It follows that

$$\Big(\sum_{i=1}^{n} \|f_i\|^p\Big)^{1/p} \leqslant K\Big(\sum_{j=1}^{m} \|g_j\|^p\Big)^{1/p}$$

and by a theorem of Lindenstrauss and Pełczyński ([5], p. 313) we conclude that $E^* \subset L^p(\mu)$ for some measure μ and the theorem is proved.

The result of Kwapień mentioned earlier is now immediate. For, if E is isomorphic to a Hilbert space then $E^* \subset L^2(\mu)$. If $T \in AS_2(E, l^2)$ then $T \in QN_2(E, l^2)$ [7] (since E is reflexive). Hence by Theorem (3.3) $T^* \in QN_2(l^2, E^*)$, implying $T^* \in AS_2(l^2, E^*)$.

Conversely, if $T \in AS_2(E, l^2) \Rightarrow T^* \in AS_2(l^2, E^*)$ then $T \in QN_2(E, l^2) \Rightarrow T \in AS_2(E, l^2) \Rightarrow T^* \in AS_2(l^2, E^*) \Rightarrow T^* \in QN_2(l^2, E^*)$ [7], and by Theorem 3.3 $E^* \subset L^2(\mu)$. It follows that E is isomorphic to a Hilbert space.

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Weak type inequalities for product operators

by

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Abstract. In this paper we prove a weak type inequality for products of sublinear operators from which a generalization of the ergodic theorems of Dunford and Schwartz is deduced. As a further application, we show how the inequality yields a simple proof of the theorem of Jessen, Marcinkiewicz and Zygmund on strong differentiability of multiple Lebesgue integrals.

INTRODUCTION

1. Preliminary definitions and statement of results. The space underlying the following exposition will be a σ-finite measure space $(\Omega, \mathfrak{F}, \mu)$.

Dunford and Schwartz have proved in [4] that if each of the linear operators $T_i (i = 1, 2, \ldots, k)$ is at the same time a contraction of L^1 and of L^∞, that is, if

$$\|T_i\|_1 \leqslant 1, \quad \|T_i\|_\infty \leqslant 1,$$

then the multiple averages

$$\frac{1}{n_1 \ldots n_k} \sum_{i_1=0}^{n_1-1} \ldots \sum_{i_k=0}^{n_k-1} T_1^{i_1} \ldots T_k^{i_k} f$$

converge almost everywhere in Ω as $n_1 \to \infty, \ldots, n_k \to \infty$ independently, provided that the function f belongs to some class L_p with $p > 1$, in which case the limit function is in L^p and the averages converge to the limit also in the L^p-norm. We denote by R_k the class of all functions f such that the integral

$$\int_{\{|f|>t\}} \frac{|f|}{t} \Big(\log \frac{|f|}{t}\Big)^k d\mu$$

is finite for every $t > 0$.

We show that this class is a vector space which contains properly, for any $k \geqslant 0$, the linear span of $\bigcup_{p>1} L^p$.

Then we prove a weak type inequality which permits to extend the result of Dunford and Schwartz to any function f in R_{k-1}, where k is, naturally, the number of operators involved.

In the following section we consider the analogous extension for the case of continuous semigroups of operators.

As a further application, we show how the inequality yields a simple proof of the theorem of Jessen, Marcinkiewicz and Zygmund on strong differentiability of multiple Lebesgue integrals.

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2. Maximal ergodic inequality. Let us consider a linear operator T defined on the class L^1+L^∞ of all functions f which can be written as the sum of a function g in L^1 and a function h in L^∞. We will assume that (i) $f \geqslant 0$ implies $Tf \geqslant 0$, (ii) $\|T\|_1 \leqslant 1$ and $\|T\|_\infty \leqslant 1$. Such an operator is usually called a positive contraction of L^1 and of L^∞. In all the problems connected with convergence in some sense of the averages

$$A_n f = \frac{f+Tf+\ldots+T^n f}{n+1}$$

an important role is played by a weak type estimate whose statement, given below, is known as the maximal ergodic theorem.

The maximal ergodic operator M is defined by the expression

$$Mf(x) = \sup_{n\geqslant 0} |A_n f(x)|.$$

THEOREM 1. *If f is a function in L^1, and for a given $\lambda > 0$ we put $E = \{Mf > \lambda\}$, then*

$$\mu(E) \leqslant \frac{1}{\lambda} \int_E |f|\, d\mu.$$

For the proof of Theorem 1 we refer to [6].

Let (X, μ) and (Y, ν) be measure spaces. We say that an operator T, mapping measurable functions from the first space into measurable functions from the second, is *sublinear* if

$$|T(f+g)| \leqslant |Tf| + |Tg|,$$
$$|T(cf)| = |c| \cdot |Tf|.$$

We say that T is of weak type $(1, 1)$ if there exists a constant C, such that

$$\nu\{|Tf| > \lambda\} \leqslant \frac{C}{\lambda} \|f\|_1$$

for any $\lambda > 0$ and any f in the domain of T.

For example, the maximal ergodic operator is clearly sublinear and Theorem 1 implies that it is also of weak type $(1, 1)$.

3. The Hardy-Littlewood maximal theorem. If for any integrable function f on the unit interval, we define

$$Mf(x) = \sup_{\alpha<x<\beta} \frac{1}{\beta-\alpha} \int_\alpha^\beta |f(y)|\, dy,$$

then the maximal theorem of Hardy–Littlewood asserts that M is of weak type $(1, 1)$; more precisely that

$$m\{Mf > t\} \leqslant \frac{2}{t} \int_0^1 |f|\, dx,$$

where m denotes the Lebesgue measure.

For a proof of this theorem we refer to [7].

Repeated use will be made of the following.

LEMMA 3. *Let $\varphi(t)$ be a non-decreasing function on the real interval $0 \leqslant t < \infty$, such that $\varphi(0) = 0$ and $\varphi(t)$ is absolutely continuous on every finite subinterval.*

Then, for any non-negative function f on a measure space $(\Omega, \mathfrak{F}, \mu)$ and any set E in $\mathfrak{F}$, we have

$$\int_E \varphi\{f(x)\}\mu(dx) = \int_0^\infty \mu(E \cap \{f > t\})\varphi'(t)\, dt.$$

Two particular cases of this formula are important:

(1) *If $\varphi(t) = t$, we obtain* $\int_E f(x)\, d\mu = \int_0^\infty \mu(E \cap \{f > t\})\, dt.$

(2) *If $\varphi(t) = t^p$ with $p > 1$, then* $\int_\Omega f^p\, d\mu = p\int_0^\infty \mu\{f > t\} \cdot t^{p-1} d\mu.$

For a proof of these results we refer again to [7].

4. The ergodic theorems of Dunford and Schwartz.

THEOREM 2. *Let T_i, $i = 1, 2, \ldots, k$ be linear operators in L^1 with $\|T_i\|_1 \leqslant 1$, and $\|T_i\|_\infty \leqslant 1$, $i = 1, 2, \ldots, k$. Then, for every f in L^p with $1 < p < \infty$, the multiple sequence*

$$\frac{1}{n_1 \ldots n_k} \sum_{i_1=0}^{n_1-1} \ldots \sum_{i_k=0}^{n_k-1} T_1^{i_1} \ldots T_k^{i_k} f$$

converges almost everywhere as $n_1 \to \infty, \ldots, n_k \to \infty$ independently.

THEOREM 3. *Let the semigroups $\{T_i(t), t \geqslant 0\}$, $i = 1, 2, \ldots, k$ be strongly measurable semigroups in L^1 with $\|T_i(t)\|_1 \leqslant 1$ and $\|T_i(t)\|_\infty \leqslant 1$. Then, for every f in L^p with $1 < p < \infty$, the functions*

$$\frac{1}{a_1 \ldots a_k} \int_0^{a_1} \ldots \int_0^{a_k} T_k(t_k) \ldots T_1(t_1) f \, dt_1 \ldots dt_k$$

converge almost everywhere as $a_1 \to \infty, \ldots, a_k \to \infty$ independently.

For the proofs see [4] or [5].

We will denote as usual by $L(\log^+ L)^a$ the class of all functions f such that $f(\log^+|f|)^a$ is integrable over the underlying space.

The Greek letter χ with a set as a subindex will indicate the characteristic function of the set. A transformation $\sigma\colon \Omega \to \Omega$ of $(\Omega, \mathfrak{F}, \mu)$ into itself is called measure-preserving if for each $E \in \mathfrak{F}$, we have

$$\sigma^{-1}(E) \in \mathfrak{F}, \quad \mu(\sigma^{-1}E) = \mu(E).$$

The Lebesgue measure of a set E will be denoted either by mE or by $|E|$.

I. WEAK TYPE INEQUALITIES FOR PRODUCT OPERATORS

1. Maximal operators and classes R_k.

Let $(\Omega, \mathfrak{F}, \mu)$ be a σ-finite measure space. Having in mind the properties of the maximal ergodic operator and those of the Hardy–Littlewood maximal operator, we shall say that an operator M, defined on $L^1 + L^\infty$, is a maximal operator if

(i) $f \geqslant 0$ implies $Mf \geqslant 0$,

(ii) M is sublinear,

(iii) $0 \leqslant f \leqslant g$ implies $Mf \leqslant Mg$,

(iv) $\|Mf\|_\infty \leqslant \|f\|_\infty$,

(v) M is of weak type (1, 1).

We shall denote by R_k ($k = 0, 1, 2, \ldots$) the class of all functions f, such that the integral

$$\int_{\{|f|>t\}} \frac{|f|}{t} \left(\log \frac{|f|}{t}\right)^k d\mu \tag{1}$$

is finite for every $t > 0$.

For $k \geqslant 1$ R_k is a subclass of $L(\log^+ L)^k$ and both classes coincide if and only if $\mu(\Omega) < \infty$. Obviously $L^1 \subset R_0$ and $L^p \subset R_k$ for any k if $p > 1$. The last statement follows from the fact that

$$(\log^+ u)^k \leqslant \text{const.} \cdot u^{p-1} \quad (u \geqslant 0).$$

The class R_k is a vector space for any $k \geqslant 0$. We consider the case $k > 0$ and remark that a similar proof is valid for R_0.

It is clear that $\lambda f \in R_k$ for any scalar λ and any f in R_k. If f and g are in R_k, then

$$\int \frac{|f+g|}{t} \left(\log^+ \frac{|f+g|}{t}\right)^k d\mu$$
$$\leqslant \left(\int_{\{|f|\geqslant|g|\}} + \int_{\{|f|<|g|\}}\right) \frac{|f|+|g|}{t} \left(\log^+ \frac{|f|+|g|}{t}\right)^k d\mu$$
$$\leqslant \int \frac{2|f|}{t} \left(\log^+ \frac{2|f|}{t}\right)^k d\mu + \int \frac{2|g|}{t} \left(\log^+ \frac{2|g|}{t}\right)^k d\mu < \infty$$

for every $t > 0$.

Among the classes R_k, we have the relations

$$R_0 \supset R_1 \supset R_2 \supset \ldots$$

In order to prove this fact, we write the integral (1) in the form

$$\left(\int_{\{t<|f|\leqslant 2t\}} + \int_{\{|f|>2t\}}\right) \frac{|f|}{t} \left(\log \frac{|f|}{t}\right)^k d\mu.$$

If f is in R_{k+1}, then the first integral is finite since $\mu\{|f| > t\} < \infty$. The second is dominated by

$$\frac{1}{\log 2} \int_{\{|f|>t\}} \frac{|f|}{t} \left(\log \frac{|f|}{t}\right)^{k+1} d\mu.$$

Hence $R_{k+1} \subset R_k$.

Since $R_0 \subset L^1 + L^\infty$, it follows that any class R_k is contained in $L^1 + L^\infty$. Finally we remark that on the real line the function $f(x) = (\log x)^{-1}$ for $x \geqslant 2$, $f(x) = 0$ for $x < 2$ is in any class R_k. However f does not belong to the vector space spanned by $\bigcup_{p>1} L^p$. To see this put $\lambda(t) = \text{meas}\{x\colon f(x) > t\}$. If it were possible to write $f = f_1+f_2+\ldots+f_m$, where $f_i \in L^{p_i}$ and $p_i > 1$, $i = 1, 2, \ldots, m$, then we would have $\lambda(t) \leqslant \sum_{i=1}^{m} \text{meas}\left\{|f_i| > \frac{t}{m}\right\} \leqslant \sum_{i=1}^{m} C_i/t^{p_i}$. On the other hand for $t < (\log 2)^{-1}$, we have $\lambda(t) = e^{1/t}-2$; hence for $x > \log 2$ we should have

$$e^x \leqslant 2+\sum_{i=1}^{m} C_i x^{p_i}$$

which is absurd.

LEMMA 1. *If M is a maximal operator and f is a non-negative function in L^1+L^∞, then for every $t > 0$*

$$\mu\{Mf > 2t\} \leqslant \frac{c}{t} \int_{\{f>t\}} f\,d\mu.$$

If the right hand member is infinite the lemma is trivial; otherwise the function $f^t = f\chi_{\{f>t\}}$ is integrable, and since $f \leqslant f^t+t$, we have $Mf \leqslant Mf^t+t$. Hence

$$\mu\{Mf > 2t\} \leqslant \mu\{Mf^t > t\} \leqslant \frac{c}{t}\int f^t\,d\mu = \frac{c}{t}\int_{\{f>t\}} f\,d\mu.$$

COROLLARY. *If $p > 1$, then*

$$\|Mf\|_p \leqslant \text{const.}\cdot\|f\|_p.$$

The proof can be given by using the interpolation theorem of Marcinkiewicz, or else directly as follows:

Assume that $f \geqslant 0$; then

$$\int [Mf]^p\,d\mu = p\int_0^\infty t^{p-1}\mu\{Mf > t\}\,dt \leqslant p\int_0^\infty dt\cdot t^{p-1}\cdot\frac{2C}{t}\int_{\{2f>t\}} f\,d\mu$$

$$= 2pC\int_\Omega d\mu f\int_0^{2f} t^{p-2}\,dt = \frac{2p\cdot p\cdot C}{p-1}\int_\Omega f^p\,d\mu.$$

2. Principal result. By induction on k, we prove the following.

THEOREM 1. *Let $M_1, \ldots, M_k$ be maximal operators. Then*

(i) *The operation $M_k\ldots M_1 f$ is well defined for any non-negative function f in R_{k-1} and*

$$\mu\{M_k\ldots M_1 f > 4t\} \leqslant C\int_{\{f>t\}} \frac{f}{t}\left(\log\frac{f}{t}\right)^{k-1} d\mu,$$

where C is a constant independent of f and of $t > 0$.

(ii) *If $f \in R_k$, then $M_k\ldots M_1 f$ is integrable over every set of finite measure, and consequently it belongs to L^1+L^∞.*

Proof. Let us put $f^t = f\chi_{\{f>t\}}$, $f_t = f\chi_{\{f\leqslant t\}}$; so that $f = f^t+f_t$, $\|f_t\|_\infty \leqslant t$.

(1) $k = 1$. If $f \in R_0$, then $M_1 f$ is well defined since $R_0 \subset L^1+L^\infty$. The inequality

$$\mu\{M_1 f > 4t\} \leqslant C\int_{\{f>t\}} \frac{f}{t}\,d\mu \tag{2}$$

is obviously implied by the previous lemma.

Suppose now that $f \in R_1$. If E is a set of finite measure in Ω, then

$$\int_E M_1 f\,d\mu = \int_0^\infty \mu(E\cap\{M_1 f > t\})\,dt \leqslant 4\mu(E)+\int_4^\infty \mu\{M_1 f > t\}\,dt.$$

Using (2) we see that the last integral is dominated by

$$C\int_4^\infty dt\int_{\{4f>t\}} \frac{4f}{t}\,d\mu = C\int_\Omega d\mu\int_4^\infty \chi_{\{4f>t\}}\frac{4f}{t}\,dt$$

$$= C\int_\Omega d\mu\int_4^{4f}\frac{4f}{t}\,dt = 4C\int_\Omega f\log^+ f\,d\mu < \infty$$

since $R_1 \subset L(\log^+ L)$, and the theorem is proved in the case $k = 1$.

(2) Assume inductively that the theorem holds as stated for $k \geqslant 1$ maximal operators and consider $k+1$ operators $M_1, M_2, \ldots, M_{k+1}$.

If $f \in R_k$, then the operation $M_{k+1}M_k\ldots M_1 f$ is well defined by virtue of (ii). Moreover, from $f = f^{2t}+f_{2t}$ we get

$$M_{k+1}\ldots M_1 f \leqslant M_{k+1}\ldots M_1 f^{2t}+2t.$$

Hence,

$$\mu\{M_{k+1}M_k\ldots M_1 f > 4t\} \leqslant \mu\{M_{k+1}M_k\ldots M_1 f^{2t} > 2t\} \tag{3}$$

$$\leqslant \frac{c}{t}\int_{\{M_k\ldots M_1 f^{2t}>t\}} M_k\ldots M_1 f^{2t}\,d\mu$$

$$= C\int_{\{M_k\ldots M_1 g>1\}} M_k\ldots M_1 g\,d\mu \quad\text{where } g = \frac{f^{2t}}{t}.$$

In terms of the distribution function

$$\lambda(r) = \mu\{M_k\ldots M_1 g > r\}$$

the last integral in (3) can be written as

$$\int_0^\infty \mu(\{M_k\ldots M_1 g > 1\}\cap\{M_k\ldots M_1 g > r\})\,dr = \lambda(1)+\int_1^\infty \lambda(r)\,dr. \tag{4}$$

Now we estimate each of the last two terms.

By our inductive hypothesis

$$\lambda(r) \leqslant 4C \int_{\{4g>r\}} \frac{g}{r}\left(\log\frac{4g}{r}\right)^{k-1} d\mu. \tag{5}$$

Hence

$$\int_1^\infty \lambda(r)\,dr \leqslant \int_1^\infty dr\,\frac{4C}{r}\int_{\{4g>r\}} g\left(\log\frac{4g}{r}\right)^{k-1} d\mu \tag{6}$$

$$= 4C\int_\Omega d\mu\, g\int_1^{4g}\left(\log\frac{4g}{r}\right)^{k-1}\frac{1}{r}\,dr = \frac{4C}{k}\int_\Omega g(\log^+ 4g)^k\,d\mu.$$

Since $\log^+(ab) \leqslant \log^+ a + \log^+ b$ and $(a+b)^k \leqslant 2^k(a^k+b^k)$ for non-negative numbers a and b, we can dominate the last integral by an expression of the form

$$A\int g\,d\mu + B\int g(\log^+ g)^k\,d\mu.$$

By the definition of g,

$$\int g\,d\mu = \int_{\{f>2t\}} \frac{f}{t}\,d\mu \leqslant \left(\frac{1}{\log 2}\right)^k \int_{\{f>2t\}} \frac{f}{t}\left(\log\frac{f}{t}\right)^k d\mu$$

and also

$$\int g(\log^+ g)^k\,d\mu = \int_{\{f>2t\}} \frac{f}{t}\left(\log\frac{f}{t}\right)^k d\mu.$$

Hence

$$\int_1^\infty \lambda(r)\,dr \leqslant C_1\int_{\{f>t\}} \frac{f}{t}\left(\log\frac{f}{t}\right)^k d\mu. \tag{7}$$

On the other hand, (5) yields

$$\lambda(1) \leqslant 4C\int_{\{4g>1\}} g(\log 4g)^{k-1}\,d\mu = 4C\int_\Omega g(\log^+ 4g)^{k-1}\,d\mu \tag{8}$$

$$\leqslant A\int g\,d\mu + B\int g(\log^+ g)^{k-1}\,d\mu \leqslant C_2\int_{\{f>t\}} \frac{f}{t}\left(\log\frac{f}{t}\right)^k d\mu.$$

From (7) and (8) we conclude that (i) is true for $k+1$ operators. To show that (ii) also holds we assume that $f \in R_{k+1}$ and compute as follows

$$\int_E M_{k+1}\ldots M_1 f\,d\mu = \int_0^\infty \mu(E\cap\{M_{k+1}\ldots M_1 f > t\})\,dt \tag{9}$$

$$\leqslant 4\mu(E) + \int_4^\infty \mu\{M_{k+1}\ldots M_1 f > t\}\,dt.$$

By what we proved before

$$\mu\{M_{k+1}\ldots M_1 f > t\} \leqslant C\int_{\{4f>t\}} \frac{4f}{t}\left(\log\frac{4f}{t}\right)^k d\mu.$$

Hence, the last integral in (9) is dominated by

$$C\int_4^\infty dt\int_{\{4f>t\}} \frac{4f}{t}\left(\log\frac{4f}{t}\right)^k d\mu = C\int_\Omega d\mu\int_4^{4f} \frac{4f}{t}\left(\log\frac{4f}{t}\right)^k dt$$

$$= \frac{4C}{k+1}\int_\Omega f(\log^+ f)^{k+1}\,d\mu$$

which is finite since $R_{k+1} \subset L(\log^+ L)^{k+1}$.

This completes the proof of the theorem.

Finally we remark that for $k \geqslant 2$ the inequality of the theorem may be written as

$$\mu\{M_k\ldots M_1 f > 4t\} \leqslant C\int_\Omega \frac{f}{t}\left(\log^+\frac{f}{t}\right)^{k-1} d\mu$$

for any f in R_{k-1}.

3. Application to almost everywhere convergence of operator averages. Suppose that we have k linear operators T_i $(i = 1, 2, \ldots, k)$, defined on the class $L^1 + L^\infty$, such that $\|T_i\|_1 \leqslant 1$, $\|T_i\|_\infty \leqslant 1$.

We also assume that each T_i is a positive operator, that is, $f \geqslant 0$ implies $T_i f \geqslant 0$.

In order to study the behavior of the multiple averages

$$A(n_1, \ldots, n_k)f = \frac{1}{n_1\ldots n_k}\sum_{i_1=0}^{n_1-1}\ldots\sum_{i_k=0}^{n_k-1}\ldots T_1^{i_1}\ldots T_k^{i_k} f$$

we introduce the "maximal function"

$$f^*(x) = \sup_{n_1,\ldots,n_k>0} |A(n_1, \ldots, n_k)f(x)|.$$

From Theorem 1, we obtain the following weak-type estimate.

THEOREM 2. *There exists a constant C, such that for every f in R_{k-1} and every $t > 0$*

$$\mu\{x\colon f^*(x) > 4t\} \leqslant C\int_{\{|f|>t\}} \frac{|f|}{t}\left(\log\frac{|f|}{t}\right)^{k-1} d\mu.$$

Proof. If we put

$$M_i f = \sup_{n>0}\left|\frac{1}{n}\sum_{j=0}^{n-1} T_i^j f\right|,$$

then each M_i is a maximal operator and

$$A(n_1, \ldots, n_k)f = \frac{1}{n_1 \ldots n_{k-1}} \sum_{i_1=0}^{n_1-1} \ldots \sum_{i_{k-1}=0}^{n_{k-1}-1} T_1^{i_1} \ldots T_{k-1}^{i_{k-1}} \left(\frac{1}{n_k} \sum_{i_k=0}^{n_k-1} T_k^{i_k} f \right)$$

$$\leqslant \frac{1}{n_1 \ldots n_{k-1}} \sum_{i_1=0}^{n_1-1} \ldots \sum_{i_{k-1}=0}^{n_{k-1}-1} T_1^{i_1} \ldots T_{k-1}^{i_{k-1}} M_k f$$

$$\leqslant \ldots \leqslant M_1 \ldots M_k f.$$

Changing f by $-f$, we get

$$A(n_1, \ldots, n_k)f \geqslant -M_1 \ldots M_k f$$

and consequently

$$|A(n_1, \ldots, n_k)f| \leqslant M_1 \ldots M_k f.$$

Hence

$$f^* \leqslant M_1 \ldots M_k f$$

and Theorem 2 follows at once from Theorem 1. Q. E. D.

From the estimate just obtained, we derive a result on pointwise convergence.

THEOREM 3. *If f is a function in R_{k-1}, then the averages $A(n_1, \ldots, n_k)f$ converge almost everywhere in Ω as $n_1 \to \infty, \ldots, n_k \to \infty$ independently.*

Proof. For any function g in $L^1 + L^\infty$, let us define

$$\omega(g) = \limsup_{n_1, \ldots, n_k \to \infty} A(n_1, \ldots, n_k)g - \liminf_{n_1, \ldots, n_k \to \infty} A(n_1, \ldots, n_k)g.$$

It is clear that ω is subadditive and that $\omega(g) \leqslant 2g^*$. Given f in R_{k-1}, we select a sequence $f_n (n = 1, 2, \ldots)$ of simple functions having support of finite measure, such that $f_n \to f$ pointwise and $|f - f_n| \leqslant |f|$ for each n.

Since $\omega(f) \leqslant \omega(f - f_n) + \omega(f_n)$, and $\omega(f_n) = 0$ by virtue of the theorem of Dunford and Schwartz, we have

$$\omega(f) \leqslant \omega(f - f_n) \leqslant 2(f - f_n)^*.$$

Hence, for every $t > 0$

$$\mu\{\omega(f) > 8t\} \leqslant \mu\{(f-f_n)^* > 4t\} \leqslant C \int_{\{|f-f_n|>t\}} \frac{|f-f_n|}{t} \left(\log \frac{|f-f_n|}{t} \right)^{k-1} d\mu$$

$$\leqslant C \int_{\{|f|>t\}} \frac{|f-f_n|}{t} \left(\log^+ \frac{|f-f_n|}{t} \right)^{k-1} d\mu.$$

But the last integral tends to zero as $n \to \infty$, by virtue of the Lebesgue dominated convergence theorem.

COROLLARY. *If $\mu(\Omega) < \infty$, then for every function f in $L(\log^+ L)^{k-1}$ the averages $A(n_1, \ldots, n_k)f$ converge almost everywhere in Ω as $n_1 \to \infty, \ldots, n_k \to \infty$ independently.*

An easy example shows that the condition $\mu(\Omega) < \infty$ cannot be removed from this corollary. Let us consider a sequence $a_0, a_1, a_2, \ldots$ of real numbers, such that for each n, $|a_n| = 1$, and the sequence of arithmetic means

$$\frac{a_0 + a_1 + \ldots + a_{n-1}}{n}$$

is divergent.

On the real line, we consider the operators

$$Tf(x) = f(x+1), \quad Sf(x) = f(x)$$

and define a function g by putting $g(x) = a_n$ if $n \leqslant x < n+1$ $(n = 0, 1, 2, \ldots)$, and $g(x) = 0$ otherwise. Clearly $g \in L\log^+ L$. On the other hand, if $0 \leqslant x < 1$,

$$A(n, m)g(x) = \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} T^i S^j g(x)$$

$$= \frac{1}{n} \sum_{i=0}^{n-1} T^i g(x) = \frac{1}{n}(a_0 + \ldots + a_{n-1})$$

diverges as $n \to \infty$. Actually the averages diverge everywhere.

4. The case of continuous semigroups. Let $\{T(t), t \geqslant 0\}$ be a semigroup of bounded linear operators in some space L^p, where $1 \leqslant p \leqslant \infty$. The underlying measure space being denoted by $(\Omega, \mathfrak{F}, \mu)$.

For all the definition and results needed here we refer to [4] in the bibliography; specifically pp. 684–687.

We say that the semigroup $T(t)$ is strongly continuous if

$$\lim_{s \to t} \|T(s)f - T(t)f\|_p = 0$$

for every f in L^p and every $t \geqslant 0$. If this condition is satisfied, the integral $(L^p) \int_0^a T(t) f dt$ can be represented as the L^p-limit of the Riemann sums

$$S_n(f) = \frac{a}{n} \sum_{k=1}^{n} T\left(\frac{ka}{n}\right) f.$$

We say that $T(t)$ is strongly integrable over every finite interval if, for each f in L^p the function $T(\cdot)f$ is integrable with respect to Lebesgue measure on every finite interval $0 \leqslant t \leqslant a$. If this condition is satisfied, we have the following.

Lemma 1. *There exists a function $g(t, x)$, measurable on the product space $[0, \infty) \times \Omega$, which is uniquely determined up to a set of measure zero in this space by the conditions*

(1) *For almost all t, $g(t, \cdot) = T(t)f$.*

(2) *For almost all x, the function $g(\cdot, x)$ is integrable over every finite interval and the integral $\int_0^a g(t, x)\,dt$ as a function of x equals $(L^p)\int_0^a T(t) f\,dt$.*

For the proof we refer to [4], page 198, Theorem 17.

The function $g(t, x)$ is called a scalar representation of $T(t)f$ and will often be denoted by $T(t)f(x)$.

From now on, we will consider a semigroup $T(t)$, $t \geqslant 0$, of positive linear operators defined in the class $L^1 + L^+$. We will also assume that

(i) $\|T(t)\|_1 \leqslant 1$ and $T(t)$ is strongly continuous when restricted to L^1.

(ii) $\|T(t)\|_\infty \leqslant 1$ and $T(t)$ is strongly integrable over every finite interval when it is restricted to L^∞.

If $f = g + h$, with $g \in L^1$ and $h \in L^\infty$, we define

$$\int_0^a T(t) f\,dt = (L^1)\int_0^a T(t) g\,dt + (L^\infty)\int_0^a T(t) h\,dt.$$

To see that this definition is consistent, it is enough to show that for any f in $L^1 \cap L^\infty$

$$(L^1)\int_0^a T(t) f\,dt = (L^\infty)\int_0^a T(t) f\,dt$$

where as before, the signs preceding the integrals indicate the norm with respect to which each integral is defined. But this fact follows immediately from Lemma 1, since the L^1 and the L^∞-scalar representations of $T(t)f$ must coincide almost everywhere on $[0, \infty) \times \Omega$.

Choosing scalar representations $T(t)g(x)$, $T(t)h(x)$ of $T(t)g$ and $T(t)h$, we get a scalar representation

$$T(t)f(x) = T(t)g(x) + T(t)h(x)$$

and the ordinary Lebesgue integral

$$\int_0^a T(t)f(x)\,dt = \int_0^a T(t)g(x)\,dt + \int_0^a T(t)h(x)\,dt$$

as a function of x equals the element $\int_0^a T(t) f\,dt$ whose definition has just been given.

For an arbitrary f in $L^1 + L^\infty$, we consider the average

$$\sigma_a f(x) = \frac{1}{a}\int_0^a T(t)f(x)\,dt$$

and define the maximal ergodic operator M by

$$Mf(x) = \sup_{a>0} \frac{1}{a}\left|\int_0^a T(t)f(x)\,dt\right|.$$

Lemma 2. *The operator M defined by the previous equation is a maximal operator.*

Proof. First we show that M does not increase the L^∞-norm of any function. Suppose that $\|f\|_\infty < \infty$, and define the set

$$E = \{(t, x) : |T(t)f(x)| > \|f\|_\infty\}.$$

For almost all t

$$|T(t)f(x)| \leqslant \|T(t)f\|_\infty \leqslant \|f\|_\infty$$

almost everywhere in Ω.

Hence

$$\int_\Omega m\{t : (t, x) \in E\}\mu(dx) = m \otimes \mu(E) = 0,$$

where m denotes the Lebesgue measure. Therefore, for almost all x

$$|T(t)f(x)| \leqslant \|f\|_\infty$$

almost everywhere in t.

Consequently

$$\frac{1}{a}\left|\int_0^a T(t)f(x)\,dt\right| \leqslant \|f\|_\infty$$

for all a and all x outside a certain set of μ-measure zero which depends only on f. This proves that $\|Mf\|_\infty \leqslant \|f\|_\infty$.

Now we prove that M is of weak type $(1, 1)$.

Let R be the set of all positive rational numbers, and consider a function f in L^1.

For every $a \in R$, we have

$$\frac{1}{a}\int_0^a T(t) f\,dt = L^1\text{-}\lim_{n\to\infty} \frac{1}{an!}\sum_{m=0}^{an!-1} T\left(\frac{m}{n!}\right) f.$$

Using the Cantor's diagonal process we find a sequence $n_1 < n_2 < n_3 < \dots$ such that

$$\lim_{i\to\infty} \frac{1}{an_i!}\sum_{m=0}^{an_i!-1} T\left(\frac{m}{n_i!}\right) f(x) = \sigma_a f(x)$$

for all $a \in R$ and all x outside a set $E(f)$ having measure zero. If we put

$$M_i f = \sup_{0<k<\infty} \frac{1}{k} \left| \sum_{m=0}^{k-1} T\left(\frac{m}{n_i!}\right) f \right|,$$

then for $i \geqslant i_0(a)$ and all x outside $E(f)$, we have

$$M_i f(x) \geqslant \frac{1}{an_i!} \left| \sum_{m=0}^{an_i!-1} T\left(\frac{m}{n_i!}\right) f(x) \right|$$

and consequently

$$\liminf_{i\to\infty} M_i f(x) \geqslant |\sigma_a f(x)|$$

for all $a \in R$ and all $x \notin E(f)$. Hence

$$Mf(x) \leqslant \liminf_{i\to\infty} M_i f(x) \qquad \big(x \notin E(f)\big).$$

On the other hand, since $T\left(\frac{m}{n}\right) = T\left(\frac{1}{n}\right)^m$,

$$\mu\{M_i f > t\} \leqslant \frac{1}{t} \|f\|_1$$

and finally, since $\{Mf > t\} \subset \liminf\{M_i f > t\} \cup E(f)$,

$$\mu\{Mf > t\} \leqslant \liminf_{i\to\infty} \mu\{M_i f > t\} \leqslant \frac{1}{t} \|f\|_1.$$

Since the remaining conditions are trivial to check, the lemma is proved.

Let $T_i(t)$, $t \geqslant 0$, $i = 1, 2, \ldots, k$ be semigroups of positive linear operators in $L^1 + L^\infty$, satisfying the conditions (i) and (ii) that follow Lemma 1. For any f in $L^1 + L^\infty$, we form the averages

$$A(a_1, \ldots, a_k) f = \frac{1}{a_1 \ldots a_k} \int_0^{a_1} \ldots \int_0^{a_k} T_1(t_1) \ldots T_k(t_k) f \, dt_1 \ldots dt_k.$$

THEOREM 4. *If we put*

$$f^*(x) = \sup_{a_1, \ldots, a_k > 0} |A(a_1, \ldots, a_k) f(x)|,$$

then there is a constant C, such that for every f in R_{k-1} and every $t > 0$

$$\mu\{x \colon f^*(x) > 4t\} \leqslant C \int_\Omega \frac{|f|}{t} \left(\log^+ \frac{|f|}{t}\right)^{k-1} d\mu.$$

Proof. Let us consider the "partial" maximal operator M_i, defined by

$$M_i f(x) = \sup_{a>0} \left| \frac{1}{a} \int_0^a T_i(t) f(x)\, dt \right|.$$

We have

$$\begin{aligned} &A(a_1, \ldots, a_k) f \\ &= \frac{1}{a_1 \ldots a_{k-1}} \int_0^{a_1} \ldots \int_0^{a_{k-1}} T_1(t_1) \ldots T_{k-1}(t_{k-1}) \left[\frac{1}{a_k} \int_0^{a_k} T_k(t_k) f \, dt_k \right] dt_1 \ldots dt_{k-1} \\ &\leqslant \frac{1}{a_1 \ldots a_{k-1}} \int_0^{a_1} \ldots \int_0^{a_{k-1}} T_1(t_1) \ldots T_{k-1}(t_{k-1}) M_k f \, dt_1 \ldots dt_{k-1} \\ &\leqslant \ldots \leqslant M_1 \ldots M_k f \end{aligned}$$

almost everywhere in Ω.

Changing f by $-f$, we get

$$A(a_1, \ldots, a_k) f \geqslant -M_1 \ldots M_k f.$$

Hence

$$|A(a_1, \ldots, a_k) f| \leqslant M_1 \ldots M_k f$$

almost everywhere, and consequently

$$f^* \leqslant M_1 \ldots M_k f$$

except possibly in a set of measure zero. Theorem 4 follows now easily from this relation and Theorem 1. From Theorem 4, we derive an individual ergodic theorem.

THEOREM 5. *If f is in R_{k-1}, then the averages $A(a_1, \ldots, a_k) f$ converge almost everywhere in Ω as $a_1 \to \infty, \ldots, a_k \to \infty$ independently.*

We omit the proof of this theorem, since the argument is the same as that in Theorem 3.

COROLLARY. *If $\mu(\Omega) < \infty$, then for every function f in $L(\log^+ L)^{k-1}$, the averages $A(a_1, \ldots, a_k) f$ converge almost everywhere in Ω as $a_1 \to \infty, \ldots$ $\ldots, a_k \to \infty$ independently.*

5. Application to strong differentiability of multiple Lebesgue integrals. Let $f(x) = f(x_1, \ldots, x_k)$ be an integrable function on the unit cell (Ω) $0 \leqslant x_i \leqslant 1$ $(i = 1, 2, \ldots, k)$.

Following the usage of [8], we shall say that the integral of the function f is strongly differentiable at the point x, if

$$\lim_{\delta(I)\to 0} \frac{1}{|I|} \int_I f(y)\, dy \tag{1}$$

exists and is finite, where I denotes any cell with sides parallel to the axis contained in Ω and containing x; $|I|$ denotes the measure, and $\delta(I)$ the diameter of I. The limit (1) will be called the *strong derivative* of the integral of f at the point x. It was proved by Saks that there is a function $f \in L^1(\Omega)$ such that its integral is nowhere strongly differentiable.

LEMMA. *If for any function f in $L^1(\Omega)$, we define*

$$f^*(x) = \sup_{I\ni x} \frac{1}{|I|} \int_I |f(y)|\,dy,$$

then for every $t > 0$

$$m\{x\colon f^*(x) > 4t\} \leqslant C \int_\Omega \frac{|f|}{t}\left(\log^+ \frac{|f|}{t}\right)^{k-1} dx,$$

where C is a constant independent of f and of t.

Proof. Let us define the operators M_i by

$$M_1 f(x) = \sup_{a<x_1<\beta} \frac{1}{\beta-a} \int_a^\beta |f(u, x_2, \ldots, x_k)|\,du,$$

$$M_2 f(x) = \sup_{a<x_2<\beta} \frac{1}{\beta-a} \int_a^\beta |f(x_1, u, \ldots, x_k)|\,du,$$

$$\ldots$$

$$M_k f(x) = \sup_{a<x_k<\beta} \frac{1}{\beta-a} \int_a^\beta |f(x_1, x_2, \ldots, u)|\,du.$$

It is clear that each M_i is a positive sublinear operator, such that $\|M_i f\|_\infty \leqslant \|f\|_\infty$. We want to show that M_i is of weak type (1, 1). Here is the proof for $i = 1$.

$$m\{x\colon M_1 f(x) > t\} = \int_0^1 \ldots \int_0^1 |\{x_1\colon M_1 f(x_1, x_2, \ldots, x_k) > t\}|\,dx_2 \ldots dx_k$$
$$\leqslant \int_0^1 \ldots \int_0^1 \left(\frac{2}{t} \int_0^1 |f(x_1, x_2, \ldots, x_k)|\,dx_1\right) dx_2 \ldots dx_k$$
$$= \frac{2}{t} \|f\|_1.$$

Hence, each M_i is a maximal operator. Finally, from Fubini's theorem,

$$f^* \leqslant M_1 \ldots M_k f$$

and the lemma follows by Theorem 1.

COROLLARY (Jessen, Marcinkiewicz and Zygmund). *If $f(\log^+|f|)^{k-1}$ is integrable over the unit cube Ω, then, at almost every point x, the integral of f is strongly differentiable and the derivative is equal to $f(x)$.*

Proof. For any function g in L^1, define

$$\omega g(x) = \limsup_{\delta(I)\to 0} \frac{1}{|I|} \int_I g(y)\,dy - \liminf_{\delta(I)\to 0} \frac{1}{|I|} \int_I g(y)\,dy.$$

Clearly, ω is subadditive, and $\omega g \leqslant 2g^*$. To prove the theorem, assume first that f is a simple function; then, there exists a sequence $\varphi_n (n = 1, 2, \ldots)$ of continuous functions such that $\varphi_n(x) \to f(x)$ almost everywhere in Ω, and such that $|\varphi_n - f| \leqslant$ const. $(n = 1, 2, \ldots)$.

Since $\omega f \leqslant \omega(f-\varphi_n) + \omega\varphi_n$, and $\omega\varphi_n = 0$, we have

$$|\{x\colon \omega f(x) > 8t\}| \leqslant |\{x\colon \omega(f-\varphi_n) > 8t\}|$$
$$\leqslant |\{x\colon (f-\varphi_n)^* > 4t\}| \leqslant C \int_\Omega \frac{|f-\varphi_n|}{t}\left(\log^+ \frac{|f-\varphi_n|}{t}\right)^{k-1} dx.$$

Since the last integral tends to zero as $n \to \infty$, for any $t > 0$, the statement of the theorem is true for any simple function f.

Suppose now merely that f is in $L(\log^+ L)^{k-1} = R_{k-1}$, and select a sequence f_n of simple functions, such that $f_n(x) \to f(x)$ almost everywhere in Ω and $|f - f_n| \leqslant |f|$. With the same reasoning as before,

$$|\{x\colon \omega f(x) > 8t\}| \leqslant C \int_\Omega \frac{|f-f_n|}{t}\left(\log^+ \frac{|f-f_n|}{t}\right)^{k-1} dx$$

and the last integral tends to zero as $n \to \infty$, for any $t > 0$. Hence the limit

$$g(x) = \lim_{\delta(I)\to 0} \frac{1}{|I|} \int_I f(y)\,dy \qquad (I \ni x)$$

exists and is finite almost everywhere in Ω. To finish the proof, it remains to show that $g(x) = f(x)$ a.e.

Let $I_n(x)$ be the cube of edge $\frac{1}{n}$ with center at x. Then the sequence

$$\frac{1}{|I_n(x)|} \int_{I_n(x)} f(y)\,dy$$

converges almost everywhere to $g(x)$ as $n \to \infty$. Since it also converges in L^1 to f, we must have $g(x) = f(x)$ a.e., and the theorem is proved.

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Invariant norms for $C(T)$

by

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Abstract. The space of continuous functions with supremum norm has a huge group of isometries. Given a subgroup of this group one can ask whether there is another algebra norm for the continuous functions having isometry group containing the given subgroup. This paper presents various constructions of algebra norms designed to accommodate several natural groups of isometries and gives conditions under which certain groups of isometries characterize the sup norm among all algebra norms.

In many calculations on function algebras an important property of the sup norm, in addition to completeness and the indispensible inequalities defining "norm", is that a particular collection of mappings of the algebra are isometric, or perhaps norm-decreasing. It is often evident that the sup norm could be replaced by any other "invariant" norm. This gives rise to a natural question: are there any other norms besides $\| \ \|_\infty$ which have a given invariance behavior, and how much invariance must be imposed in order to characterize $\| \ \|_\infty$ among all norms? The purpose of this note is to exhibit several distinct norms which are invariant under large collections of mappings and to give conditions sufficient to ensure that a norm must be identical with the sup norm. For simplicity let us consider $C(T)$, where T is the circle. Generalizations to $C(G)$, G a compact abelian group, and in some cases to $C(X)$, X a compact Hausdorff space, will be apparent.

If $\| \ \|$ is an algebra norm for $C(T)$, then $\|f\| \geqslant \|f\|_\infty$, by a theorem of Kaplansky ([1], Theorem 6.2). An algebra norm is complete if and only if $\| \ \|_\infty \leqslant \| \ \| \leqslant K \| \ \|_\infty$, for some $K < \infty$. A theorem of Bade and Curtis ([2], Theorem 4.1) asserts that $\|f\| \leqslant K\|f\|_\infty$ for all f vanishing on a neighborhood of a certain finite set, which may be empty. If $\| \ \|$ is translation-invariant $\big(\|f(t+s)\| = \|f(t)\|\big)$, then it immediately follows

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