sup norm. Theorem 4 shows that one can greatly enlarge the class $f \to gf$ subject to restrictions imposed by Theorem 3. On the other hand, the second class may be greatly enlarged (Theorem 5); however, this will enormously restrict the first class (Theorem 6).

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References

[1] I. Kaplansky, *Normed algebras*, Duke Math. J. 16 (1949), pp. 399–418.

[2] W. G. Bade, and P. C. Curtis, Jr., *Homomorphisms of commutative Banach algebras*, Amer. J. Math. 82 (1960), pp. 589–608.

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An example of infinite dimensional reflexive Banach space non-isomorphic to its Cartesian square(*)

by

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Abstract. The paper gives the first example of an infinite dimensional reflexive Banach space X non-isomorphic to X^2. The sense of this non-isomorphism depends on a difference between the structure of finite dimensional subspaces of X and those of X^2. The proof involves certain properties of subspaces of l_p^n, some of them seeming to be new.

Introduction. The problem whether every infinite dimensional Banach space X is isomorphic to its square X^2 was raised in Banach's monograph and remained unsolved until 1959. The first counterexamples were the space J of R. C. James and some related spaces (cf. [2]) and the space $C(\Gamma_{\omega_1})$ (cf. [12]).

The proofs of these non-isomorphisms were based on certain additive, isomorphic invariants $\delta(X)$ characterizing the natural embedding $\varkappa\colon X \to X^{**}$ (additive in the sense that $\delta(X_1 \oplus X_2) = \delta(X_1) + \delta(X_2)$ for any Banach spaces X_1, X_2).

Putting e.g. $\delta(X) = \dim\big(X^{**}/\varkappa(X)\big)$ we obtain such an invariant and since $\delta(J) = 1 \neq 2\delta(J) = \delta(J^2)$ we get the non-isomorphism $J \not\approx J^2$.

The question stated nowadays by several authors (cf. [1], [2]), whether X can, in addition, be reflexive required other methods and remained unsolved until today.

Below we show that spaces with such properties can be constructed from familiar spaces l_p^n, i.e. spaces of all n-tuples $a = (a_1, \ldots, a_n)$ of numbers with $\|a\| = \big(\sum_{i=1}^{n} |a_i|^p\big)^{1/p}$ if $1 \leqslant p < \infty$ and $\|a\| = \max_{1 \leqslant i \leqslant n} |a_i|$ if $p = \infty$. (Letters i, j, k, m, n will always denote in the sequel positive integers.)

(*) The paper is a part of the author's Ph. D. thesis written at the Warsaw University under the supervision of Professor A. Pełczyński.

Let $((X_k, \|\cdot\|_k))_{k=1}^{\infty}$ be a sequence of Banach spaces and let $1 \leqslant p < \infty$. The space of all sequences $x = (x_k)_{k=1}^{\infty}$ such that $x_k \in X_k$ for $k = 1, 2, \ldots$ and $\|x\| = (\sum_{k=1}^{\infty} \|x_k\|_k^p)^{1/p} < \infty$, will be denoted by $(\sum_{k=1}^{\infty} \oplus X_k)_p$.

We shall prove that for every $p > 1$ and each decreasing sequence $(p_i)_{i=1}^{\infty}$ of reals with $\lim p_i \geqslant \max(2, p)$ there exists a sequence $(n_i)_{i=1}^{\infty}$ such that the space $X = (\sum_{i=1}^{\infty} \oplus l_{p_i}^{n_i})_p$ is not isomorphic to X^2 (1).

All the spaces we shall consider are supposed to be over the same (real or complex) field.

The spaces we have described above as well as their duals have certain positive properties which the previous (non-reflexive) counter-examples could not possess. They have unconditional bases and for every n are isomorphic to the nth power of a space of similar type. They are also uniformly convex, moreover it follows from Propositions 1 and (K) that if $p > 2$ then the dual X^* of such an X has the Orlicz property (i.e. for every unconditionally convergent series $\sum_{k=1}^{\infty} x_k^*$ in X^*, $\sum_{k=1}^{\infty} \|x_k^*\|^2 < \infty$).

These spaces are also interesting in connection with a problem of contractibility of linear groups of Banach spaces. Similarly as J and $C(\Gamma_{\omega_1})$ they do not satisfy the conditions of, worked up by Mitjagin [9], general scheme of all known contractibility proofs. However, the non-contractibility of $GL(X)$ cannot be established as simply as that of $GL(J)$ or $GL(C(\Gamma_{\omega_1}))$ (cf. [10]). It seems that the homotopy type of $GL(X)$ is different than the previously known ones.

Using some recent results of Mankiewicz [8] one can prove that there exists no Lipschitzian embedding of X^2 into X. The question of existence of uniformly continuous embeddings and, in particular, of uniform homeomorphisms between X^2 and X remains open.

The main result. In the present and the following sections let $(p_i)_{i=1}^{\infty}$ be a fixed, strictly decreasing sequence of real numbers greater than 2, and p be a fixed number from the interval $(1, \lim p_i]$.

The main theorem reads as follows.

THEOREM. *There exists a sequence $(n_i)_{i=1}^{\infty}$ such that the space $X = (\sum_{i=1}^{\infty} \oplus l_{p_i}^{n_i})_p$ has the following property:*

For any n there exists no isomorphic embedding of X^{n+1} into X^n. (In particular any two different positive powers of X are not isomorphic.)

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(1) These results have been presented at the Colloque International d'Analyse Fonctionelle held in Bordeaux in April 1971. Their statements will be published in the Proceedings of the Conference.

We deduce this theorem from two propositions to be proved in the sequel.

Let $d(Y, Z)$, Y, Z being isomorphic Banach spaces, denote the greatest lower bound of numbers $\|T\| \, \|T^{-1}\|$, where T is an invertible linear operator from Y onto Z.

Now we can formulate the results we need.

PROPOSITION A. *There exists a sequence $(m_i)_{i=1}^{\infty}$ such that for every i and every m_i-dimensional subspace Z of the space $(\sum_{j=1}^{\infty} \oplus l_{p_{i+j}})_p$ the distance $d(Z, l_{p_i}^{m_i})$ is greater than i.*

PROPOSITION B. *For every sequence $(m_i)_{i=1}^{\infty}$ there exists a sequence $(n_i)_{i=1}^{\infty}$ such that*

$$n_i > i \sum_{j<i} n_j \quad \text{for } i = 1, 2, \ldots \tag{1}$$

and each subspace $Y \subset l_{p_i}^{in_i}$ with $\dim Y > \frac{1}{2} n_i$ contains a subspace Z such that $\dim Z = m_i$ and $d(Z, l_{p_i}^{m_i}) < 2$.

We shall prove that for the sequence defining X a sequence corresponding to $(m_i)_{i=1}^{\infty}$ in Proposition A can be taken.

Indeed, suppose a contrario that for some n there exists an isomorphic embedding of X^{n+1} into X^n. Then, denoting by $X^{(k)}$ the space $(\sum_{i=1}^{\infty} \oplus l_{p_i}^{kn_i})_p$ and using obvious isomorphism $X^k \simeq X^{(k)}$, where $k = n$, $n+1$, we get a linear operator $T: X^{(n+1)} \to X^{(n)}$ such that for some positive constants a, b and every $x \in X^{(n+1)}$

$$a\|x\| \leqslant \|Tx\| \leqslant b\|x\|. \tag{2}$$

Let j be an index greater than $2\max(n, a^{-1}b)$. Let I_j be the natural embedding of $l_{p_j}^{(n+1)n_j}$ into $X^{(n+1)}$ and T_i, $i = 1, \ldots, j$, be a superposition of T and the projection of $X^{(n)}$ onto $l_{p_i}^{nn_i}$. Consider the subspace Y of $l_{p_j}^{(n+1)n_j}$

$$Y = \{x \in l_{p_j}^{(n+1)n_j} : T_i I_j x = 0 \quad \text{for } 1 \leqslant i \leqslant j\}.$$

Using (1) we get

$$\dim Y \geqslant (n+1) n_j - n \sum_{i \leqslant j} n_i > n_j - \tfrac{1}{2} j \sum_{i<j} n_i > \tfrac{1}{2} n_j.$$

Since $l_{p_j}^{(n+1)n_j}$ is isometrically isomorphic to a subspace of the space $l_{p_j}^{jn_j}$ we infer using Proposition B that there exists a subspace $Z \subset Y$ such that $\dim Z = m_j$ and

$$d(Z, l_{p_j}^{m_j}) < 2.$$

The operator $S = TI_j|_Z$ defines an isomorphic embedding of Z into the subspace of $X^{(n)}$ consisting of those sequences which have zeros on first j places. Treating the last as a subspace of the space $(\sum_{j=1}^{\infty} \oplus l_{p_{j+i}})_p$ and using Proposition I we obtain an inequality

$$d(S(Z), l_{p_j}^{m_j}) > j. \tag{3}$$

On the other hand it follows from (2) that

$$d(Z, S(Z)) \leqslant a^{-1}b < \tfrac{1}{2}j,$$

hence

$$d(S(Z), l_{p_j}^{m_j}) \leqslant d(S(Z), Z)\, d(Z, l_{p_j}^{m_j}) < j,$$

which contradicts (3). This completes the proof of the theorem.

Proof of Proposition A. We shall need the estimate of the modulus of convexity of the l_p-sum $(\sum_{i=k}^{\infty} \oplus l_{p_i})_p$ closer than that in [4].

We recall that the *modulus of convexity* $\delta_Y(\varepsilon)$ of a Banach space Y is defined by formula

$$\delta_Y(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in Y, \|x\| = \|y\| = 1, \|x-y\| \geqslant \varepsilon\right\}.$$

The estimate we need is a consequence of the following proposition.

PROPOSITION 1. *Let p, q, L be positive numbers such that $1 < q < p \geqslant 2$. Then there exists $aK = K(p, q, L) > 0$ such that if $X = (\sum_{i=1}^{\infty} \oplus X_i)_q$, $(X_i)_{i=1}^{\infty}$ being a sequence of Banach spaces with $\delta_{X_i}(\varepsilon) \geqslant L\varepsilon^p$ for $0 \leqslant \varepsilon \leqslant 2$ and $i = 1, 2, \ldots$, then the modulus of convexity of X admits the estimate*

$$\delta_X(\varepsilon) \geqslant K\varepsilon^p \quad \textit{for } 0 \leqslant \varepsilon \leqslant 2.$$

We shall use the lemma.

LEMMA 1. *If p, q, L are as above then there exists a positive constant $K_1 = K_1(p, q, L)$ such that for every Banach space Y with $\delta_Y(\varepsilon) \geqslant L\varepsilon^p$ and every $x, y \in Y$ such that*

$$\|x\|^q + \|y\|^q = 2,$$

the following inequality holds

$$\tfrac{1}{2}\|x+y\| \leqslant 1 - K_1(\tfrac{1}{2}\|x-y\|)^p.$$

Using this lemma we can prove that in Proposition 1 one can take

$$K(p, q, L) = 2^{-p}K_2 = 2^{-p}\min\left(\frac{p-q}{q(p-1)}, K_1(p, q, L)\right).$$

Indeed, let $x, y \in X$, $\|x\| = \|y\| = 1$, i.e. $x = (x_i)_{i=1}^{\infty}$, $y = (y_i)_{i=1}^{\infty}$, $x_i, y_i \in X_i$ for $i = 1, 2, \ldots$, and

$$\sum_{i=1}^{\infty} \|x_i\|^q = \sum_{i=1}^{\infty} \|y_i\|^q = 1.$$

Let

$$r_i = \tfrac{1}{2}(\|x_i\|^q + \|y_i\|^q) \quad (i = 1, 2, \ldots).$$

If $r_i \neq 0$, then, using Lemma 1 for vectors $r_i^{-1/q}x_i, r_i^{-1/q}y_i$, we obtain the following inequality

$$2^{-q}\|x_i+y_i\|^q \leqslant r_i\left(1 - K_1\left(\frac{\|x_i-y_i\|}{2^q r_i}\right)^{p/q}\right)^q,$$

which is a fortiori valid with K_1 replaced by K_2 defined above.

One can easily verify that if $0 < M \leqslant \dfrac{p-q}{q(p-1)}$, then the function $g(t) = (1 - Mt^{p/q})^q$ is concave on the segment $[0, 1]$. Hence denoting by $\sum'$ the sum over all indices i such that $r_i \neq 0$ and using the obvious relation

$$\sum_{i=1}^{\infty} r_i = \sum{}' r_i = 1,$$

we get

$$2^{-q}\sum{}' \|x_i+y_i\|^q \leqslant \sum{}' r_i\left(1 - K_2\left(\frac{\|x_i-y_i\|^q}{2^q r_i}\right)^{p/q}\right)^q$$
$$\leqslant \left(1 - K_2\sum{}'\left(r_i\frac{\|x_i-y_i\|^q}{2^q r_i}\right)^{p/q}\right)^q,$$

which is equivalent to

$$\tfrac{1}{2}\|x+y\| \leqslant 1 - K_2\, 2^{-p}\|x-y\|^p.$$

This concludes the proof of the proposition.

We pass to the proof of the lemma. Let $a = \|x\|$, $b = \|y\|$, $c = \frac{1}{2}\|x-y\|$. We may certainly assume that $a \geqslant b$. Consider two cases.

Case 1. $a - b \geqslant c^{p/2}$. Then, since $a + b \leqslant 2$, we have $b \leqslant 1 - \frac{1}{2}c^{p/2}$ and therefore there exists an $s \leqslant \frac{1}{2}p$ such that $b = 1 - \frac{1}{2}c^s$. Consider the function

$$f(t) = \left(\sqrt[q]{2 - (1-t)^q} - 1 - t\right)/t^2 \qquad 0 < t \leqslant 1.$$

An elementary computation shows that

$$\sup\{f(t) : 0 < t \leqslant 1\} = -a < 0,$$

or

$$\sqrt[q]{2 - (1-t)^q} \leqslant 1 + t - at^2 \quad \text{for } 0 < t \leqslant 1.$$

This inequality holds also for $t=0$, hence we get

$$a = \sqrt[q]{2-(1-\tfrac{1}{2}c^s)^q} \leqslant 1+\tfrac{1}{2}c^s-\tfrac{1}{4}ac^{2s}.$$

Consequently,

$$\|x+y\| \leqslant a+b \leqslant 2-\tfrac{1}{4}ac^{2s} \leqslant 2(1-\tfrac{1}{8}ac^p). \tag{4}$$

Case 2. $0 \leqslant a-b < c^{p/2}$.

Let $z = \dfrac{b}{a}x$. Then $\|z\| = \|y\|$, $\|x-z\| = a-b$, hence

$$\|z-y\| \geqslant \|x-y\| - \|x-z\| = 2c-a+b > 2c-c^{p/2} \geqslant c.$$

Inasmuch as $b \leqslant 1$ and $\delta_Y(\varepsilon) \geqslant L\varepsilon^p$ we get

$$\|z+y\| = b\left\|\frac{z}{b}+\frac{y}{b}\right\| \leqslant 2b\left(1-L\left\|\frac{z}{b}-\frac{y}{b}\right\|^p\right)$$
$$\leqslant 2b-2bL\left(\frac{c}{b}\right)^p \leqslant 2b-2Lc^p,$$

and finally

$$\|x+y\| \leqslant \|x-z\|+\|z+y\| \leqslant a-b+2b-2Lc^p \leqslant 2(1-Lc^p). \tag{5}$$

Inequalities (4) and (5) show that one can take for $K_1(p, q, L)$ the number $\min(L, \tfrac{1}{8}a)$.

Since the modulus of convexity of the space l^p $(p \geqslant 2)$ may be estimated from below by $p^{-1}2^{-p}\varepsilon^p$ ([3], [5]), Proposition 1 implies the following corollary.

Corollary. *The modulus of convexity of the space* $Y = (\sum_{i=1}^{\infty} \oplus l_{p_{i+j}})_p$ *admits the estimate*

$$\delta_Y(\varepsilon) \geqslant K_i\varepsilon^{p_{i+1}}, \quad \text{for } 0 \leqslant \varepsilon \leqslant 2,$$

where K_i is a positive constant.

In our special case the theorem of Kadec [6] can be read as follows:

(K) *If Y is a Banach space with $\delta_Y(\varepsilon) \geqslant K\varepsilon^p$, K being a positive constant, then there exists an $L > 0$ such that for every $x_1, \ldots, x_n \in Y$*

$$\Big(\sum_{i=1}^{n}\|x_i\|^p\Big)^{1/p} \leqslant L\max_{\varepsilon_i=\pm 1}\Big\|\sum_{i=1}^{n}\varepsilon_i x_i\Big\|.$$

One can take $L = \dfrac{1}{2}\left(\dfrac{3}{K}\right)^{1/p}$.

Proposition A is an easy consequence of just formulated results.

Indeed, let i be fixed and let Z be an n-dimensional subspace of the space $(\sum_{j=1}^{\infty}\oplus l_{p_{i+j}})_p$. Let $T\colon l_{p_i}^n \to Z$ be an isomorphism such that

$$\|T\| = d(Z, l_{p_i}^n), \qquad \|T^{-1}\| = 1.$$

It follows from Corollary that

$$\delta_Z(\varepsilon) \geqslant K_i\varepsilon^{p_{i+1}}, \quad \text{for } 0 \leqslant \varepsilon \leqslant 2.$$

Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $l_{p_i}^n$. Using (K) we get the estimate

$$n^{1/p_{i+1}} = \Big(\sum_{j=1}^{n}\|e_j\|^{p_{i+1}}\Big)^{1/p_{i+1}} \leqslant \Big(\sum_{j=1}^{n}\|Te_j\|^{p_{i+1}}\Big)^{1/p_{i+1}}$$
$$\leqslant L_i\max_{\varepsilon_j=\pm 1}\Big\|\sum_{j=1}^{n}\varepsilon_j Te_j\Big\| \leqslant L_i\|T\|\max_{\varepsilon_j=\pm 1}\Big\|\sum_{j=1}^{n}\varepsilon_i e_j\Big\|$$
$$= L_i d(Z, l_{p_i}^n)n^{1/p},$$

or

$$d(Z, l_{p_i}^n) \geqslant L_i^{-1}n^{(p_i-p_{i+1})/p_ip_{i+1}}.$$

The last inequality shows that it suffices to take

$$m_i > (iL_i)^{p_ip_{i+1}/(p_i-p_{i+1})},$$

and, since i was arbitrary, this completes the proof of Proposition A.

Proof of Proposition B. It is well known that every infinite dimensional subspace of l_p contains a subspace isomorphic to l_p. The proof of Proposition II requires an analogous fact for finite dimensional spaces l_p^n.

Below we get such a result in the case of $p > 2$. Our proof was influenced by [7]. The methods of Kadec and Pełczyński essentially depend on infinite dimensionality of spaces they consider. We could avoid the infinity arguments using some properties of projection constants.

We recall that the *projection constant* $\lambda(B)$ of a finite dimensional Banach space B can be defined as the smallest number λ such that whenever B is embedded in a Banach space Z, there exists a projection P of Z onto B with $\|P\| \leqslant \lambda$. Let us notice that

(6) *If Q is a linear projection of a space B_1 onto its subspace B_2, then*

$$\lambda(B_2) \leqslant \|Q\|\lambda(B_1).$$

(7) *If B_1 and B_2 are isomorphic, then*

$$\lambda(B_1) \leqslant d(B_1, B_2)\lambda(B_2).$$

In particular, since for every n, $\lambda(l_\infty^n) = 1$, we get

$$\lambda(l_p^n) \leqslant d(l_p^n, l_\infty^n) \leqslant n^{1/p}, \quad \text{for } p \geqslant 1. \tag{8}$$

The last inequality we need, stating that

$$\lambda(l_2^m) \geqslant \sqrt{\frac{m}{2\pi}}, \quad \text{for } m = 1, 2, \ldots, \tag{9}$$

is a consequence of Rutovitz's results [11].

The crucial point in the proof of Proposition B is the following fact:

PROPOSITION 2. *Let p, c satisfy inequalities*

$$p > 2, \quad 0 < c < 1,$$

and let n be a positive integer. Then in every linear subspace $Z \subset l_p^n$ with $\dim Z = m \geqslant cn$ there exists a vector x_0 such that $\|x_0\| = 1$ and

$$\mathrm{Card}\big(\{j \in \{1, \ldots, n\}: |x_0(j)| \geqslant \varepsilon n^{-1/p}\}\big) < \varepsilon n,$$

where $\varepsilon = (2\pi/c)^{1/6} n^{(2-p)/6p}$.

Proof. Let $|\cdot|$ denote the Euclidean norm in l_p^n, i.e.

$$|x| = \Big(\sum_{j=1}^{n} |x(j)|^2\Big)^{1/2}, \quad \text{for } x \in l_p^n,$$

and let

$$u = \inf\{|x|: x \in Z, \|x\| = 1\}.$$

Hence for $x \in Z$

$$u\|x\| \leqslant |x| \leqslant n^{(p-2)/2p}\|x\|.$$

In particular we get

$$d(Z, l_2^m) \leqslant u^{-1} n^{(p-2)/2p}.$$

Let P be the orthogonal projection (in the Euclidean norm) of l_p^n onto Z. Obviously $|P| = 1$, hence for every $x \in l_p^n$

$$\|Px\| \leqslant u^{-1}|Px| \leqslant u^{-1}|x| \leqslant u^{-1} n^{(p-2)/2p}\|x\|.$$

We have thus proved that

$$\|P\| \leqslant u^{-1} n^{(p-2)/2p}.$$

Using (6) and (8) we infer that

$$\lambda(Z) \leqslant \lambda(l_p^n)\|P\| \leqslant u^{-1} n^{1/2}.$$

Applying (9) and (7) we get

$$\sqrt{\frac{m}{2\pi}} < \lambda(l_2^m) \leqslant \lambda(Z)\, d(Z, l_2^m) \leqslant u^{-2} n^{(p-1)/p}.$$

Recalling that $m \geqslant cn$ we obtain the estimate

$$u < (2\pi/c)^{1/4} n^{(p-2)/4p}.$$

By the definition of u there exists an $x_0 \in Z$ such that $\|x_0\| = 1$ and $|x_0| \leqslant u$. Let

$$s = \mathrm{Card}\big(\{j \in \{1, \ldots, n\}: |x_0(j)| \geqslant \varepsilon n^{-1/p}\}\big).$$

Since

$$(s\varepsilon^2 n^{-2/p})^{1/2} \leqslant |x_0| \leqslant u < (2\pi/c)^{1/4} n^{(p-2)/4p},$$

after easy calculations we get that $s < \varepsilon n$, which concludes the proof.

The last proposition enables us to apply an analogon of a usual "gliding hump" procedure in the next proof.

PROPOSITION 3. *Let $p > 2$, $0 < c < d < 1$, $0 < \beta < 1$. Then there exist positive constants N, a such that, for every $n > N$, every linear subspace $Z \subset l_p^n$ with $\dim Z \geqslant dn$ contains a subspace Y such that*

$$m = \dim Y > an^{(p-2)/6p},$$
$$d(Y, l_p^m) \leqslant (1+\beta)/(1-\beta).$$

Proof. Denote for $0 \neq x \in l_p^n$ by $S(x)$ the set

$$\{j \in \{1, \ldots, n\}: |x(j)| \geqslant \varepsilon n^{-1/p}\|x\|\},$$

where ε denotes again the number $(2\pi/c)^{1/6} n^{(2-p)/6p}$.

For n large enough we have $\varepsilon < d - c$. Let m be an arbitrary positive integer less than $(d-c)/\varepsilon$.

We define inductively a sequence $y_1, \ldots, y_m$ of elements of Z such that

$$\|y_i\| = 1,$$
$$\mathrm{Card}\big(S(y_i)\big) < \varepsilon n,$$
$$S(y_i) \cap \bigcup_{j<i} S(y_j) = \emptyset, \quad \text{for } 1 \leqslant i \leqslant m.$$

The existence of y_1 is guaranteed by Proposition 2.

Having defined $y_1, \ldots, y_k$, where $k < m$, we consider the subspace

$$Z_k = \{z \in Z: z(j) = 0 \quad \text{for } j \in \bigcup_{i=1}^{k} S(x_i)\}.$$

Since

$$\dim Z_k > \dim Z - k\varepsilon n \geqslant (d - k\varepsilon)\, n \geqslant cn,$$

we may apply Proposition 2 to the space Z_k and find in it a y_{k+1}.

Define vectors $w_1, \ldots, w_m \in l_p^n$ by formulae

$$w_i(j) = \begin{cases} y_i(j) & \text{for } j \in S(y_i), \\ 0 & \text{otherwise.} \end{cases}$$

These vectors have disjoint "supports", hence the subspace W they span in l_p^n is isometrically isomorphic to l_p^m. Since $\|y_i - w_i\| < \varepsilon$, for $1 \leqslant i \leqslant m$, we have $\|w_i\| > 1-\varepsilon$ and therefore if $w = \sum_{i=1}^{m} c_i w_i \in W$, then

$$|c_i| \leqslant (1-\varepsilon)^{-1}\|w\| \quad \text{for } 1 \leqslant i \leqslant m.$$

Consider the operator T mapping W into the subspace Y of Z spanned by $y_1, \ldots, y_m$ given by the formula

$$T\Big(\sum_{i=1}^{m} c_i w_i\Big) = \sum_{i=1}^{m} c_i y_i.$$

For any $w = \sum_{i=1}^{m} c_i w_i$ we have

$$\|Tw - w\| = \Big\|\sum_{i=1}^{m} c_i(y_i - w_i)\Big\| \leqslant \sum_{i=1}^{m} |c_i|\,\|y_i - w_i\| \leqslant m\varepsilon(1-\varepsilon)^{-1}\|w\|.$$

Suppose that

$$m\varepsilon(1-\varepsilon)^{-1} \leqslant \beta, \tag{10}$$

then we have

$$\|T\| \leqslant 1+\beta, \quad \|T^{-1}\|^{-1} \geqslant 1-\beta,$$

hence

$$d(Y, l_p^m) \leqslant (1+\beta)/(1-\beta).$$

Since, by (10), for m the integer part of the number

$$\min\big((d-c)/\varepsilon,\ \beta(\varepsilon^{-1}-1)\big)$$

can be taken, and this number admits for n large enough the estimate by $an^{(p-2)/6p}$ from below, where a is a positive constant independent of n, the proof is complete.

Proposition B follows from the previous result by easy induction.

The author is greatly obliged to Professor A. Pełczyński for suggestion of the problem and valuable discussions.

Added in proof. It should be noticed that Proposition 1 remains true when q is *mutatis mutandis* replaced by p.

Indeed, it is easy to see that the proof of Lemma 1 depends in fact only on assumption that $p \geqslant 2$, $q > 1$. Hence, in the notation of this lemma, we have, for every $x, y \in Y$ such that $\|x\|^p + \|y\|^p = 2$,

$$\tfrac{1}{2}\|x+y\| \leqslant 1 - K_1(\tfrac{1}{2}\|x-y\|)^p,$$

where K_1 is a positive constant depending only on p and L. Let $K_2 = \min(K_1, 1)$.

Inasmuch as $\|x-y\| \leqslant 2$ we get

$$\begin{aligned}(\tfrac{1}{2}\|x+y\|)^p &\leqslant (1 - K_2(\tfrac{1}{2}\|x-y\|)^p)^p \\ &\leqslant 1 - K_2(\tfrac{1}{2}\|x-y\|)^p \\ &= \tfrac{1}{2}(\|x\|^p + \|y\|^p) - K_2(\tfrac{1}{2}\|x-y\|)^p.\end{aligned}$$

Clearly, this inequality holds for arbitrary $x, y \in Y$.

Now let $x = (x_i)_{i=1}^{\infty}$ and $y = (y_i)_{i=1}^{\infty}$ be arbitrary elements of the unit sphere in the space $X = (\sum_{i=1}^{\infty} \oplus X_i)_p$. We have

$$\begin{aligned}\|\tfrac{1}{2}(x+y)\|^p &= \sum_{i=1}^{\infty}(\tfrac{1}{2}\|x_i+y_i\|_i)^p \\ &\leqslant \sum_{i=1}^{\infty}\big(\tfrac{1}{2}(\|x_i\|_i^p + \|y_i\|_i^p) - K_2(\tfrac{1}{2}\|x_i-y_i\|_i)^p\big) \\ &= 1 - K_2 2^{-p}\|x-y\|^p,\end{aligned}$$

and therefore, by Bernoulli's inequality,

$$\begin{aligned}\|\tfrac{1}{2}(x+y)\| &\leqslant (1 - K_2 2^{-p}\|x-y\|^p)^{1/p} \\ &\leqslant 1 - K_2 2^{-p}p^{-1}\|x-y\|^p.\end{aligned}$$

Hence we infer that $\delta_X(\varepsilon) \geqslant K_2 2^{-p}p^{-1}\varepsilon^p$, which completes the proof.

References

[1] R. D. Anderson, T. A. Chapman and R. M. Schori, *Problems in the topology of infinite-dimensional spaces and manifolds*, Amsterdam 1971.

[2] C. Bessaga and A. Pełczyński, *Banach spaces non-isomorphic to their Cartesian squares. I*, Bull. Acad. Polon. Sci., sér. sci. math., astr. et phys. 8 (1960), pp. 77–80.

[3] J. A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. 40 (1936), pp. 396–414.

[4] M. M. Day, *Some more uniformly convex spaces*, Bull. Amer. Math. Soc. 47 (1941), pp. 504–507.

[5] O. Hanner, *On the uniform convexity of L^p and l^p*, Ark. Mat. 3 (1955), pp. 239–244.

[6] M. I. Kadec, *Unconditionally convergent series in a uniformly convex space*, Uspehi Mat. Nauk (N. S.) 11 (1956), pp. 185–190 (Russian).

[7] — and A. Pełczyński, *Bases, lacunary sequences and complemented subspaces in the spaces L_p*, Studia Math. 21 (1962), pp. 161–176.

[8] P. Mankiewicz, *On Lipschitz mappings between Frechet spaces*, Studia Math. 41 (1972), pp. 225–241.

[9] B. S. Mitjagin, *The homotopy structure of the linear group of a Banach space*, Uspehi Mat. Nauk (N. S.) 25 (1970), pp. 63–107 (Russian).

[10] B. S. Mitjagin and I. S. Edel'šteĭn, *The homotopy type of the linear groups of two classes of Banach spaces*, Funkcional. Anal. i Priložen, 4 (1970), pp. 61–72 (Russian).

[11] D. Rutovitz, *Some parameters associated with finite-dimensional Banach spaces*, J. London Math. Soc. 40 (1965), pp. 241–255.

[12] Z. Semadeni, *Banach spaces non-isomorphic to their Cartesian squares. II*, Bull. Acad. Polon. Sci., sér. sci. math., astr. et phys. 8 (1960), pp. 81–86.

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