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A characterization of subspaces of $L^p(\mu)$

by

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Abstract. Generalizing results of Kwapień and Cohen (Studia Math. (38), pp. 271–278) which characterize Hilbert space in terms of the behavior of absolutely 2-summing operators, the following theorem is proved:

If E is a Banach space then the following are equivalent:

(i) $E^* \subset L^p(\mu)$ *for some* μ $(1 \leqslant p < +\infty)$.

(ii) *If* $T \in QN_p(E, l^p)$ *then* $T^* \in QN_p(l^q, E^*)$.

§ 1. Motivated by a result of Cohen [1], Kwapień has proved the following characterization of Hilbert space.

THEOREM [4]. *If E is a Banach space then the following are equivalent:*

(i) *E is isomorphic to a Hilbert space.*

(ii) *If* $T \in AS_2(E, l^2)$ *then* $T^* \in AS_2(l^2, E^*)$.

In this paper we show that the following more general result concerning subspaces of $L^p(\mu)$ is valid.

THEOREM. *If E is a Banach space then the following are equivalent:*

(i) $E^* \subset L^p(\mu)$ *for some* μ $(1 \leqslant p < +\infty)$.

(ii) *If* $T \in QN_p(E, l^p)$ *then* $T^* \in QN_p(l^q, E^*)$.

The proof is based on a result concerning the adjoints of p-quasi-nuclear maps on $\mathscr{L}^q$-spaces (which extends a theorem of Persson [7]) and on a characterization of subspaces of $L^p(\mu)$ in terms of a domination property of sequences which has been given by Lindenstrauss and Pełczyński [5]. The result of Kwapień follows immediately from our theorem.

§ 2. Notation. Throughout the paper E and F will denote Banach spaces and $\mathscr{L}(E, F)$ the space of all continuous linear operators from E to F. The term "operator" or "map" will *always* mean an element of $\mathscr{L}(E, F)$.

Recall that an operator $T \in \mathscr{L}(E, F)$ is said to be

(i) *p-nuclear* [8] if T has the representation $T = \sum_i f_i \otimes y_i$ where $(f_i) \subset E^*$, $(y_i) \subset F$ and $\sum_i \|f_i\|^p < +\infty$, $\sup_{\|g\| \leqslant 1, g \in F^*} \sum_i |g(y_i)|^q < +\infty$ $(p^{-1}+q^{-1} = 1)$. The *p-nuclear norm* of T, denoted $N_p(T)$, is defined by

$$N_p(T) = \inf\Big\{\Big(\sum_i \|f_i\|^p\Big)^{1/p} \sup_{\|g\| \leqslant 1} \Big(\sum_i |g(y_i)|^q\Big)^{1/q}\Big\},$$

where the inf is taken over all such representations of T.

(ii) *p-quasinuclear* [8] if there is a sequence $(f_i) \subset E^*$ for which $\sum_i \|f_i\|^p < +\infty$ and $\|Tx\| \leqslant \big(\sum_i |f_i(x)|^p\big)^{1/p}$ for all $x \in E$. The *p-quasinuclear norm* of T, denoted by $QN_p(T)$, is defined by

$$QN_p(T) = \inf\Big\{\Big(\sum_i \|f_i\|^p\Big)^{1/p}\Big\},$$

where the inf is taken over all such (f_i).

(iii) *p-absolutely summing* [5] if there is a $K \geqslant 1$ such that for every $(x_i)_{i=1}^n \subset E$, $\big(\sum_{i=1}^n \|Tx_i\|^p\big)^{1/p} \leqslant K \sup_{\|f\| \leqslant 1, f \in E^*} \big(\sum_{i=1}^n |f(x_i)|^p\big)^{1/p}$. The *p-absolutely summing norm of* T, denoted $AS_p(T)$, is defined by

$$AS_p(T) = \inf K, \text{ where } K \text{ is as above.}$$

We refer the reader to [7] and [8] (among others) for results concerning properties of these operators.

If E is isomorphically embedded in F we write $E \subset F$. If E and F are isomorphic under a mapping $T\colon E \leftrightarrow F$ for which $\|T\|\,\|T^{-1}\| \leqslant \lambda$ then we say E and F are *λ-isomorphic* and that T is a *λ-isomorphism.*

A Banach space E is called an $\mathcal{L}_\lambda^p$-space $(1 \leqslant p \leqslant +\infty)$ [5] if given any finite dimensional subspace $F \subset E$ there is a finite dimensional subspace $F_0 \subset E$ such that $F \subset F_0$ and F_0 is λ-isomorphic to l_r^p (where $r = \dim F_0$). It is well known that $L^p(\mu)$ is an $\mathcal{L}_\lambda^p$-space (for some λ) for any measure μ [5].

Throughout the paper we assume that $p^{-1}+q^{-1} = 1$ wherever these symbols occur.

§ 3. Adjoints of *p*-quasinuclear maps. Our main theorem is a consequence of the following result which extends a theorem of Persson [7].

THEOREM 3.1. *Let E be an $\mathcal{L}_\lambda^q$-space $(1 < q \leqslant +\infty)$ and let $T\colon E \to F$ be in $QN_p(E, F)$ (where $p^{-1}+q^{-1} = 1$). Then $T^*\colon F^* \to E^*$ is in $N_p(F^*, E^*)$.*

Proof. Since E is an $\mathcal{L}_\lambda^q$-space it follows from results of Persson and Pietsch [8] that T is the limit in p-absolutely summing norm of a sequence of finite dimensional maps and hence may be written $T = \sum_i x_i \otimes y_i$ where $(x_i) \subset E^*$, $(y_i) \subset F$ and the series converges in p-absolutely summing norm (use essentially the same proof as ([10], p. 94)).

Let $\varepsilon > 0$ and N so large that $m, n \geqslant N$ implies $\big\|\sum_{i=m}^n x_i \otimes y_i\big\|_{AS_p} < \varepsilon$. Let $m, n \geqslant N$. Since E is an $\mathcal{L}_\lambda^q$-space, E^* is an $\mathcal{L}_\lambda^p$-space [6] $(p^{-1}+q^{-1} = 1)$ and so there is a subspace $X_r \subset E^*$ such that X_r is λ-isomorphic to l_r^p under the mapping $S\colon l_r^p \to X_r$ and $(x_i)_{i=m}^n \subset X_r$.

Then for each $i = m, m+1, \ldots, n$, $x_i = \sum_{j=1}^r a_{ij} S e_j$ (where $(e_j)_{j=1}^n$ is the unit vector basis for l_r^p) and hence

$$\Big\|\sum_{j=1}^r Se_j \otimes \sum_{i=m}^n a_{ij} y_i\Big\|_{AS_p} = \Big\|\sum_{i=m}^n \Big(\sum_{j=1}^r a_{ij} Se_j\Big) \otimes y_i\Big\|_{AS_p} = \Big\|\sum_{i=m}^n x_i \otimes y_i\Big\|_{AS_p} < \varepsilon.$$

Therefore since S is a λ-isomorphism it easily follows that

$$\Big\|\sum_{j=1}^r e_j \otimes \Big(\sum_{i=m}^n a_{ij} y_i\Big)\Big\|_{AS_p} < \lambda \cdot \varepsilon,$$

where we are considering $\sum_{j=1}^r e_j \otimes \big(\sum_{i=m}^n a_{ij} y_i\big)$ as an operator from l_r^q to F. However we may also consider $\sum_{j=1}^r e_j \otimes \big(\sum_{i=m}^n a_{ij} y_i\big)$ as an operator from F^* to l_r^p and under this identification we have by definition of the p-nuclear norm and the fact that (e_j) is the unit vector basis for l_r^p,

$$\Big\|\sum_{j=1}^r e_j \otimes \Big(\sum_{i=m}^n a_{ij} y_i\Big)\Big\|_{N_p} \leqslant \Big(\sum_{j=1}^r \Big\|\sum_{i=m}^n a_{ij} y_i\Big\|^p\Big)^{1/p}.$$

But

$$\begin{aligned}\Big(\sum_{j=1}^r \Big\|\sum_{i=m}^n a_{ij} y_i\Big\|^p\Big)^{1/p} &= \Big(\sum_{k=1}^r \Big\|\Big\langle \sum_{j=1}^r e_j \otimes \Big(\sum_{i=m}^n a_{ij} y_i\Big), e_k\Big\rangle\Big\|^p\Big)^{1/p} \\ &\leqslant \Big\|\sum_{j=1}^r e_j \otimes \Big(\sum_{i=m}^n a_{ij} y_i\Big)\Big\|_{AS_p} \\ &< \lambda \cdot \varepsilon.\end{aligned}$$

Therefore we have

$$\Big\|\sum_{j=1}^r e_j \otimes \Big(\sum_{i=m}^n a_{ij} y_i\Big)\Big\|_{N_p} < \lambda \cdot \varepsilon$$

and since S is a λ-isomorphism we again have

$$\Big\|\sum_{j=1}^r Se_j \otimes \Big(\sum_{i=m}^n a_{ij} y_i\Big)\Big\|_{N_p} < \lambda^2 \cdot \varepsilon.$$

Since $\sum_{j=1}^{r} Se_j \otimes \left(\sum_{i=m}^{n} a_{ij} y_i\right) = \sum_{i=m}^{n} x_i \otimes y_i$ we then have, *considering* $\sum_{i=m}^{n} x_i \otimes y_i$ *as an operator from* F^* *to* E^*,

$$\left\|\sum_{i=m}^{n} x_i \otimes y_i\right\|_{N_p} < \lambda^2 \cdot \varepsilon,$$

and therefore $\sum_{i=1}^{\infty} x_i \otimes y_i$ converges in $N_p(F^*, E^*)$. But since $\sum_{i=1}^{\infty} x_i \otimes y_i$, as an element of $\mathscr{L}(F^*, E^*)$, is simply the adjoint of $T = \sum_{i=1}^{\infty} x_i \otimes y_i$ considered as an element of $\mathscr{L}(E, F)$, we have proved that $T^* \in N_p(F^*, E^*)$.

As a corollary to Theorem 3.1 we have the following result which extends Persson's theorem that every p-absolutely summing operator from $L^q(\mu)$ to $L^p(\nu)$ is p-nuclear [7].

COROLLARY 3.2. *Let* $1 < q < +\infty$, *E an $\mathscr{L}^q$-space, F an $\mathscr{L}^p$-space and $T: E \to F$ in $\mathscr{L}(E, F)$. Then the following are equivalent:*

(i) *T is p-absolutely summing.*

(ii) *T^* is p-nuclear.*

(iii) *T is p-nuclear.*

(iv) *T^* is p-absolutely summing.*

Proof. (i) $\Rightarrow$ (ii) follows directly from Theorem 3.1 since for $1 < q < +\infty$ an $\mathscr{L}^q$-space is reflexive [5] and every p-absolutely summing map on a reflexive space is p-quasinuclear [7].

(ii) $\Rightarrow$ (iii): If F is an $\mathscr{L}^p$-space then F^* is an $\mathscr{L}^q$-space [6]. Hence if $T^*: F^* \to E^*$ is p-nuclear then T^* is certainly p-quasinuclear and so by Theorem (3.1) $T^{**}: E^{**} \to F^{**}$ is p-nuclear. But $T^{**} = T$ and so (iii) holds.

(iii) $\Rightarrow$ (i) is well known [8].

(ii) $\Leftrightarrow$ (iv) is clear from the proof of (i) $\Leftrightarrow$ (iii).

We are now ready to prove the theorem announced in the introduction.

THEOREM 3.3. *If E is a Banach space then the following are equivalent:*

(i) *$E^* \subset L^p(\mu)$ for some measure μ $(1 \leqslant p < +\infty)$.*

(ii) $T \in QN_p(E, l^p) \Rightarrow T^* \in QN_p(l^q, E^*)$.

Proof. (i) $\Rightarrow$ (ii): First, suppose $E^* \subset L^p(\mu)$ for $1 < p < +\infty$. Then E^*, and hence E, is reflexive [5] and there is a mapping $Q: L^q(\mu) \overset{\text{onto}}{\to} E^{**} = E$. Let $T \in QN_p(E, l^p)$. Then $T \circ Q: L^q(\mu) \overset{\text{onto}}{\to} E \to l^p$ is in $QN_p(L^q(\mu), l^p)$ [8] and hence by Theorem (3.1) the adjoint $(T \circ Q)^* \in N_p(l^q, L^p(\mu))$. But since $(T \circ Q)^* = Q^* \circ T^*$ where Q^* is an isomorphism it easily follows that $T^* \in QN_p(l^q, E^*)$.

For the remaining case suppose $E^* \subset L^1(\mu)$. Then there is a mapping $Q: L^\infty(\mu) \overset{\text{onto}}{\to} E^{**}$. By Kakutani's theorem $L^\infty(\mu)$ is isomorphic to some $\mathscr{C}(A)$ (where A is a compact Hausdorff space) and thus there exists an operator $Q_1: \mathscr{C}(A) \overset{\text{onto}}{\to} E^{**}$. Let $T \in \mathscr{L}(l^1, E^{**})$. Since l^1 has the lifting property [3] there is then a mapping $S: l^1 \to \mathscr{C}(A)$ such that T factors as

$$T: l^1 \to \mathscr{C}(A) \to E^{**}$$

that is, $\mathscr{L}(l^1, E^{**}) = I_\infty(l^1, E^{**})$ [8].

Since l^1 has the metric approximation property of Grothendieck [2] it is known that $\mathscr{L}(l^1, E^{**}) = N(E, l^1)^*$ [2] and $I_\infty(l^1, E^{**}) = QN(E, l^1)^*$ [8]. Hence (since the identifications of these dual spaces are accomplished in exactly the same way in each case) we have that $QN(E, l^1) = N(E, l^1)$. Therefore if $T \in QN(E, l^1)$ then $T^* \in N(l^\infty, E^*)$ and so certainly $T^* \in QN(l^\infty, E^*)$.

(ii) $\Rightarrow$ (i): By assumption, if $T \in QN_p(E, l^p)$ then $\|T^*\|_{QN_p} < +\infty$. Therefore by the Baire category theorem and the fact that $QN_p(E, l^p)$ is complete [8] there is a number $K \geqslant 1$ such that if $T \in QN_p(E, l^p)$ then $\|T^*\|_{QN_p} \leqslant K \|T\|_{QN_p}$. In particular $\|T^*\|_{AS_p} \leqslant K \|T\|_{QN_p}$ [8].

Now let $(f_i)_{i=1}^n$ and $(g_j)_{j=1}^m$ be sequences in E^* such that $\sum_{i=1}^{n} |F(f_i)|^p \leqslant \sum_{j=1}^{m} |F(g_j)|^p$ for all $F \in E^{**}$. Then in particular $\sum_{i=1}^{n} |f_i(x)|^p \leqslant \sum_{j=1}^{m} |g_j(x)|^p$ for all $x \in E$. Define the operator $T: E \to l^p$ by

$$T(x) = \sum_{i=1}^{n} f_i(x) e_i.$$

Then $T \in QN_p(E, l^p)$ and by definition of $\| \ \|_{QN_p}$ we have

$$\|T\|_{QN_p} \leqslant \left(\sum_{j=1}^{m} \|g_j\|^p\right)^{1/p}$$

(since $\|Tx\| = \left(\sum_{i=1}^{n} |f_i(x)|^p\right)^{1/p} \leqslant \left(\sum_{j=1}^{m} |g_j(x)|^p\right)^{1/p}$ for all x).

Therefore by the above we have

$$\|T^*\|_{AS_p} \leqslant K\left(\sum_{j=1}^{m} \|g_j\|^p\right)^{1/p}.$$

But clearly $T^* = \sum_{i=1}^{n} e_i \otimes f_i: l^q \to E^*$ so

$$\|T^*\|_{AS_p} \geqslant \left(\sum_{i=1}^{n} \|f_i\|^p\right)^{1/p}.$$

It follows that

$$\Big(\sum_{i=1}^{n} \|f_i\|^p\Big)^{1/p} \leqslant K\Big(\sum_{j=1}^{m} \|g_j\|^p\Big)^{1/p}$$

and by a theorem of Lindenstrauss and Pełczyński ([5], p. 313) we conclude that $E^* \subset L^p(\mu)$ for some measure μ and the theorem is proved.

The result of Kwapień mentioned earlier is now immediate. For, if E is isomorphic to a Hilbert space then $E^* \subset L^2(\mu)$. If $T \in AS_2(E, l^2)$ then $T \in QN_2(E, l^2)$ [7] (since E is reflexive). Hence by Theorem (3.3) $T^* \in QN_2(l^2, E^*)$, implying $T^* \in AS_2(l^2, E^*)$.

Conversely, if $T \in AS_2(E, l^2) \Rightarrow T^* \in AS_2(l^2, E^*)$ then $T \in QN_2(E, l^2) \Rightarrow T \in AS_2(E, l^2) \Rightarrow T^* \in AS_2(l^2, E^*) \Rightarrow T^* \in QN_2(l^2, E^*)$ [7], and by Theorem 3.3 $E^* \subset L^2(\mu)$. It follows that E is isomorphic to a Hilbert space.

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Weak type inequalities for product operators

by

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Abstract. In this paper we prove a weak type inequality for products of sublinear operators from which a generalization of the ergodic theorems of Dunford and Schwartz is deduced. As a further application, we show how the inequality yields a simple proof of the theorem of Jessen, Marcinkiewicz and Zygmund on strong differentiability of multiple Lebesgue integrals.

INTRODUCTION

1. Preliminary definitions and statement of results. The space underlying the following exposition will be a σ-finite measure space $(\Omega, \mathfrak{F}, \mu)$.

Dunford and Schwartz have proved in [4] that if each of the linear operators $T_i (i = 1, 2, \ldots, k)$ is at the same time a contraction of L^1 and of L^∞, that is, if

$$\|T_i\|_1 \leqslant 1, \quad \|T_i\|_\infty \leqslant 1,$$

then the multiple averages

$$\frac{1}{n_1 \ldots n_k} \sum_{i_1=0}^{n_1-1} \ldots \sum_{i_k=0}^{n_k-1} T_1^{i_1} \ldots T_k^{i_k} f$$

converge almost everywhere in Ω as $n_1 \to \infty, \ldots, n_k \to \infty$ independently, provided that the function f belongs to some class L_p with $p > 1$, in which case the limit function is in L^p and the averages converge to the limit also in the L^p-norm. We denote by R_k the class of all functions f such that the integral

$$\int_{\{|f|>t\}} \frac{|f|}{t}\Big(\log\frac{|f|}{t}\Big)^k d\mu$$

is finite for every $t > 0$.

We show that this class is a vector space which contains properly, for any $k \geqslant 0$, the linear span of $\bigcup_{p>1} L^p$.