For $2m_p+1 \leqslant i \leqslant k$, where k is the largest odd integer $\leqslant (n_p/n_{p-1})-2$, each term in the first sum in zero, where now Proposition 2 (i) has been applied to F_p and F_{p-1}. For $i \geqslant k$, the left hand side of (6) is $\leqslant$

$$\omega\Big(\sum_{j=1}^{p-1} F_j,\ \pi/n_p\Big),$$

which, by (5), is $< p^{-1} (\log n_{p-1})^{-1}$. We conclude from (6), and from Proposition 2 (ii) applied to F_p that

$$W_{n_p}(0) \geqslant \sum_i i^{-1} a_i^{(p)} - 2p^{-1} (\log n_{p=1})^{-1} \sum_{i=k+1}^{n_p-2} i^{-1}.$$

Since $k \geqslant (n_p/n_{p-1})-4$, the second sum on the right is $O(\log n_{p-1})$. The first sum on the right is > 1. Hence $W_{n_p}(0) > 1-o(1)$ $(p \to \infty)$ and thus the Fourier series of F diverges at 0.

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Convergence of convolution operators

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Abstract. In this paper a locally convex topology is defined on the space of convolution operators over a general test space of functions. When the test space is the space $\mathscr{S}$ of rapidly decreasing functions, convergence in this topology coincides with the convergence introduced in $\mathcal{O}_c'$ by L. Schwartz. The topology is studied in some detail, and then the special case when the test space is a $K\{M_p\}$ space is considered.

In [8], L. Schwartz defined a class of convolution operators between certain spaces of distributions and introduced a topology on this space of operators. In this approach emphasis is placed on considering convolution by a fixed distribution as a linear operator between spaces of distributions. In Gelfand and Shilov [2], a somewhat different approach is taken. Gelfand and Shilov define a convolution operator on an arbitrary test space with continuous translation and then consider a few examples of such operators, some in very general test spaces. There is no topology defined on the space of convolution operators although one sequential limit theorem is proven ([2], III. 3.5).

In this paper we consider the approach of Gelfand and Shilov and introduce a locally convex topology on the space of convolution operators on a test space with continuous translation. In the first section some of the properties of this topology are studied and we compare this topology with the topology introduced by L. Schwartz in [8]. In the second section we consider this topology for a certain type of $K\{M_p\}$ space ([2], II. 2.1). Our results yield the characterization of sequential convergence in $\mathcal{O}_c'(K_1, K_1)$ as given in [12] and also a characterization of sequential convergence in the space $\mathcal{O}_c'$ of L. Schwartz ([7], VII. 5).

Our terminology and notation will basically be that of Gelfand and Shilov [2]. A *test space* is a vector space Φ of infinitely differentiable functions on R^k equipped with a locally convex Hausdorff topology such that

(i) $\mathscr{D} \subseteq \Phi$ with the injection continuous and $\mathscr{D}$ dense in Φ,

(ii) if the net $\{\varphi_\nu\}$ converges to 0 in Φ, then for each $x \in R^k$ $\varphi_\nu(x) \to 0$

(iii) if $P(D)$ is a partial differential operator with constant coefficients, then the map $\varphi \to P(D)\varphi$ is continuous on Φ into Φ.

1. A topology for the space of convolution operators. Throughout this section Φ will denote a test space with continuous translation. That is, for each $h \in R^n$ the map $\varphi \to \tau_h\varphi$ is continuous from Φ back into Φ. (Here $\tau_h\varphi$ denotes the translation of φ through h, $\tau_h\varphi(x) = \varphi(x+h)$.) If $T \in \Phi'$ and $\varphi \in \Phi$, $T*\varphi$ is the function defined by $T*\varphi(h) = \langle T, \tau_h\varphi\rangle$. A generalized function $T \in \Phi'$ is said to be a *convolution operator* on Φ (convolute in [2]) if for each $\varphi \in \Phi$, $T*\varphi \in \Phi$ and the map $\varphi \to T*\varphi$ is continuous on Φ. If T is a convolution operator on Φ and $S \in \Phi'$, the convolution of T and S, $T*S$, is defined to be $\langle T*S, \varphi\rangle = \langle S, T*\varphi\rangle$ for $\varphi \in \Phi$. The subspace of Φ' consisting of all convolution operators on Φ is denoted by $\mathscr{O}'_c(\Phi)$.

We define a locally convex topology on $\mathscr{O}'_c(\Phi)$ in the following manner. If p is a continuous semi-norm on Φ and B is a bounded subset of Φ, we define a semi-norm $q_{p,B}$ on $\mathscr{O}'_c(\Phi)$ by

$$q_{p,B}(T) = \sup\{p(T*\varphi):\ \varphi \in B\}. \tag{1}$$

Then $\mathscr{O}'_c(\Phi)$ is equipped with the locally convex Hausdorff topology generated by the family of semi-norms $q_{p,B}$, where p runs over the family of continuous semi-norms on Φ and B runs over the family of bounded subsets of Φ. Of course, the same locally convex topology is generated if p is allowed to run over some family of semi-norms on Φ which generate the topology of Φ.

We have the following obvious criteria for convergence in $\mathscr{O}'_c(\Phi)$.

PROPOSITION 1. *A net $\{T_\nu\}$ in $\mathscr{O}'_c(\Phi)$ converges to 0 in $\mathscr{O}'_c(\Phi)$ iff for each bounded subset B of Φ, $T_\nu*\varphi \to 0$ in Φ uniformly for $\varphi \in B$.*

Remark 1. For the familiar spaces $\Phi = \mathscr{S}$ and $\mathscr{D}$ we have $\mathscr{O}'_c(\mathscr{S}) = \mathscr{O}'_c$ and $\mathscr{O}'_c(\mathscr{D}) = \mathscr{E}'$, and it will follow from Proposition 9 that the topology defined above agrees with the topologies defined on these spaces by L. Schwartz [8].

We recall that a test space Φ is said to have a *differentiable translation* if the limit relation $(1/h_j)\big(\varphi(x+h_j) - \varphi(x)\big) \to \dfrac{\partial\varphi}{\partial x_j}\big(h_j = (0, \ldots, h_j, \ldots 0) \to 0\big)$ holds in Φ for each $\varphi \in \Phi$ ([2], III, 3.3). An immediate corollary of Proposition 1 is then

COROLLARY 2. *If Φ has a differentiable translation and $P(D)$ is a linear partial differential operator with constant coefficients, then the map $T \to P(D)T$ is continuous from $\mathscr{O}'_c(\Phi)$ into $\mathscr{O}'_c(\Phi)$.*

Proof. If $T \in \mathscr{O}'_c(\Phi)$, then $P(D)T \in \mathscr{O}'_c(\Phi)$ by the theorem in § III. 3.3 of [2]. Moreover, for $\varphi \in \Phi$, $P(D)(T*\varphi) = T*P(D)\varphi$ so the corollary follows from Proposition 1 and the fact that $\varphi \to P(D)\varphi$ is continuous on Φ.

We now consider some of the continuity properties of the convolution operation. First, we note

PROPOSITION 3. *The bilinear map $(T, \varphi) \to T*\varphi$ from $\mathscr{O}'_c(\Phi) \times \Phi$ into Φ is $\mathscr{B}$-hypocontinuous, where $\mathscr{B}$ is the family of all bounded subsets of Φ.*

Proof. If $B \subseteq \Phi$ is bounded and the net $\{T_\nu\}$ converges to 0 in $\mathscr{O}'_c(\Phi)$, then $T_\nu*\varphi \to 0$ in Φ uniformly for $\varphi \in B$ by Proposition 1.

If $T \in \mathscr{O}'_c(\Phi)$ is fixed, the map $\varphi \to T*\varphi$ is continuous on Φ by the definition of convolution operator.

From the definition of the semi-norms in (1), we observe the following criteria for boundedness in $\mathscr{O}'_c(\Phi)$.

LEMMA 4. *A subset $A \subset \mathscr{O}'_c(\Phi)$ is bounded iff for every bounded subset B of Φ, $\{T*\varphi:\ T \in A,\ \varphi \in B)$ is bounded in Φ.*

Using this fact, we obtain,

PROPOSITION 5. *The bilinear map $(T, S) \to T*S$ from $\mathscr{O}'_c(\Phi) \times \Phi'_b$ into Φ'_b is $\mathscr{A}$-hypocontinuous, where $\mathscr{A}$ is the family of all bounded subsets of $\mathscr{O}'_c(\Phi)$ and Φ'_b denotes Φ' equipped with the strong topology* ([10], II. 19).

Proof. If $S \in \Phi'$, $\{T_\nu\}$ is a net in $\mathscr{O}'_c(\Phi)$ which converges to 0 and B is a bounded subset of Φ, then $T_\nu*\varphi \to 0$ in Φ uniformly for $\varphi \in B$ by Proposition 1. Thus $T_\nu*S \to 0$ in Φ'_b since $\langle T_\nu*S, \varphi\rangle = \langle S, T_\nu*\varphi\rangle$.

If $A \subseteq \mathscr{O}'_c(\Phi)$ is bounded, $B \subseteq \Phi$ is bounded and $\{S_\nu\}$ is a net which converges to 0 in Φ'_b, then $\langle T*S_\nu, \varphi\rangle = \langle S_\nu, T*\varphi\rangle \to 0$ uniformly for $\varphi \in B$ and $T \in A$ by Lemma 4.

Similarly, we have

PROPOSITION 6. *If $S, T \in \mathscr{O}'_c(\Phi)$, then $S*T \in \mathscr{O}'_c(\Phi)$ and the bilinear map $(S, T) \to S*T$ from $\mathscr{O}'_c(\Phi) \times \mathscr{O}'_c(\Phi)$ into $\mathscr{O}'_c(\Phi)$ is hypocontinuous with respect to bounded sets.*

Proof. Note for $\varphi \in \Phi$, $(S*T)*\varphi = S*(T*\varphi)$ and then apply Proposition 1 and Lemma 4.

COROLLARY 7. *If $U \in \Phi'$ and $S, T \in \mathscr{O}'_c(\Phi)$, then $(S*T)*U = S*(T*U)$.*

In order to compare this approach to convolution with that given in [8], we make the following observations. If $T \in \mathscr{O}'_c(\Phi)$, then T induces a continuous linear operator $\hat{T}: \Phi \to \Phi$ defined by $\hat{T}(\varphi) = T*\varphi$. The linear map $T \to \hat{T}$ from $\mathscr{O}'_c(\Phi)$ into $L(\Phi, \Phi)$, the vector space of continuous linear operators on Φ, is one-one since if $T \neq 0$, there is a $\varphi \in \Phi$ such that $\langle T, \varphi\rangle \neq 0$ and then $T*\varphi(0) = \hat{T}(\varphi)(0) \neq 0$. Hence $\mathscr{O}'_c(\Phi)$ may be identified with a linear subspace of $L(\Phi, \Phi)$. Let τ be the topology induced on $\mathscr{O}'_c(\Phi)$ by $L_b(\Phi, \Phi)$, the topology of uniform convergence on bounded subsets of Φ ([10], II. 32).

Proposition 8. *The topology of $\mathcal{O}'_c(\Phi)$ coincides with the topology τ.*

Proof. Let B be bounded in Φ and $\{T_\nu\}$ be a net in $\mathcal{O}'_c(\Phi)$. Then $T_\nu \to 0$ in $\mathcal{O}'_c(\Phi)$ iff $T_\nu * \varphi \to 0$ in Φ uniformly for $\varphi \in B$ iff $\hat{T}_\nu \to 0$ in $L_b(\Phi, \Phi)$.

In a similar fashion if $T \in \mathcal{O}'_c(\Phi)$, then T induces a continuous linear operator $T^\#$ from Φ'_b into Φ'_b defined by $T^\#(S) = T * S$ (Proposition 5). Again the linear map $T \to T^\#$ from $\mathcal{O}'_c(\Phi)$ into $L(\Phi'_b, \Phi'_b)$ is one-one since if $T \in \mathcal{O}'_c(\Phi)$ and $T \neq 0$, then there exists $\varphi \in \Phi$ such that $T * \varphi \neq 0$ and the Hahn–Banach Theorem insures that there exists $S \in \Phi'$ such that $\langle S, T*\varphi\rangle = \langle T*S, \varphi\rangle \neq 0$. Thus $\mathcal{O}'_c(\Phi)$ may be identified with a linear subspace of $L(\Phi', \Phi')$. Let τ' be the topology induced on $\mathcal{O}'_c(\Phi)$ by the space of linear operators $L(\Phi', \Phi'_b)$ equipped with the topology of uniform convergence on equicontinuous subsets of Φ'.

Proposition 9. *The topology of $\mathcal{O}'_c(\Phi)$ coincides with the topology τ'.*

Proof. Let B be bounded in Φ and A be an equicontinuous subset of Φ'. If $\{T_\nu\}$ is a net in $\mathcal{O}'_c(\Phi)$, then $T_\nu \to 0$ in $\mathcal{O}'_c(\Phi)$ iff $T_\nu * \varphi \to 0$ in Φ uniformly for $\varphi \in B$ iff $\langle S, T_\nu * \varphi\rangle = \langle T_\nu * S, \varphi\rangle = T^\#_\nu(S)(\varphi) \to 0$ uniformly for $\varphi \in B$ and $S \in A$ ([6], Prop. 3 of III) iff $T^\#_\nu \to 0$ in $L(\Phi', \Phi'_b)$ with respect to the topology of uniform convergence on equicontinuous subsets.

Remark 2. From [8], § 11 it follows that the topology defined here coincides with the topology introduced by L. Schwartz for the spaces $\mathcal{O}'_c(\mathcal{D}) = \mathcal{E}'$ and $\mathcal{O}'_c(\mathcal{S}) = \mathcal{O}'_c$.

Remark 3. This is also the case when Φ' is the space of distributions of exponential order (see [12], Theorem 4).

The results in Propositions 8 and 9 can be used to deduce certain topological properties of $\mathcal{O}'_c(\Phi)$ inherited from Φ.

Corollary 10. *If Φ is complete and barreled, and Φ'_b is nuclear, then $\mathcal{O}'_c(\Phi)$ is nuclear.*

Proof. Under the hypothesis, $L_b(\Phi, \Phi)$ is nuclear ([10], III. 50.5), and thus $\mathcal{O}'_c(\Phi)$ is nuclear ([10], III. 50.1).

Corollary 11. *If Φ is bornological and complete, then $\mathcal{O}'_c(\Phi)$ is complete.*

Proof. Suppose $\{T_\nu\}$ is a Cauchy net in $\mathcal{O}'_c(\Phi)$. Then $\{\hat{T}_\nu\}$ is a Cauchy net in $L_b(\Phi, \Phi)$, and since $L_b(\Phi, \Phi)$ is complete ([10], Corollary 1 of II. 32.2), there exists $S \in L_b(\Phi, \Phi)$ such that $\hat{T}_\nu \to S$.

Define $T \in \Phi'$ by $\langle T, \varphi\rangle = S(\varphi)(0)$. For $\varphi \in \Phi$, $S(\varphi) = T * \varphi$ since for $h \in R^n$,

$$S(\varphi)(h) = \lim T_\nu * \varphi(h) = \lim T_\nu * (\tau_h\varphi)(0) = = S(\tau_h\varphi)(0) = \langle T, \tau_h\varphi\rangle = T*\varphi(h).$$

Thus for $\varphi \in \Phi$, $T*\varphi \in \Phi$ and the map $\varphi \to T*\varphi$ is continuous. That is, $T \in \mathcal{O}'_c(\Phi)$ and, moreover, $T_\nu \to T$ in $\mathcal{O}'_c(\Phi)$ since $\hat{T} = S$.

By applying Corollary 2 of Theorem II. 34.2 of [10] and using the same technique as above, we may also obtain

Corollary 12. *If Φ is barreled, then $\mathcal{O}'_c(\Phi)$ is quasi-complete.*

This method of course has its shortcomings. For example, properties such as $\mathcal{O}'_c(\Phi)$ being barreled (bornological) cannot be deduced by this method since subspaces of barreled (bornological) spaces aren't necessarily barreled (bornological). However, many of the familiar spaces of convolution operators are barreled and bornological. (See [4], Theorem 16 for $\mathcal{O}'_c(\mathcal{S})$ and [12], Corollary 1 of Theorem 9, for the space of convolution operators on the distributions of exponential order.) It would be interesting to determine what properties of Φ imply $\mathcal{O}'_c(\Phi)$ is barreled or bornological.

2. Convergence of convolution operators for certain $K\{M_p\}$ spaces. Let $\{M_p\}$ be a sequence of continuous functions defined on R^k such that $1 \leqslant M_1(x) \leqslant M_2(x) \leqslant \ldots$ $x \in R^k$.

The space $K\{M_p\}$ is defined to be the vector space of all infinitely differentiable functions φ such that $M_p D^\alpha \varphi$ is bounded on R^k for every positive integer p and $|\alpha| \leqslant p$. The vector space $K\{M_p\}$ is given a locally convex Hausdorff topology by means of the semi-norms

$$(2) \qquad \|\varphi\|_p = \sup\{M_p(x)|D^\alpha\varphi(x)|:\ x \in R^k,\ |\alpha| \leqslant p\} \qquad (p = 1, 2, \ldots).$$

In this section we will only consider $K\{M_p\}$ spaces which satisfy the condition (P) ([2], II. 2.1). The sequence $\{M_p\}$ satisfies condition (P) if: (P) for each p there exists $p' > p$ such that

$$\lim_{|x|\to\infty} M_p(x)/M_{p'}(x) = 0.$$

We will give examples of such spaces at the end of this section.

If $\Phi = K\{M_p\}$ and the sequence $\{M_p\}$ satisfies condition (P), then Φ is a test space. Indeed, it is easily checked that properties (ii) and (iii) for a test space hold and property (i) follows from Theorem 1 of II. 2.5 in [2]. Thus any generalized function $T \in \Phi'$ can be identified with a distribution.

We have the following result pertaining to sequential convergence in $\mathcal{O}'_c(\Phi)$.

Proposition 13. *If $T_n \to 0$ in $\mathcal{O}'_c(\Phi)$, then for each positive integer p there is a positive integer n_p and bounded continuous functions $f_{n,j}$ $(0 \leqslant |j| \leqslant n_p)$ such that*

(i) $T_n = \sum_{|j|\leqslant n_p} D^j f_{n,j}$,

(ii) *each function $M_p f_{n,j}$ is bounded and for every j, $\lim_n M_p f_{n,j}(t) = 0$ uniformly for t in R^k.*

Proof. Since $T_n \to 0$ in $\mathcal{O}_c'(\Phi)$, $D = \{M_p(h)\tau_{-h}T_n\colon\ h \in R^k,\ n \geqslant 1\}$ is strongly bounded in $\mathcal{D}'$. Consider D as a net in $\mathcal{D}'$ directed by $(n, h) \geqslant (n', h')$ iff $n \geqslant n'$. Then D is a bounded net which converges to 0 in $\mathcal{D}'$ with respect to the strong topology since $\mathcal{D} \subseteq \Phi$ and the injection is continuous (Proposition 3). Therefore there is a compact neighborhood K of 0 in R^k and a positive integer m such that if $\psi \in \mathcal{D}_K^m$, then the net $\{(M_p(h)\tau_{-h}T_n)*\psi\colon\ h \in R^k,\ n \geqslant 1\}$ of bounded continuous functions converges to 0 uniformly on K ([7], Ch. VI, § 7, Th. XXIII). The elementary solution E of Δ^N is m-times continuously differentiable for large N so if we take $\gamma \in \mathcal{D}_K$ such that γ is equal to 1 on a neighborhood of the origin, then $\gamma E \in \mathcal{D}_K^m$ and $\delta = \Delta^N(\gamma E) - \varphi$ where $\varphi \in \mathcal{D}$. Thus

$$T_n = T_n * \delta = \Delta^N(T_n * \gamma E) - T_n * \varphi. \tag{3}$$

Since the net D converges to 0 in $\mathcal{D}'$,

$$\langle M_p(h)\tau_{-h}T_n, \varphi\rangle = M_p(h)T_n*\varphi(h) \to 0 \quad \text{as} \quad n \to \infty \text{ uniformly} \quad \text{for } h \in R^k \tag{4}$$

and each function $T_n*\varphi$ is bounded and continuous. Since $\gamma E \in \mathcal{D}_K^m$, the net of continuous functions $\{M_p(h)\tau_{-h}T_n*\gamma E\colon\ h \in R^k,\ n \geqslant 1\}$ converges to 0 uniformly on K. Therefore each function $T_n*\gamma E$ is bounded and continuous and

$$\lim_n M_p(h)T_n*(\gamma E)(h) = 0 \tag{5}$$

uniformly for $h \in R^k$ since $0 \in K$.

Formulas (3), (4) and (5) give conditions (i) and (ii).

If the sequence $\{M_p\}$ satisfies some additional conditions, the conclusion of Proposition 13 can be reformulated. In particular, $\{M_p\}$ is said to satisfy conditions (M) and (N) if:

(M) the functions M_p are quasi-monotonic in each coordinate, i.e., if $|X_j'| \leqslant |X_j''|$, then $M_p(X_1, \ldots, X_j', \ldots, X_k) \leqslant M_p(X_1, \ldots, X_j'', \ldots, X_k)$ for each fixed point $(X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_k)$.

(N) for each p there is $p' > p$ such that the ratio $M_p(x)/M_{p'}(x) = m_{pp'}(x)$ tends to 0 as $|x| \to \infty$ and the function $m_{pp'}$ is Lebesgue summable on R^k ([2], II. 4.2).

Corollary 14. *Let $\{M_p\}$ satisfy conditions* (M) *and* (N). *If $T_n \to 0$ in $\mathcal{O}_c'(\Phi)$, then for each p there exists $n(p)$ and continuous functions $f_{n,j} \in L^1(R^k)$ such that*

(I) $T_n = \sum_{|j| \leqslant n(p)} D^j f_{n,j}$,

(II) *each $M_p f_{n,j} \in L^1(R^k)$ and for every j, $M_p f_{n,j} \to 0$ in $L^1(R^k)$ as $n \to \infty$.*

Proof. Pick $p' > p$ as in condition (N) and then set $n(p) = n_p$, as in Proposition 13. With the notation as in Proposition 13, we have $|M_p f_{n,j}| \leqslant M_{p'}|f_{n,j}|m_{pp'} \leqslant Cm_{pp'}$, since the sequence $\{M_{p'}f_{n,j}\}_{n=1}^{\infty}$ is uniformly bounded. Since $\lim_n M_p(t)f_{n,j}(t) = 0$, $M_p f_{n,j} \to 0$ in $L^1(R^k)$ as $n \to \infty$ by Lebesgue's Dominated Convergence Theorem.

We now seek converses to the two statements above. We show that such converses do exist in certain $K\{M_p\}$ spaces.

The sequence $\{M_p\}$ satisfies condition (F) if:

(F) each M_p is symmetric, i.e., $M_p(x) = M_p(-x)$, and for each p there is a $p' > p$ and $C_{p'} > 0$ such that $M_p(x+h) \leqslant C_{p'}M_{p'}(x)M_{p'}(h)$ for all $x, h \in R^k$ ([9]).

We recall that if $\{M_p\}$ satisfies conditions (M) and (N), then the semi-norms

$$\|\varphi\|_p' = \sup\Big\{\int M_p(x)|D^\alpha\varphi(x)|\,dx\colon\ |\alpha| \leqslant p\Big\} \qquad (1 \leqslant p < \infty) \tag{6}$$

generate the same locally convex topology on $K\{M_p\}$ as the sequence in (2) ([2], II. 4.2). (Throughout $\int f$ denotes the integral of f over R^k.) Thus in this case the topology of $\mathcal{O}_c'(\Phi)$ is generated by the semi-norms

$$T \to \sup\Big\{\int M_p(t)|D^\alpha T*\varphi(t)|\,dt\colon\ |\alpha| \leqslant p,\ \varphi \in B\Big\}, \tag{7}$$

where $1 \leqslant p < \infty$ and B is a bounded subset of Φ (equation (1)).

Proposition 15. *Let $\{M_p\}$ satisfy conditions* (M), (N) *and* (F). *If the sequence $\{T_n\} \subseteq \Phi'$ satisfies the conditions* (I) *and* (II) *of Corollary* 14, *then each $T_n \in \mathcal{O}_c'(\Phi)$ and $T_n \to 0$ in $\mathcal{O}_c'(\Phi)$.*

Proof. First we show that if $T \in \Phi'$ satisfies conditions (I) and (II) of Corollary 14, then $T \in \mathcal{O}_c'(\Phi)$. Let q be a positive integer and $|\alpha| \leqslant q$. Set $p = q'$ as in condition (F), and apply the hypothesis of (I), (II) in Corollary 14 to the integer p. Then for $\varphi \in \Phi$,

$$\begin{aligned}
\int M_q(h)|D^\alpha T*\varphi(h)|\,dh &= \int M_q(h)|\langle T, \tau_h D^\alpha\varphi\rangle|\,dh \\
&\leqslant \int M_q(h) \sum_{|j| \leqslant n(p)} \int |f_j(x) D^{\alpha+j}\varphi(x+h)|\,dx\,dh \\
&= \sum_{|j| \leqslant n(p)} \int |f_j(x)|\, M_q(h)|D^{\alpha+j}\varphi(x+h)|\,dh\,dx \\
&\leqslant \sum_{|j| \leqslant n(p)} C_p \int |f_j(x)|M_p(x)\,dx \int M_p(u)|D^{\alpha+j}\varphi(u)|\,du \\
&\leqslant C_p \sum_{|j| \leqslant n(p)} \int |f_j|M_p \|\varphi\|_{p+n(p)}'.
\end{aligned} \tag{8}$$

Since Φ has a differentiable translation ([9], Cor. 2), each $T*\varphi$ is infinitely differentiable, and it follows from (8) and (6) that $T*\varphi \in \Phi$ and the map $\varphi \to T*\varphi$ is continuous. Hence $T \in \mathcal{O}_c'(\Phi)$.

If the sequence $\{T_n\} \subset \Phi'$ satisfies conditions (I) and (II) of Corollary 14, then equation (8) applied to each T_n implies that $T_n \to 0$ in $\mathcal{O}_c'(\Phi)$ by (7).

COROLLARY 16. *Let* $\{M_p\}$ *satisfy condition* (M), (N) *and* (F). *If* $\{T_n\} \subseteq \Phi'$ *satisfies the conditions* (i) *and* (ii) *of Proposition* 13, *then* $\{T_n\} \subseteq \mathcal{O}_c'(\Phi)$ *and* $T_n \to 0$ *in* $\mathcal{O}_c'(\Phi)$.

Proof. Let p be a positive integer and choose p' as in condition (N). Apply the representation in (i), (ii) of Proposition 13 to the integer p'. Then $M_p|f_{n,j}| \leqslant M_{p'}|f_{n,j}|m_{pp'} \leqslant Cm_{pp'}$ since the family of functions $\{M_{p'}f_{n,j}\colon\ n \geqslant 1,\ 0 \leqslant |j| \leqslant n_{p'}\}$ is uniformly bounded by hypothesis. Since $\lim_n M_p(t)f_{n,j}(t) = 0$, the Lebesgue Dominated Convergence Theorem implies $M_pf_{n,j} \to 0$ in $L^1(R^k)$. The corollary now follows from Proposition 15.

Summarizing, we have

THEOREM 17. *Let* $\{M_p\}$ *satisfy conditions* (M), (N) *and* (F). *For a sequence* $\{T_n\} \subseteq \Phi'$, *the following are equivalent*:

(a) $\{T_n\} \subseteq \mathcal{O}_c'(\Phi)$ *and* $T_n \to 0$ *in* $\mathcal{O}_c'(\Phi)$,

(b) *conditions* (i), (ii) *of Proposition* 13,

(c) *conditions* (I), (II) *of Corollary* 14.

Remark 4. The argument in Corollary 14 shows that if $T \in \Phi'$ is such that for each p, T is a finite sum, $T = \Sigma D^j f_j$ with M_pf_j bounded, then

(9) *for each* p, T *is a finite sum*, $T = \Sigma D^j g_j$, *with* $M_pg_j \in L^1(R^k)$.

The argument in Proposition 15 shows that if $T \in \Phi'$ satisfies (9), then $T \in \mathcal{O}_c'(\Phi)$. Therefore, by Theorem 2 of [9], $T \in \mathcal{O}_c'(\Phi)$ iff (9) holds.

By the same type of arguments employed above, we can obtain

THEOREM 18. *Let* $\{M_p\}$ *satisfy conditions* (M), (N) *and* (F). *For a subset* $B = \{T_a\colon a \in A\} \subseteq \Phi'$, *the following are equivalent*:

(a) B *is bounded in* $\mathcal{O}_c'(\Phi)$;

(b) *for each positive integer* p *there exist* $n(p)$ *and continuous functions* $f_{a,j}\big(a \in A,\ 0 \leqslant |j| \leqslant n(p)\big)$ *such that* $T_a = \sum_{|j| \leqslant n(p)} D^j f_{a,j}$ *and the family of continuous functions* $\{M_pf_{a,j}\colon\ a \in A,\ 0 \leqslant |j| \leqslant n(p)\}$ *is uniformly bounded on* R^k;

(c) *for each positive integer* p *there exist* $n(p)$ *and continuous functions* $f_{a,j}\big(a \in A,\ 0 \leqslant |j| \leqslant n(p)\big)$ *with* $T_a = \sum_{|j| \leqslant n(p)} D^j f_{a,j}$ *and*

$$\{M_pf_{a,j}\colon\ a \in A,\ 0 \leqslant |j| \leqslant n(p)\} \quad \textit{bounded in } L^1(R^k).$$

We now consider these results for some of the more familiar spaces of generalized functions.

EXAMPLE 1. The space $\mathscr{S}$ ([7], VII. 3) is a $K\{M_p\}$ space with $M_p(x) = (1+|x|^2)^p$ and $\{M_p\}$ satisfies (M), (N) and (F) ([9]). Theorems 17 and 18 therefore apply to $\mathscr{S}$. These characterizations of convergent sequences and bounded sets in $\mathcal{O}_c'$ do not seem to appear in Schwartz' book [7] but can be derived by using the method of proof employed in Theorem 3.59 of [1].

EXAMPLE 2. The space of distributions of exponential type ([5], [11], [12]) is the dual of the $K\{M_p\}$ space, where $M_p(x) = \exp\big(p\gamma(x)\big)$, $\gamma(x) = \sqrt{1+|x|^2}$. The sequence $\{M_p\}$ in this case also satisfies (M), (N) and (F) so that theorems 17 and 18 apply. A form of Theorem 18 for this case is given in [11], Proposition 12.

EXAMPLE 3. The $W_{M,a}$ spaces of Gelfand and Shilov ([3], I. 1.1) are also $K\{M_p\}$ spaces satisfying (M), (N) and (F) ([9]). Hence Theorems 17 and 18 are also applicable to these spaces.

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