gamma(E)$ generated by a compact Hausdorff space E. There is a canonical embedding $\theta\colon E \to \Gamma(E)$ such that if $i\colon E \to G$ is a continuous map of E into a compact commutative group then there is a homomorphism $\Gamma(E) \xrightarrow{\pi} G$, $\pi = \Gamma(i)$, such that $i = \pi\circ\theta$. The character group $\Gamma^\wedge(E)$ is the discrete group of continuous maps from E to π with the operation of pointwise multiplication.

The next step is

PROPOSITION 2. *Let* E *be a totally disconnected compact Hausdorff space and* $\theta\colon E \to \Gamma(E)$ *the canonical embedding in the free compact commutative group generated by* E. *Then the conclusion of Theorem 2′ holds for* $(\theta E, \Gamma(E))$.

Proposition 2 is quite easy to prove. Using Lemma 2 one gets as a corollary

THEOREM 2′V. *Theorem 2′ holds for* E *totally disconnected and* G *compact.*

This is a result of Varopoulos [3; Theorem 1] except that he has $8\varepsilon^{-1}$ in place of $\omega(\varepsilon)$.

The enormous complication of Varopoulos' proof arises from the fact that he has nothing like Theorem 1 at his disposal. He is unable to derive Theorem 2′V directly from Proposition 2 and has to replace $\Gamma(E)$ by a group whose character group is $C(E, \boldsymbol{T})$, the continuous maps from E to $\boldsymbol{T}$ in the topology of uniform convergence; thus he has to leave the category of locally compact groups.

After Drury's work appeared, Varopoulos ([3], Theorem 4) indicated how to pass from Theorem 2′V to Theorem 2 for E metrizable and G compact with β^2 replaced by an unspecified quantity and $\omega(\varepsilon)$ by ε^{-1}.

As Varopoulos pointed out, Lemma 1 allows us to conclude

COROLLARY TO THEOREM 2′V. *Theorem 2′ holds for* E *metrizable and* G *compact.*

His method stops here, but, using a trick and Theorem 1 again, we get

PROPOSITION 3. *Theorem 2′ holds for* G *compact, in particular for* $(\theta E, \Gamma(E))$, *where* E *is an arbitrary compact Hausdorff space.*

Now if we use Lemma 2 with $(E^{\sim}, H) = (\theta E, \Gamma(E))$ we get Theorem 2 for G compact. If G is not compact look at its Bohr compactification bG. If (E, G) is Helson a then (E, bG) is Helson a by Theorem 0. The conclusion of Theorem 2 for (E, bG) is *a fortiori* valid for (E, G) by Lemma 1 and the fact that the inclusion $M(G) \subset M(bG)$ is isometric for both the variation and pseudomeasure norms.

1. "Drury's Lemma" and the proof of Theorem 1. It will be convenient to deal with Banach-valued functions. Let BAN be the category of linear transformations of norm $\leqslant 1$ of (complex) Banach spaces. Given a locally compact group G and a Banach space B we denote by $L_p(G; B)$, $1 \leqslant p < \infty$, the completion of the continuous functions of compact support $u\colon G \to B$ for the norm $\|u\|_p = \{\int_G |u(x)|_B^p\,dx\}^{1/p}$, where dx indicates integration with respect to the (left-invariant) Haar measure on G. The limiting case, $p = \infty$, corresponds to $C_0(G; B)$. The crucial remark is that $L_p(G; \cdot)$ and $C_0(G; \cdot)$ are endofunctors of BAN.

There is a tensor product $\otimes$ for BAN (see appendix) which we use in two places. First there is the identification of $L_1(G; B)$ with $L_1(G; \boldsymbol{C})\otimes B$; the needed part of the Fubini theorem is a canonical isomorphism $L_1(G\times H; \boldsymbol{C})$ with $L_1(G; \boldsymbol{C})\otimes L_1(H; \boldsymbol{C})$. The second place involves the group structure in an essential way. We shall denote the group by $H^{\hat{}}$ because this is the notation which occurs in the applications. Let B be a Banach algebra, i.e. a Banach space with a "multiplication" morphism $B\otimes B \to B$ which is associative. Then writing $*$ for convolution and $v^{\check{}}(\eta) = v(\eta^{-1})$ we state

LEMMA A. *If B is a Banach algebra then $u\otimes v \mapsto u*v^{\check{}}$ gives a morphism $L_2(H^{\hat{}}; B)\otimes L_2(H^{\hat{}}; B) \to C_0(H^{\hat{}}; B)$.*

The explicit formula when u and v are continuous B-valued functions of compact support is

$$u*v^{\check{}}(\eta) = \int_{H^{\hat{}}} u(\zeta)v(\eta^{-1}\zeta)\,d\zeta = \int_{H^{\hat{}}} u(\eta\zeta)v(\zeta)\,d\zeta.$$

The multiplication is that of B. The proof of Lemma A is the estimate $|u*v^{\check{}}(\eta)|_B \leqslant \int_{H^{\hat{}}} |u(\eta\zeta)|_B |v(\zeta)|_B\,d\zeta$ followed by the Schwarz inequality.

In the special case $B = \boldsymbol{C}$ the coimage of $L_2(H^{\hat{}})\otimes L_2(H^{\hat{}}) \to C_0(H^{\hat{}})$ is the Fourier algebra $A(H^{\hat{}})$, i.e. the elements of $A(H^{\hat{}})$ are viewed as continuous functions on $H^{\hat{}}$ but the norm is the quotient norm from $L_2(H^{\hat{}})\otimes L_2(H^{\hat{}})$. In particular, if $H^{\hat{}}$ is the character group of the locally compact commutative group H then the Fourier transform, which is an isomorphism of $L_2(H^{\hat{}})$ with $L_2(H)$ takes

$$L_2(H)\otimes L_2(H) \to L_1(H);\ f\otimes g \mapsto f\cdot g^{\check{}}$$

isomorphically over to

$$L_2(H^{\hat{}})\otimes L_2(H^{\hat{}}) \to A(H^{\hat{}});\ u\otimes v \mapsto u*v^{\check{}}.$$

In this section $*$ refers to convolution over the group $H^{\hat{}}$ only. In the applications we shall have $v = \bar{u}$ in which case $v^{\check{}}$ is u^*. We use Lemma A directly with $B = A(G)$.

The next is obvious and the whole point is to state it.

LEMMA B. *If G is a locally compact Hausdorff space and B_1 and B_2 are Banach spaces then pointwise $\otimes$ multiplication gives a morphism*

$$C_0(G;\ B_1)\otimes C_0(G;\ B_2) \to C_0(G;\ B_1\otimes B_2).$$

Explicitly $f_1\otimes f_2 \mapsto f$ where $f(x) = f_1(x)\otimes f_2(x)$.

In our application we have $B_1 = B_2 = L_2(H^{\hat{}})$. Since $C_0(G; \cdot)$ is a functor we have

COROLLARY. *Convolution over $H^{\hat{}}$, gives a morphism*

$$C_0(G;\ L_2(H^{\hat{}}))\otimes C_0(G;\ L_2(H^{\hat{}})) \to C_0(G;\ A(H^{\hat{}})),$$

where $u\otimes v \mapsto u*v^{\check{}}$.

Observe that the corollary corresponds to the formula

$$u*v^{\check{}}(x,\eta) = \int_{H^{\hat{}}} u(x, \eta\zeta)v(x, \zeta)\,d\zeta.$$

Since $L_2(H^{\hat{}}; \cdot)$ is a functor, the "inclusion" $A(G) \to C_0(G)$ gives a morphism $L_2(H^{\hat{}};\ A(G)) \to L_2(H^{\hat{}};\ C_0(G))$, but there is also a morphism $L_2(H^{\hat{}};\ C_0(G)) \to C_0(G;\ L_2(H^{\hat{}}))$ coming from the inequality

$$\sup_{x\in G}\int_{H^{\hat{}}} |\omega(x, \eta)|^2\,d\eta \leqslant \int_{H^{\hat{}}} \sup_{x\in G}|\omega(x, \eta)|^2\,d\eta.$$

If we put these morphisms together with the Corollary to Lemma B we have that $u\otimes v \mapsto u*v^{\check{}}$ gives a morphism

$$L_2(H^{\hat{}};\ A(G))\otimes L_2(H^{\hat{}};\ A(G)) \to C_0(G;\ A(H^{\hat{}}))$$

which is what we need below.

DRURY'S LEMMA. *Let (E, G) be Helson a with E compact, and $\theta\colon E \to H$ a continuous map into a locally compact commutative group. Then given $f\in A(G\times H)$, $\beta > a$, and $\varepsilon > 0$ there exists $g\in A(G)$ such that*

(i) $\|g\|_A \leqslant \beta^2\|f\|_A$,

(ii) $|f(x, \theta(x)) - g(x)| < \varepsilon$ *for all* $x\in E$,

(iii) $|g(x)| \leqslant \beta^2 \sup_{y\in H}|f(x, y)|$ *for all* $x\in G$.

Proof. For simplicity assume $\|f\|_A \leqslant 1$. We may write

$$f(x, y) = \int_{H^{\hat{}}} \langle y, \eta\rangle \varphi(x, \eta)\,d\eta$$

where φ is viewed as an element of $L_1(H^\wedge; A(G))$ with $\|\varphi\|_1 = \|f\|_A$. There exists a continuous function of compact support $k: H^\wedge \to \boldsymbol{C}$ such that $\|k\|_2 \leqslant 1$ and

$$\int_{H^\wedge} |1-k*k^*(\eta)|\, |\varphi(\cdot, \eta)|_A\, d\eta < \tfrac{1}{3}\varepsilon.$$

Since θE is a compact set in H, the set

$$U = \{\eta \in H^\wedge: \sup_{x\in E}|1-\langle\theta(x), \eta\rangle| < \tfrac{1}{3}\varepsilon\}$$

is a neighborhood of the identity in $H^\wedge$. Since $K = \operatorname{supp} k$ is compact there is a finite collection $\{\eta_1, \ldots, \eta_n\} \subset H^\wedge$ such that $K \subset \bigcup_{i=1}^n \eta_i U$. Put $K_1 = K \cap \eta_1 U$ and $K_{i+1} = K \cap \eta_{i+1} U \cap \complement[K_1 \cup \ldots \cup K_i]$. Thus K is the disjoint union of the Borel sets $K_1, \ldots, K_n$. Since (E, G) is Helson α, for each i there exists $l_i \in A(G)$ with $\|l_i\|_A \leqslant \beta > \alpha$ such that $l_i(x) = \langle\theta(x), \eta_i\rangle$ for $x \in E$. Let l be the Borel-measurable $A(G)$-valued function defined on $H^\wedge$ by $l(\eta) = l_i$ for $\eta \in K_i$, $l(\eta) = 0$ for $\eta \notin K$. Then $kl \in L_2(H^\wedge; A(G))$. This implies that $\psi = (kl)*(kl)^*$ is an element both of $C_0(H^\wedge; A(G))$ and of $C_0(G; A(H^\wedge))$, in both cases with norm $\leqslant \beta^2$ by Lemmas A and B. Viewing ψ as an element of $C_0(G; A(H^\wedge))$ we have $\psi(x, \eta) = \int_H \overline{\langle y, \eta\rangle}\, h(x, y)\, dy$ where $h \in C_0(G; L_1(H))$. Define g by

$$g(x) = \int_H f(x, y) h(x, y)\, dy.$$

Since $f \in C_0(G; C_0(H))$ we obviously have

$$|g(x)| \leqslant \sup_{y\in H}|f(x, y)| \int_H |h(x, z)|\, dz$$

which gives assertion (iii). On the other hand, for fixed $x \in G$ the Parseval formula gives

$$\int_H f(x, y) h(x, y)\, dy = \int_{H^\wedge} \varphi(x, \eta)\psi(x, \eta)\, d\eta$$

since $h(x, \cdot) \in L_1(H)$ with $\psi(x, \cdot)$ as Fourier transform and $\varphi(x, \cdot) \in L_1(H^\wedge)$ with $f(x, \cdot)$ as Fourier transform. Thus we have

$$g(x) = \int_{H^\wedge} \varphi(x, \eta)\psi(x, \eta)\, d\eta.$$

Regard φ as an element of $L_1(H^\wedge; A(G))$ and ψ as an element of $C_0(H^\wedge; A(G))$. Then we see that $g \in A(G)$ and $\|g\|_A \leqslant \|\varphi\|_1\|\psi\|_\infty \leqslant \|f\|_A\beta^2$. This gives assertion (i). Finally we must estimate

$$f(x, \theta(x)) - g(x) = \int_{H^\wedge} [\langle\theta(x), \eta\rangle - \psi(x, \eta)]\varphi(x, \eta)\, d\eta$$

for $x \in E$. To do this note that $|\langle\theta(x), \eta\rangle - l(x, \eta)| < \frac{1}{3}\varepsilon$ for $\eta \in \operatorname{supp} k$, $x \in E$. Hence, writing $\vartheta(x, \eta) = \langle\theta(x), \eta\rangle$ we have

$$\begin{aligned}|\psi - \vartheta k*(\vartheta k)^*| &= |lk*(lk)^* - \vartheta k*(\vartheta k)^*| \\ &\leqslant |(l-\vartheta)k*(lk)^*| + |\vartheta k*[(l-\vartheta)k]^*| \\ &< \tfrac{1}{3}\varepsilon\|k\|_2\|lk\|_2 + \tfrac{1}{3}\varepsilon\|\vartheta k\|_2\|k\|_2 \leqslant \tfrac{2}{3}\varepsilon.\end{aligned}$$

On the other hand

$$\begin{aligned}\vartheta k*(\vartheta k)^*(\eta) &= \int_{H^\wedge} \langle\theta(x), \eta\zeta\rangle k(\eta\zeta)\, \overline{\langle\theta(x), \zeta\rangle k(\zeta)}\, d\zeta \\ &= \langle\theta(x), \eta\rangle k*k^*(\eta).\end{aligned}$$

The conclusion is that for $x \in E$ and all η

$$\begin{aligned}|\langle\theta(x), \eta\rangle - \psi(x, \eta)| &< |\langle\theta(x), \eta\rangle - \langle\theta(x), \eta\rangle k*k^*(\eta)| + \tfrac{2}{3}\varepsilon \\ &= |1-k*k^*(\eta)| + \tfrac{2}{3}\varepsilon\end{aligned}$$

which yields

$$|f(x, \theta(x)) - g(x)| \leqslant \int_{H^\wedge} [1-k*k^*(\eta) + \tfrac{2}{3}\varepsilon]|\varphi(x, \eta)|\, d\eta < \varepsilon;$$

this is assertion (ii).

Remark. In case G is discrete one can pass to Fourier–Stieltjes transforms and take $g \in B(G)$ so that $\|g\|_B \leqslant \alpha^2\|f\|_A$, $g(x) = f(x, \theta(x))$ for $x \in E$, and $|g(x)| \leqslant \alpha^2\sup_{y\in H}|f(x, y)|$ for all $x \in G$. In this situation E is a finite set. The particular case $H = \boldsymbol{Z}_2^E$ with $\theta: E \to H$ the obvious map in the Lemma of [2].

Proof of Theorem 1. Let $H \xrightarrow{\pi} G$ be a morphism of locally compact commutative groups. Then there is a BAN-morphism $A(G)\otimes A(H) \to A(G\times H)$ given by $k\otimes h \mapsto f$ where $f(x, y) = k(x\pi y^{-1})h(y)$. Given a neighborhood V of the identity in G let us fix $k \in A(G)$ with $\|k\|_A = 1$, $k(1) = 1$, and $k = 0$ outside V^{-1}. Thus $f(x, y) = 0$ unless $\pi y \in Vx$. Now apply Drury's Lemma.

2. Proof of Lemma 1 and related estimates. In this section the group G is kept fixed. We always have $\alpha \geqslant 1$ and $0 < \varepsilon \leqslant 1$.

For each t with $0 \leqslant t < 1$ we consider a statement about the closed set $E \subset G$.

($\mathrm{C}^t_{\alpha,\varepsilon}$) *For each $\varphi \in C_0(E)$ with $\|\varphi\|_\infty \leqslant 1$, each $\beta > \alpha$, and each closed $F \subset G$ disjoint from E there exists $f \in A(G)$ such that*

(i) $$\|f\|_A \leqslant (1-t)\beta^2\omega(\varepsilon),$$

(ii) $$|\varphi - f| \leqslant t \text{ on } E,$$

(iii) $$|f| \leqslant (1-t)\beta^2\varepsilon \text{ on } F.$$

It is easy to see that the truth-value of $C^t_{a,\varepsilon}$ is independent of t. The conclusion of Theorem 2 is: $(\forall\varepsilon)C^0_{a,\varepsilon}$.

We also consider the statement

$(I_{a,\varepsilon})$ *For each* $\mu\in M(G)$ *we have*

$$\int_E |d\mu| \leqslant a^2(1+a^2\varepsilon)^{-1}[\omega(\varepsilon)\|\mu\|_{PM}+\varepsilon\|\mu\|_M].$$

In order to prove Lemma 1 it suffices to prove that $C_{a,\varepsilon}$ and $I_{a,\varepsilon}$ are equivalent. Before doing this let us observe that if we take $\varepsilon^{-1} = \lambda a^2$ then $I_{a,\varepsilon}$ gives

$$(\lambda+1)\int_E |d\mu| \leqslant \lambda a^2\omega(1/\lambda a^2)\|\mu\|_{PM}+\|\mu\|_M,$$

where $\lambda > 0$ is arbitrary. Now if (E_1, G) is Helson a and (E_2, G) is Helson β and we suppse $\operatorname{supp}\mu\subset E_1\cup E_2$ then the above inequality gives

$$\|\mu\|_M \leqslant \lambda(\lambda-1)^{-1}[a^2\omega(1/\lambda a^2)+\beta^2\omega(1/\lambda\beta^2)]\|\mu\|_{PM}.$$

An estimate of the form $\omega(\varepsilon)\leqslant \Omega_j\varepsilon^{-1/2j}$ and $\lambda = 2j+1$ shows that the union of a Helson a and a Helson β is Helson $H(a,\beta)\leqslant (2j)^{-1}(2j+1)^{1+1/2j}\Omega_j(a^{2+1/j}+\beta^{2+1/j})$. For a and β near 1 we take $j=1$ and $\Omega_1 = 1$. For large a and β better estimates are obtained by taking j near $\log a$ and using estimates for Ω_j given below. For fixed a and large β it is better to use the $I_{a,\varepsilon}$ inequality only. In this way the Corollary following Theorem 3 and the subsequent remarks are proved.

If we weaken $I_{a,\varepsilon}$ to $\int_E |d\mu|\leqslant a^2[\omega(\varepsilon)\|\mu\|_{PM}+\varepsilon\|\mu\|_M]$ and use $\omega(\varepsilon)\leqslant\Omega_j\varepsilon^{-1/2j}$ then there is no need to restrict $\varepsilon\leqslant 1$. The best choice of ε gives

COROLLARY TO THEOREMS 2 AND 3. *If* (E, G) *is Helson* a *then for all* $\mu\in M(G)$ *we have*

$$\int_E |d\mu| \leqslant a^2\Omega_j^\theta(2j)^{1-\theta}\|\mu\|_{PM}^{1-\theta}\|\mu\|_M^\theta, \quad \textit{where } \theta = (2j+1)^{-1}.$$

We now turn to the proof of Lemma 1.

Proof that $C_{a,\varepsilon}\Rightarrow I_{a,\varepsilon}$. It suffices to establish $I_{a,\varepsilon}$ for a dense subset of $M(G)$, in particular for μ's such that $\operatorname{supp}\mu = E\cup F$ where F is a closed set disjoint from E. In this case $\int_E |d\mu| = \sup|\int\varphi\,d\mu|$ where $\varphi\in C_0(E\cup F)$, $\|\varphi\|_\infty\leqslant 1$ and $\varphi = 0$ on F. Now for $f\in A(G)$

$$\left|\int\varphi\,d\mu\right|\leqslant\left|\int f\,d\mu\right|+\left|\int(\varphi-f)\,d\mu\right|\leqslant\|f\|_A\|\mu\|_{PM}+\|\varphi-f\|_\infty\|\mu\|_M,$$

where $\|\varphi-f\|_\infty$ is the supremum on $E\cup F$. Thus from $C^t_{a,\varepsilon}$ we get

$$\left|\int\varphi\,d\mu\right|\leqslant(1-t)\beta^2\omega(\varepsilon)\|\mu\|_{PM}+\max[t, (1-t)\beta^2\varepsilon]\|\mu\|_M \quad \text{for all } \beta > a,$$

and hence

$$\left|\int\varphi\,d\mu\right|\leqslant(1-t)a^2\omega(\varepsilon)\|\mu\|_{PM}+\max[t, (1-t)a^2\varepsilon]\|\mu\|_M.$$

Take $t = a^2\varepsilon(1+a^2\varepsilon)^{-1}$.

Proof that $I_{a,\varepsilon}\Rightarrow C_{a,\varepsilon}$. Fix $t = a^2\varepsilon(1+a^2\varepsilon)^{-1}$ and put $\tau = \varepsilon/\omega(\varepsilon)$. Consider the Banach space $C_0(E\cup F)\times A(G)$ which consists of the pairs (ψ, g) with $\psi\in C_0(E\cup F)$, $g\in A(G)$ and $\|(\psi, g)\| = \max(\|\psi\|_\infty, \|g\|_A)$. Let S be the subspace consisting of elements of the form $(f, -\tau f)$ for $f\in A(G)$. Given $\varphi\in C_0(E)$ let $\tilde\varphi\in C_0(E\cup F)\times A(G)$ be the element $\tilde\varphi = (\varphi, 0)$ where φ is extended to be 0 on F. Then $C^t_{a,\varepsilon}$ is the statement: distance $(\tilde\varphi, S)\leqslant t\|\varphi\|_\infty$. This number can be computed as $\sup|L(\tilde\varphi)|$ where L ranges over the linear functional of norm $\leqslant 1$ which vanish on S. The dual space to $X\times Y$ is $X'+Y'$. In this instance it is $M(E\cup F)+PM(G)$ which consists of pairs (μ, T) with $\mu\in M(E\cup F)$, $T\in PM(G)$ and $\|(\mu, T)\| = \|\mu\|_M+\|T\|_{PM}$. If (μ, T) vanishes on S then $\int f\,d\mu-\tau\langle f, T\rangle = 0$ for all $f\in A(G)$. This says $T = \tau^{-1}d\mu$. Thus $\sup|L(\tilde\varphi)| = \sup|\int_E\varphi\,d\mu|$ taken over $\mu\in M(E\cup F)$ with $\|\mu\|_M+\tau^{-1}\|\mu\|_{PM}\leqslant 1$. By $I_{a,\varepsilon}$ this inequality implies $\int_E|d\mu|\leqslant t$.

3. Proof of Lemma 2. The deduction of Lemma 2 from Theorem 1 is straightforward and the reader may skip the details.

We suppose that (E, G) is Helson a with E compact and that $\varphi\in C_0(E)$, $\beta > a$, F a closed subset of G disjoint from E, and ε with $0<\varepsilon\leqslant 1$ are given. For simplicity we suppose $\|\varphi\|_\infty = 1$.

Fix $\gamma > 1$ such that $\gamma^4a^2 < \beta^2$, and fix $\sigma > 0$ such that $a\sigma < (\beta^2-\gamma^4a^2)\varepsilon$. Let V be a neighborhood of the identity in G such that $E\cap\overline{V}F = \emptyset$.

Since $(E^\sim, H)$ is supposed to satisfy the conclusion of Theorem 2′, given $F^\sim$ a closed subset of H disjoint from $E^\sim$ there exists $h_0\in A(H)$ with $\|h_0\|_A < \gamma\omega(\varepsilon)$, $h_0 = 1$ on $E^\sim$, $|h|_0\leqslant\varepsilon$ on $F^\sim$. Put $\tilde\varphi = \varphi\circ\pi$ where $H\xrightarrow{\pi}G$ is the given morphism. Then since $(E^\sim, H)$ is Helson 1 there exists $h_1\in A(H)$ with $\|h_1\|_A<\gamma$ such that $h_1 = \tilde\varphi$ on $E^\sim$. We put $h = h_0h_1$, $F^\sim = \pi^{-1}(\overline{V}F)$ and apply Theorem 1. This gives the existence of $g\in A(G)$ with $\|g\|_A\leqslant(\gamma a)^2\|h\|_A\leqslant\gamma^4a^2\omega(\varepsilon)$ such that $|\varphi-g|<\sigma$ on E, $|g|\leqslant(\gamma a)^2\gamma\varepsilon = \gamma^3a^2\varepsilon$ on F. Since E is Helson a and $\varphi-g$ considered as an element of $C(E)$ has norm $<\sigma$, there exists $g_1\in A(G)$ with $\|g_1\|_A\leqslant a\sigma$ such that $g_1 = \varphi-g$ on E. Put $f = g+g_1$. Then f satisfies the conclusion of Theorem 2.

4. Proof of Proposition 1 and description of the behavior of ω. Given $t>0$ define $p\in A(\mathbf{Z}^n)$ by $p(x) = \exp(-\frac{1}{2}t|x|^2)$ where $|x|^2 = x_1^2+\ldots+x_n^2$. Then $\|p\|_A = 1$. The set $H_1 = \{x\in\mathbf{Z}^n\colon x_1+\ldots+x_n = 1\}$ is a coset. Hence there is a Fourier–Stieltjes transform $h\in B(\mathbf{Z}^n)$ with $\|h\|_B = 1$ such that $h = 1$ on H_1 and $h = 0$ elsewhere. Put $f = e^{\frac{1}{2}t}ph$; then $\|f\|_A\leqslant e^{\frac{1}{2}t}\|p\|_A\|h\|_B\leqslant e^{\frac{1}{2}t}$ and $f(x) = 0$ if $x\notin H_1$. If $x\in H_1$ we have

$f(x) = \exp\{-tk(x)\}$ where
$$2k(x)+1 = |x_1|^2+\ldots+|x_n|^2.$$
It is clear that $k(x)$ is always a positive integer and $k(x) = 0$ iff $x\in E_n$. Taking $t = \log 1/\varepsilon$ the f just constructed meets the requirements of Proposition 1.

By the definition of ω_n and the weak* compactness of the unit ball in $B(\boldsymbol{Z}^n)$, given ε there exists a Fourier–Stieltjes transform $f\in B(\boldsymbol{Z}^n)$ with $\|f\|_B = \omega_n(\varepsilon)$ such that $f = 1$ on E_n and $|f|\leqslant\varepsilon$ elsewhere. In this context we can pass to $n=\infty$. Thus $\omega(\varepsilon) = \inf\|f\|_B$ for $f\in B(\boldsymbol{Z}^\infty)$ with $f=1$ on E_∞ and $|f|\leqslant\varepsilon$ elsewhere. It is immediate that
$$\omega\left(\frac{\delta+\varepsilon}{2}\right)\leqslant\tfrac{1}{2}[\omega(\delta)+\omega(\varepsilon)] \text{ and } \omega(\delta\epsilon)\leqslant\omega(\delta)\,\omega(\epsilon).$$
In particular, ω is convex and hence continuous.

We have already established the estimate $\omega(\varepsilon)\leqslant\varepsilon^{-1/2}$. I don't see how to improve this when $\frac14\leqslant\varepsilon\leqslant 1$, but we shall now show that for $j = 1, 2, \ldots$ there are bounds $\omega(\epsilon)\leqslant\Omega_j\varepsilon^{-1/2j}$. In fact, we shall construct suitable Ω_j with $\log\Omega_j\leqslant\frac14(\log j)^2+C\log j$ where C is some constant. For a given small ε we can choose $j = [\log 1/\varepsilon(\log\log 1/\varepsilon)^{-1}]$ which gives
$$\log\omega(\varepsilon)-\tfrac14(\log\log 1/\varepsilon)^2\to-\infty \quad \text{as } \varepsilon\to 0.$$

Our initial construction gave us a function $f\in B(\boldsymbol{Z}^\infty)$ of the form $f(x) = \varphi_1(\varepsilon, k(x))$ where $\varphi_1(u,k) = u^k$; we shall always assume $0<u<1$ and $k(x) = +\infty$ if $x\in H_1$. Now suppose we have a function φ_j with the properties

(a) $\varphi_j(\cdot, 0) = 1$

(b) $\varphi_j(\cdot, k) = 0$ for $0<j<k$,

(c) $\varphi_j(u^j, k = C_{j,k}u^k$ for $k\geqslant j$ where $|C_{j,k}|\leqslant 1$,

(d) $\|\varphi_j(u^j, k(\cdot))\|_B\leqslant\Omega_j u^{-1/2}$.

Choose $\lambda_j\geqslant 1$ and put
$$\varphi_{j+1}(u^{j+1}, k) = \lambda_j^{-1}\{(\lambda_j+1)\varphi_j[(\lambda_j+1)^{-1}u^jk]-\varphi_j(u^j, k)\}.$$
Then φ_{j+1} has the above properties with
$$\Omega_{j+1} = \left[\left(1+\frac{1}{\lambda_j}\right)(\lambda_j+1)^{1/2j}+\frac{1}{\lambda_j}\right]\Omega_j.$$
If we put $f(x) = \varphi_j(\varepsilon^{1/j}, k(x))$ then $f=1$ on E_∞ and $|f|\leqslant\varepsilon$ elsewhere while $\|f\|_B\leqslant\Omega_j\varepsilon^{-1/2j}$. Starting with $\Omega_1 = 1$ the Ω_j' s are determined recursively from the sequence $\{\lambda_j\}$. The choice $\lambda_j = j$ gives the type of estimate previously claimed.

5. The free compact commutative group and the proof of Proposition 2. Let CA designate the category of continuous homomorphisms of compact commutative groups and COMP the category of continuous maps of compact Hausdorff spaces. Let U: CA $\to$ COMP be the forgetful functor to the underlying space. Then U has a left-adjoint Γ: COMP $\to$ CA, and there is a natural transformation Id $\xrightarrow{\theta}$ $U\Gamma$ of endofunctors of COMP. The explicit description is this. Given a compact Hausdorff space E we put $\Gamma^\wedge(E)$ for the discrete group of continuous maps $\chi\colon E\to\boldsymbol{T}$. If $i\colon E\to G$ is a continuous map into a compact commutative group then $\chi\to\chi\circ i$ gives a homomorphism $G^\wedge\to\Gamma^\wedge(E)$. By duality we have a CA-morphism $\Gamma(E)\to G$ designated by $\Gamma(i)$. The map θ associates to $x\in E$ the character of $\Gamma^\wedge(E)$ given by $\langle\theta x, \chi\rangle = \chi(x)$. It is easy to see that θ is a homeomorphism.

The functor Γ, being a left-adjoint, preserves inductive limits. It does not, in general, preserve projective limits; and, in particular, it does not preserve the type of inverse union we use below.

Let COMP$_0$ designate the category of continuous maps of totally disconnected compact Hausdorff spaces and CA$_0$ the category of continuous homomorphisms of profinite groups. The forgetful functor U_0: CA$_0\to$ COMP$_0$ has a left-adjoint Γ_0: COMP$_0\to$ CA$_0$ giving the *free profinite commutative group*. Given a totally disconnected compact Hausdorff space E, the group $\Gamma_0(E)$ has for character group $\Gamma_0^\wedge(E)$, the discrete group of all continuous maps $\chi\colon E\to\boldsymbol{Q}/\boldsymbol{Z}$. This time, if $E = \varprojlim E_i$ is the presentation of E as the inverse union of its finite quotient spaces we have $\Gamma_0(E) = \varprojlim\Gamma_0(E_i)$. Once again the natural transformation Id $\xrightarrow{\theta_0}$ $U_0\Gamma_0$ is a homeomorphism.

Put $\psi = \Gamma(\theta_0)$, i.e. ψ is a morphism $\Gamma(E)\to\Gamma_0(E)$ such that $\psi\circ\theta = \theta_0$. Let Δ be the kernel of ψ. Then we have

LEMMA 3. *Let E be a totally disconnected compact Hausdorff space and F a closed subset of $\theta E\cdot\Delta$ disjoint from θE. Then given $\delta>0$ there exists $k\in A(\Gamma(E))$ with $\|k\|_\Delta = 1$ such that $|1-k|<\delta$ on θE and $|k|<\delta$ on F.*

Proof. The map $E\times\Delta\to\theta E\cdot\Delta$ given by $(x,y)\mapsto\theta(x)y$ is a homeomorphism since $\theta(x_1)y_1 = \theta(x_2)y_2$ implies $\theta_0(x_1) = \theta_0(x_2)$ and hence $x_1 = x_2$. It follows that there exists a closed $K\subset\Delta$, disjoint from the unit element, such that $F\subset\theta E\cdot K$. Choose $p\in A(\Gamma(E))$ such that $\|p\|_\Delta = 1$, $p(1) = 1$, and $p = 0$ on K. Then $p = \sum p^\wedge(\chi)\chi$, $\chi\in\Gamma^\wedge(E)$, where each $p^\wedge(\chi)\geqslant 0$ and $\sum p^\wedge(\chi) = 1$. For each χ we can find $\varphi_\chi\in\Gamma^\wedge{}_0(E)$ such that $|\chi-\varphi_\chi|<\delta$ as functions on E; here the fact that E is totally disconnected is used. Put $k = \sum p^\wedge(\chi)\overline{\varphi}_\chi\,\chi$. For $x\in E$, $y\in\Delta$ we have $k(\theta(x)y) = \sum p^\wedge(\chi)\overline{\varphi}_\chi(x)\chi(x)\langle y,\chi\rangle$; and so $|k(\theta(x)y)-p(y)|<\sum p^\wedge(\chi)\delta = \delta$.

Proof of Proposition 2. Given $\gamma>1$ and ε with $0<\varepsilon\leqslant 1$ we can find ε_0 with $0<\varepsilon_0<\varepsilon$ such that $\gamma\omega(\varepsilon)>\omega(\varepsilon_0)$. Fix $\delta>0$ so that

$\delta \leqslant \varepsilon/4\omega(\varepsilon_0)$, $(1+\delta)\varepsilon_0 \leqslant \varepsilon$, and $(1+\delta)^2\omega(\varepsilon_0) \leqslant \gamma\omega(\varepsilon)$. Let F be a closed subset of $\Gamma(E)$ disjoint from θE. Using Lemma 3 and the fact that $(\theta E, \Gamma(E))$ is Helson 1, we find that there exists $f_1 \in A(\Gamma(E))$ such that $\|f_1\|_A < 1+\delta$, $f_1 = 1$ on E, and $|f_1| < 2\delta$ on $F \cap (\theta E \cdot \Delta)$. Put $F_0 = F \cap \{|f_1| \geqslant 2\delta\}$. Then $\theta_0 E$ and ψF_0 are disjoint subsets of $\Gamma_0(E)$. From the fact that $\Gamma_0(E) = \varprojlim \Gamma_0(E_i)$ where the E_i range over the finite quotient spaces of E, it follows that there is a finite space E' such that under the canonical homomorphism $\Gamma_0(E) \xrightarrow{\pi'} \Gamma_0(E')$ the sets $\theta_0 E'$ and πF_0, where $\pi = \pi' \circ \psi$, are disjoint. Now $(\theta_0 E', \Gamma_0(E'))$ is Helson 1, so the first sentence of Theorem 2D applies. Thus there exists $f_0 \in A(\Gamma_0(E'))$ with $\|f_0\|_A < (1+\delta)\omega(\varepsilon_0)$ such that $f_0 = 1$ on $\theta_0 E'$, $|f_0| \leqslant \varepsilon_0$ on πF_0. Then $f = (f_0 \circ \pi) f_1$ meets the requirements of the assertion of Proposition 2.

6. Proof of Proposition 3. If E is a compact metric space and μ is a Radon measure on E, then given $\varepsilon > 0$ there is a closed totally disconnected subset $K_\varepsilon \subset E$ such that $\int_{E/K_\varepsilon} |d\mu| < \varepsilon$, — to see this it is sufficient to map E homeomorphically into $[0,1]^\infty$ and preform an explicit construction. If (E, G) is Helson 1 then Theorem 2′ applies to (K, G) where K is any compact totally disconnected subset of E. Lemma 1 now shows that Theorem 2′ holds for (E, G) with E metrizable and G compact.

Let G be an arbitrary compact commutative group and E and F two disjoint compact subsets of G. Then there exists a finite set $S_1 \subset G\hat{}$ which distinguishes E and F, i.e. given $x \in E$, $y \in F$ there exists $\chi \in S_1$ with $\langle x, \chi\rangle \neq \langle y, \chi\rangle$. Let S_0 be a subset of $G\hat{}$ such that $S_0 \cup S_1$ is a set of generators for the abelian group $G\hat{}$. Let $\iota_j \colon G \to G_j = \boldsymbol{T}^{S_j}$ be the morphisms obtained by evaluating elements of G at the points of S_j. Then $(\iota_0 \times \iota_1)\colon G \to G_0 \times G_1$ is a monomorphism with G_1 a finite-dimensional torus. Put $E_1 = \iota_1 E$, $F_1 = \iota_1 F$ and let V be a neighborhood of the identity in G_1 such that $E_1 \cap \overline{V} F_1 = \emptyset$. Let $\theta_1 \colon E_1 \to \Gamma(E_1)$ be the natural map. Then Theorem 2′ applies to $(\theta_1 E_1, \Gamma(E_1))$ because E_1 is metrizable. Hence given $\gamma_1 > 1$ and $0 < \varepsilon_1 \leqslant 1$ there exists $f_1 \in A(\Gamma(E_1))$ such that $\|f_1\|_A < \gamma_1\omega(\varepsilon_1)$, $f_1 = 1$ on $\theta_1 E_1$, $|f_1| \leqslant \varepsilon_1$ on $\pi_1^{-1}(\overline{V} F_1)$, where $\Gamma(E_1) \xrightarrow{\pi_1} G_1$ is the natural morphism. Put $H = G_0 \times \Gamma(E_1)$ and define $h \in A(H)$ by $h(t_0, y) = f_1(y)$. We have a morphism $H \xrightarrow{\psi} G_0 \times G_1$ given by $\psi = G_0 \times \pi_1$. Let $E\tilde{} = (\iota_0 \times \iota_1)E$ be the included image of E in $G_0 \times G_1$. Then $(E\tilde{}, G_0 \times G_1)$ is Helson 1 whenever (E, G) is. Moreover $\theta\tilde{}\colon E\tilde{} \to H$ defined by $\tilde{\theta}(\iota_0 x, \iota_1 x) = (\iota_0 x, \theta_1 \iota_1 x)$ is a section. It follows from Theorem 1 that given $\delta > 0$ there exists $g \in A(G_0 \times G_1)$ with $\|g\|_A < \gamma_1^3\omega(\varepsilon_1)$ such that $|1 - g(\iota_0 x, \iota_1 x)| < \delta$ for $x \in E$ (this is a rewriting of $|1-g| < \delta$ on $E\tilde{}$ and $|g| < \gamma_1^3 \epsilon$ on $G_0 \times F_1$). A slight adjustment of the function $g \circ (\iota_0 \times \iota_1)$ gives the required f.

Appendix on tensor products. For Banach spaces A and C we define $\mathrm{HOM}(A, C)$ as the Banach space of bounded linear transformations. This gives an endofunctor $\mathrm{HOM}(A, \cdot)$ of BAN. The tensor product $A \otimes \cdot$ is defined as the left adjoint of $\mathrm{HOM}(A, \cdot)$. This means that (assuming existence) $A \otimes B$ is a Banach space such that the morphisms $A \otimes B \to C$ are in natural one-to-one correspondence with the morphisms $B \to \mathrm{HOM}(A, C)$. This description gives the elementary properties of tensor products immediately.

Given a Banach space B let us write $\bar{B}$ for the set constituted by the closed unit ball of B ($B \to \bar{B}$ is the good forgetful functor for BAN). Then the morphisms $A \otimes B \to C$ are in one-to-one correspondence with the maps $\bar{A} \times \bar{B} \to \bar{C}$ which preserve the obvious linearity relations. To prove the existence of the tensor product the easiest thing to do is to verify that $l_1(\bar{A} \times \bar{B})/R$ has the required universal property, where R is the subspace of $l_1(\bar{A} \times \bar{B})$ generated by the elements of the forms

$$(t_1 a_1 + t_2 a_2, b) - t_1(a_1, b) - t_2(a_2, b); \ |t_1| + |t_2| \leqslant 1, \ a_1, a_2 \in \bar{A}, \ b \in \bar{B}$$

and the similar expressions with the roles of A and B reversed. One can also present $A \otimes B$ as the completion of the vector space tensor product for the greatest crossed norm. All presentations of $A \otimes B$ must be naturally isomorphic; this follows from the universal definition.

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