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Examples of nuclear systems

by

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Abstract. The purpose of this paper is to present a detailed study of some concrete examples of nuclear systems, whose general theory has been presented by the author in previous papers. There are three classes of examples. First we consider nuclear systems generated either by a sequence of commuting normal operators or a sequence of permutations. Next, certain matrices which are zero everywhere except on the main diagonal and the diagonal directly above it are considered. Finally a very simple type of lower triangular matrix is discussed. In most cases it is shown that the resulting nuclear Fréchet space has a Schauder basis, but an example is constructed in which all of our methods fail to yield a basis.

The theory of nuclear systems (insofar as it has been developed) was presented in [1], [2]. This theory provides a method of constructing nuclear Fréchet spaces which in principle produces all such spaces whose topology is defined by norms and in practice permits the construction of examples not previously studied. Moreover, several criteria for the existence of Schauder bases have been established. It is the purpose of this paper to study in detail some of the examples of nuclear Fréchet spaces provided by nuclear systems, in most cases proving the existence of a Schauder basis.

We recall now the definitions and results which will be used. Proofs and further explanations are to be found in [1] and [2].

A *nuclear system* is a sequence (A_k) of injective nuclear operators in l_2 with dense range. The *associated space*, written

$$\hat{E} = \{(x_k): x_k \in l_2,\ x_k = A_k(x_{k+1}),\ k = 1, 2, \ldots\},$$

is a subset of the countable product of copies of l_2 so it may be equipped with the subspace topology whence it becomes a nuclear Fréchet space

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whose topology is defined by a sequence of norms, and all such spaces are obtained in this way ([1], p. 373). The *projection operators*, $P_k\colon \hat{E} \to l_2$ $k = 1, 2, \ldots$ are defined by setting $P_k(x) = x_k$. Each P_k is continuous, linear and has dense range ([1], p. 376). We define B_0 to be the identity map on l_2 and we set $B_k = A_1 \ldots A_k$, $k = 1, 2, \ldots$

We recall that a *Schauder basis* in a topological vector space E is a sequence (b_n) with the property that for each $x \in E$ there is a unique sequence (ξ_n) of scalars such that $x = \lim\limits_n \sum\limits_{i=1}^{n} \xi_i b_i$. We shall call a sequence (b_n) in a topological vector space *total* if the vector subspace it generates is dense.

For each positive integer n we denote by $e^n \in l_2$ the sequence which is 1 at the nth term and 0 elsewhere and by Π_n the projection of an element of l_2 onto its nth coordinate. We denote by φ the subset of l_2 consisting of those sequences all but finitely many of whose terms are 0.

A continuous linear map $D\colon l_2 \to l_2$ is a *diagonal map* with *diagonal element* (λ_n) if $D(e^n) = \lambda_n e^n$, $n = 1, 2, \ldots$ The identity map on l_2 will be denoted by I.

The following results from [1], [2] will be used quite often throughout the paper, so we quote them here for easy reference. Proposition A is essentially proved in [1], p. 378 and appears in the following revised form as Theorem 3 in [2]. Propositions B, C are proved as Propositions 4, 5 respectively in [2].

PROPOSITION A. *The associated space of a nuclear system (A_k) has a Schauder basis if and only if there exist diagonal nuclear maps $D_k\colon l_2 \to l_2$, and continuous linear maps $f_k\colon l_2 \to l_2$, $k = 1, 2, \ldots$ such that*

(i) $A_k f_{k+1} = f_k D_k$, $k = 1, 2, \ldots$

(ii) f_1 *maps* $\bigcap\limits_{k=1}^{\infty} D_1 \ldots D_k(l_2)$ *injectively onto* $\bigcap\limits_{k=1}^{\infty} A_1 \ldots A_k(l_2)$.

PROPOSITION B. *If (A_k) is a nuclear system, then $\hat{E}$ has a Schauder basis if and only if there exists a linear injective map $S\colon \varphi \to \bigcap\limits_k B_k(l_2)$ with $B_k^{-1}S(\varphi)$ dense in l_2 for each $k \geqslant 0$ and such that for each $k \geqslant 0$ there exists $j \geqslant k$ such that*

$$\sup_p \left\| B_k^{-1} S \Pi_p S^{-1} B_j \big|_{B_j^{-1}S(\varphi)} \right\| < \infty .$$

In this case, if we consider $\hat{E}$ to be represented by $\bigcap\limits_k B_k(l_2)$ (via $P_1(\hat{E})$), then the basis is the sequence $(S(e^\nu))_\nu$.

PROPOSITION C. *Let (A_k) be a nuclear system and (b_n) a total, linearly independent sequence in $\hat{E}$. Then (b_n) is a Schauder basis for $\hat{E}$ if and only if for each $k \geqslant 0$, there exists $j \geqslant k$ such that*

$$\sup_p \left\| B_k^{-1} S \Pi_p S^{-1} B_j \big|_{B_j^{-1}S(\varphi)} \right\| < \infty ,$$

where $S\colon \varphi \to l_2$ is defined by $S(e^n) = P_1 b_n$.

In Section 1, we consider nuclear systems with the property that the eigenvectors of each A_k can be easily computed and have a relatively transparent dependence on k. In Section 2, we consider nuclear systems generated by a single operator, A, that is, $A_k = A$ for all k, where A is a matrix whose terms are 0 everywhere except the main diagonal and the diagonal just above it. We are able to give sufficient conditions for the existence of a basis and also construct an example in which two methods for obtaining a basis fail. This leads to an example of a Markuschevitch basis in a Fréchet nuclear space which is not a Schauder basis. In Section 3 we consider A_k to be a matrix which is 0 except on the main diagonal and the first N columns (N independent of k). Here $\hat{E}$ always has a basis.

1. Normal operators and permutations. In the next proposition we give a generalization of the result in [4]. The idea is that if the operators in a nuclear system all have the same set of eigenvectors, then this set can be used to construct a basis for the associated space.

PROPOSITION 1. *Let (A_k) be a nuclear system in which each A_k is normal and $A_k A_{k+1} = A_{k+1} A_k$ for all k. Then the associated space possesses a Schauder basis.*

Proof. Let λ_1 be an eigenvector of A_1 with eigenspace E_1 which is finite dimensional since A_1 is compact. If $x \in E_1$ then $A_1 A_2(x) = A_2 A_1(x) = \lambda_1 A_2(x)$ so $A_2(x) \in E_1$. Thus $A_2(E_1) \subset E_1$ and $A_2 | E_1$ is a normal operator on E_1 so we can choose a maximal eigenspace $E_2 \subset E_1$ whose dimension is positive. Repeating the process indefinitely, we obtain a decreasing sequence (E_k) of finite dimensional spaces with positive dimension and hence there exists k_0 such that $E_k = E_{k_0}$ for all $k \geqslant k_0$. It then follows that an orthonormal basis for E_{k_0} is a non-empty set whose elements are eigenvectors for each A_k, $k = 1, 2, \ldots$

Repeating the process a number of times at most equal to the dimension of E_1, we obtain an orthonormal basis for E_1 whose elements are eigenvectors for each A_k. Again repeating for each eigenvalue of A_1, we obtain an orthonormal basis (x_n) for l_2 such that each x_n is an eigenvector of each A_k.

Finally, define $S\colon l_2 \to l_2$ by $S(e^n) = x_n$ and apply Proposition B. Clearly $S(\varphi) \subset B_k(l_2)$ and $B_k^{-1}S(\varphi)$ is dense in l_2 for each k. Moreover we have $A_k = SD_kS^*$ where D_k is a diagonal operator so $B_k = SD_1 \dots D_kS^*$ and so we have,

$$\sup_p \|B_k^{-1}S\Pi_pS^{-1}B_k\| = \sup_p \|S(D_1 \dots D_k)^{-1}\Pi_pD_1 \dots D_kS^*\| \leqslant \|SS^*\| = 1$$

so Proposition B applies to yield the desired result. ■

In view of Proposition 1 one might try to construct a nuclear Fréchet space without a basis by making the eigenvectors of the maps A_k different for each k. This is perhaps also suggested by the proof of Proposition 2 of [2]. However this does not seem to work as is indicated by the next result in which we construct a basis in cases for which the eigenvectors are quite different.

PROPOSITION 2. *Let (σ^k) be a sequence of permutations of the natural numbers and let (a^k) be a sequence of elements of l_1 each of which has no 0 terms. Define $A_k\colon l_2 \to l_2$ by*

$$A_k(e^n) = a_n^k e^{\sigma^k(n)}, \quad n, k = 1, 2, \dots$$

Then (A_k) is a nuclear system whose associated space has a Schauder basis.

Proof. It is clear that (A_k) is a nuclear system. To see that it has a basis, we apply Proposition A. Let $\tau^k = \sigma^1 \dots \sigma^k$, $\tau^0 =$ identity and define diagonal operators $D_k\colon l_2 \to l_2$ and continuous operators $f_k\colon l_2 \to l_2$ by

$$D_k(e^{\tau^k(n)}) = a_n^k e^{\tau^k(n)}, \quad f_k(e^{\tau^{k-1}(n)}) = e^n, \quad k, n = 1, 2, \dots$$

Then we have

$$A_kf_{k+1}(e^{\tau^k(n)}) = A_k(e^n) = a_n^k e^{\sigma^k(n)} = a_n^k f_k(e^{\tau^{k-1}(\sigma^k(n))})$$
$$= f_k(a_n^k e^{\tau^k(n)}) = f_kD_k(e^{\tau^k(n)}),$$

so that $A_kf_{k+1} = f_kD_k$, $k = 1, 2, \dots$

Finally, define $\beta^k \in l_1$, $k = 1, 2, \dots$ by $\beta^k_{\tau^k(n)} = a_n^k$ so that $D_k(l_2) = \beta^k l_2$, $\beta^k = (\beta_n^k)_n$.

Then we have,

$$A_1 \dots A_k(e^n) = a^1_{\sigma^2 \dots \sigma^k(n)} \dots a^{k-1}_{\sigma^k(n)} a_n^k e^{\tau^k(n)} = \beta^1_{\tau^k(n)} \dots \beta^k_{\tau^k(n)} e^{\tau^k(n)}.$$

Hence, using the fact that l_2 is invariant under permutations,

$$A_1 \dots A_k(l_2) = \Big\{\sum_{n=1}^{\infty} \xi_n A_1 \dots A_k(e^n)\colon \xi = (\xi_n) \in l_2\Big\}$$
$$= \Big\{\sum_{n=1}^{\infty} \xi_n \beta^1_{\tau^k(n)} \dots \beta^k_{\tau^k(n)} e^{\tau^k(n)}\colon \xi \in l_2\Big\}$$
$$= \Big\{\sum_{n=1}^{\infty} \xi_n \beta_n^1 \dots \beta_n^k e^n\colon \xi \in l_2\Big\} = \beta^1 \dots \beta^k(l_2) = D_1 \dots D_k(l_2).$$

Hence, $f_1\big(\bigcap_{k=1}^{\infty} D_1 \dots D_k(l_2)\big) = \bigcap_{k=1}^{\infty} D_1 \dots D_k(l_2) = \bigcap_{k=1}^{\infty} A_1 \dots A_k(l_2)$ so the conditions of Proposition A are satisfied and we may conclude that the associated space has a basis.

Remark 1. A slight improvement of Proposition 1 is possible. If we assume that each A_k is similar to a diagonal matrix via the same similarity transformation; that is, there exists an isomorphism $S\colon l_2 \to l_2$ such that each $S^{-1}A_kS$ is a diagonal matrix, then the last half of the proof of Proposition 1 will still work showing that $\hat{E}(A_k)$ was a basis.

2. $\mu-\nu$ matrices. Let μ, ν be elements of l_1 with $0 < |\nu_n| \leqslant |\mu_n|$ and define $A\colon l_2 \to l_2$ by

$$A(e^1) = \mu_1 e^1, \quad A(e^n) = \mu_n e^n - \nu_{n-1}e^{n-1}, \quad n > 1.$$

Then $A(x) = 0$ if and only if

$$x_{n+1} = \frac{\mu_n}{\nu_n} x_n = \dots = \frac{\mu_n \dots \mu_1}{\nu_n \dots \nu_1} x_1,$$

so that $|x_n| \geqslant |x_1|$ for all n so if $x \in l_1$, then $x = 0$. Thus A is injective. Moreover, it is obvious that $A(\varphi) = \varphi$ so A has dense range. Thus A generates a nuclear system. We now wish to study the existence of a basis with various additional restrictions on μ, ν.

PROPOSITION 3. *Let $\mu = \nu$ and suppose that for each $k \geqslant 0$, $(|\mu_p|^{2k}p)_p$ is unbounded. Let (b_n) be the sequence in $\hat{E}$ defined by taking $b_n = (b^{k,n})_k$, where $b^{k,n} = A^{-k+1}(e^n)$, $n, k = 1, 2, \dots$ Then (b_n) is a total, linearly independent sequence in $\hat{E}$ which is not a Schauder basis for $\hat{E}$.*

Proof. Since $A(\varphi) = \varphi$ it follows that the given expression establishes (b_n) as a sequence in $\hat{E}$. If

$$\sum_{i=1}^{n} t_i b_i = 0$$

then it follows by taking $k = 1$ and evaluating the sum of sequences at its first coordinate that

$$\sum_{i=1}^{n} t_i e^i = 0$$

which implies that $t_1 = \dots t_n = 0$. Hence (b_n) is linearly independent. Moreover, for each k, the vector subspace generated by $(b^{k,n})_n$ is $A^{-k+1}(\varphi) = \varphi$ which is dense in l_2 so it follows that $(b_n)_n$ is total in $\hat{E}$.

To show that (b_n) is not a basis we apply Proposition C. Clearly we have $B_k = A^k$, S is the identity and $B_k^{-1}S(\varphi) = \varphi$ so we must show that for some $k \leqslant j$ that by choosing a suitable index p we can make $\|A^{-k}\Pi_pA^j|_\varphi\|$

arbitrarily large. We take $k = 1$ and we compute the matrix representation,

$$A^{-1}\Pi_p A = \left[\begin{matrix} \left.\begin{matrix} \overbrace{\begin{matrix} 1 & 0 & 0 & \dots & 0 \end{matrix}}^{p} & -1 \\ \begin{matrix} 0 & 1 & 0 & \dots & 0 \end{matrix} & -1 \\ \begin{matrix} 0 & 0 & 1 & \dots & 0 \end{matrix} & -1 \\ \cdot \quad \cdot \quad \cdot & \cdot \\ \cdot \quad \cdot \quad \cdot & \cdot \\ \cdot \quad \cdot \quad \cdot & \cdot \\ \begin{matrix} 0 & \dots & 0 & 1 \end{matrix} & -1 \end{matrix}\right\} p & 0 \\ 0 & 0 \end{matrix}\right],$$

or, expressed in more compact form,

$$A^{-1}\Pi_p A(e^n) = \begin{cases} e^n, & n \leqslant p, \\ -(e^1+\ldots+e^p), & n = p+1, \\ 0, & n > p+1. \end{cases}$$

From this it follows that $\|A^{-1}\Pi_p A|_\varphi\| \geqslant \|e^1+\ldots+e^p\| = \sqrt{p}$ so that the sequence is unbounded if we choose $j = 1$.

Now choose any $j \geqslant 1$. We shall show that $\sup\limits_p \|A^{-1}\Pi_p A^{j+1}|_\varphi\| = \infty$ which will complete the proof. Let the matrix of A^j be (a^j_{st}), and take $p > j$. Then we claim

$$a^j_{s,p+1} = \begin{cases} 0 & \text{if } s \leqslant p-j, \\ 0 & \text{if } s > p+1, \\ \mu^j_{p+1} & \text{if } s = p+1. \end{cases}$$

Indeed, if $j = 1$ this is immediate from the definition. Suppose that it holds for some $j-1$. Then to compute the $(p+1)$-st column of (a^j_{st}) we multiply each row of (a^1_{st}) by the $(p+1)$-st column of (a^{j-1}_{st}). Thus we have,

$$a^j_{p+1,p+1} = \sum_{n=1}^{\infty} a^1_{p+1,n} a^{j-1}_{n,p+1} = a^1_{p+1,p+1} a^{j-1}_{p+1,p+1} + a^1_{p+1,p+2} a^{j-1}_{p+2,p+1}$$
$$= \mu_{p+1}\mu^{i-1}_{p+1} - \mu_{p+1}\cdot 0 = \mu^j_{p+1}.$$

If $s > p+1$,

$$a^j_{s,p+1} = \sum_{n=1}^{\infty} a^1_{s,n} a^{j-1}_{n,p+1} = \sum_{n=s}^{s+1} a^1_{s,n} a^{j-1}_{n,p+1} = 0$$

and if $s \leqslant p-j$,

$$a^j_{s,p+1} = \sum_{n=1}^{\infty} a^1_{s,n} a^{j-1}_{n,p+1} = \mu^1_s a^{j-1}_{s,p+1} - \mu^1_s a^{j-1}_{s+1,p+1}$$

and $s \leqslant p-j$ implies that $s \leqslant p-j+1$ and $s+1 \leqslant p-j+1$ so by the induction hypothesis, both terms vanish and $a^j_{s,p+1} = 0$. Thus the claim is proved.

From this it follows that for some scalars $c_{p-j+1}, \ldots, c_p$ we have

$$A^j(e^{p+1}) = \sum_{s=p-j+1}^{p} c_s e^s + \mu^j_{p+1} e^{p+1}$$

so that

$$A^{-1}\Pi_p A^{j+1}(e^{p+1}) = \sum_{s=p-j+1}^{p} c_s e^s - \mu^j_{p+1}\Big(\sum_{s=1}^{p} e^s\Big)$$
$$= -\sum_{s=1}^{p-j} \mu^j_{p+1} e^s + \sum_{s=p-j+1}^{p} (c_s - \mu^j_{p+1}) e^s.$$

Therefore,

$$\|A^{-1}\Pi_p A^{j+1}|_\varphi\| \geqslant \Big[\sum_{j=1}^{p-j} (\mu^j_{p+1})^2 + \sum_{s=p-j+1}^{p} (c_s - \mu^j_{p+1})^2\Big]^{\frac{1}{2}} \geqslant |\mu^j_{p+1}| \sqrt{p-j}$$
$$= |\mu_{p+1}|^j \sqrt{p+1}\,\frac{\sqrt{p-j}}{\sqrt{p+1}}.$$

By our hypotheses, this last term is unbounded in p so we are finished.

Remark 1. The hypothesis of Proposition 3 is easily obtained, for instance if we choose $\mu \in l_1$ such that

$$\mu_{2^{2^n}} = \frac{1}{2^n}, \quad n = 1, 2, \ldots$$

Then if we take $p = 2^{2^n}$, we obtain

$$|\mu_p|^{2j} p = \frac{2^{2^n}}{2^{2nj}} \to \infty \quad \text{as } n \to \infty.$$

Remark 2. In a sense one may consider that Proposition 3 shows that the most naive approach to constructing a basis in a nuclear Fréchet space cannot in general succeed. Indeed we may think of $\hat{E}$ as a dense subspace of l_2 with a sequence of (Hilbertable) norms on it, the first being the original l_2 norm. Then we have taken a complete orthonormal sequence (with respect to the first norm) and shown that it was total and linearly independent in $\hat{E}$. Such a procedure is exactly what worked in the cases treated in Section 1, but it fails here.

Actually, we can show more. Recall that a *Markuschevich basis* in a linear topological space E is a sequence (b_n, f_n), where (b_n) is total in E, (f_n) is total in $E'[\mathfrak{T}_s(E)]$ and $f_n(b_m) = \delta_{nm}$. In [3] it is shown that if E and its strong dual are separable then a Markuschevich basis always exists. A Markuschevich basis is not necessarily a Schauder basis. For an example in the case of Banach spaces, see [5]. We know of no previous example for Fréchet nuclear spaces, so we show that $\mu-\nu$ matrices provide such examples.

PROPOSITION 4. *With the hypotheses of Proposition 3 there is a sequence* (f_n) *in* $\hat{E}'$ *such that* (b_n, f_n) *is a Markuschevich basis but not a Schauder basis.*

Proof. For each $n = 1, 2, \ldots$ let $f_n = \Pi_n P_1$. Since Π_n and P_1 are continuous it follows that $f_n \in \hat{E}'$. Also,

$$f_n(b_m) = \Pi_n P_1(b_m) = \Pi_n(b^{1,m}) = \Pi_n(e^n) = \delta_{nm}.$$

Finally, if $x \in \hat{E}$ and $f_n(x) = 0$ for all n, then $P_1 x \in l_2$ so we can write

$$P_1 x = \sum_{m=1}^{\infty} \xi_m e^m, \qquad \xi \in l_2 \text{ and convergence in } l_2.$$

Hence, for each n,

$$0 = f_n(x) = \Pi_n P_1(x) = \sum_{m=1}^{\infty} \xi_m g_n(e^m) = \xi_n$$

so it follows that $P_1(x) = 0$ and since P_1 is injective, $x = 0$. Thus (f_n) is total in $E'[\mathfrak{T}_s(E)]$ and we are finished.

We can try to find a basis for $\hat{E}$ using other methods. Let us assume for the rest of this section that $\mu_n \neq \mu_m$ for $n \neq m$. Then it is possible to (formally) diagonalize A. Let D be the diagonal matrix with diagonal elements (μ_n) and let U be the following matrix:

$$\begin{bmatrix} 1 & \dfrac{\nu_1}{\mu_1-\mu_2} & \dfrac{\nu_1\nu_2}{(\mu_1-\mu_3)(\mu_2-\mu_3)} & \dfrac{\nu_1\nu_2\nu_3}{(\mu_1-\mu_4)(\mu_2-\mu_4)(\mu_3-\mu_4)} & \cdots \\ 0 & 1 & \dfrac{\nu_2}{\mu_2-\mu_3} & \dfrac{\nu_2\nu_3}{(\mu_2-\mu_4)(\mu_3-\mu_4)} & \cdots \\ 0 & 0 & 1 & \dfrac{\nu_3}{\mu_3-\mu_4} & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix}.$$

One can easily verify by direct computation that $AU = UD$. Moreover, U has a (formal) inverse, U^{-1} given by the following matrix:

$$\begin{bmatrix} 1 & \dfrac{\nu_1}{\mu_2-\mu_1} & \dfrac{\nu_1\nu_2}{(\mu_2-\mu_1)(\mu_3-\mu_1)} & \dfrac{\nu_1\nu_2\nu_3}{(\mu_2-\mu_1)(\mu_3-\mu_1)(\mu_4-\mu_1)} & \cdots \\ 0 & 1 & \dfrac{\nu_2}{\mu_3-\mu_2} & \dfrac{\nu_2\nu_3}{(\mu_3-\mu_2)(\mu_4-\mu_2)} & \cdots \\ 0 & 0 & 1 & \dfrac{\nu_3}{\mu_4-\mu_3} & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix},$$

and hence we can write $A = UDU^{-1}$. Indeed all of the above statements are rigorously true if we only apply the operators to φ. It would be quite easy (Remark 1) to conclude that $\hat{E}$ has a basis if we knew that U was an isomorphism on l_2. Unfortunately, as we shall see below (Remark 3) this is not necessarily the case. However we can give a simple condition under which U is an isomorphism. More detailed computations could lead to sharper results than the following.

PROPOSITION 5. *Let* μ, ν *be such that* $\mu_n \neq \mu_m$ *for* $n \neq m$ *and*

$$r = \sup_{i<n} \left| \frac{\nu_i}{\mu_i - \mu_n} \right| < \frac{1}{2}.$$

Then $\hat{E}$ *has a basis which is obtained by applying* P_1^{-1} *to each of the columns of* U *(as an element of* l_2*).*

Proof. In view of Remark 1, we need only show that U is an isomorphism. The explicit description of the basis follows from the application of Proposition B.

To show that U is an isomorphism, let $S\colon l_2 \to l_2$ be the operator defined by $S(e^n) = e^{n+1}$ and let $E_n\colon l_2 \to l_2$ be the diagonal operator whose diagonal is given by the sequence

$$\left(\frac{\nu_m \cdot \nu_{m+1} \cdots \nu_{m+n-1}}{(\mu_m - \mu_{m+n}) \cdots (\mu_{m+n-1} - \mu_{m+n})} \right)_{m=1}^{\infty}.$$

Then clearly we have $U = I + \sum_{n=1}^{\infty} S^n E_n = I + U_0$. It suffices to show that $\|U_0\| < 1$. But $\|S\| = 1$ and $\|E_n\|$ is the maximum of the moduli in the given sequence, that is,

$$\|E_n\| = \sup_m \left| \frac{\nu_m \cdots \nu_{m+n-1}}{(\mu_m - \mu_{m+n}) \cdots (\mu_{m+n-1} - \mu_{m+n})} \right| = \sup_m \prod_{i=m}^{m+n-1} \left| \frac{\nu_i}{\mu_i - \mu_{m+n}} \right| \leqslant r^n.$$

Hence we have,

$$\|U_0\| \leqslant \sum_{n=1}^{\infty} \|S^n E_n\| \leqslant \sum_{n=1}^{\infty} \|E_n\| \leqslant \sum_{n=1}^{\infty} r^n < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Remark 3. There are some simple cases in which the hypotheses of Proposition 5 are satisfied:

(i) $\mu_n = \dfrac{1}{n^2}$, $\quad \sup\limits_i \dfrac{i^2(i+1)^2}{2i+1} |\nu_i| < \dfrac{1}{2}$,

(ii) $\mu_n = \dfrac{1}{2^n}$, $\quad \sup\limits_i 2^i |\nu_i| < \dfrac{1}{4}$.

Returning now to the situation in Proposition 3, we may well ask if the method of Proposition 5 will always give a basis. The answer is no, and this will give us an example in which U is not an isomorphism.

PROPOSITION 6. *Assume that $\mu = \nu$, $0 < \mu_n \neq \mu_m$ for $n \neq m$ and for each integer $j \geqslant 0$, the sequence*

$$\left(\frac{\mu_p \mu_{p+1}^j}{\mu_p - \mu_{p+1}}\right)_{p=1}^{\infty}$$

is unbounded. Then the sequence (b_n) given by $b_n = P_1^{-1}(b^{1,n})$, where $b^{1,n}$ is the sequence in the n-th column of U, is a total, linearly independent sequence in $\hat{E}$ which is not a Schauder basis.

Proof. Clearly $U(\varphi) = \varphi$ and U is invective on φ. Hence $U(e^n) \in U(\varphi) = UD^k U^{-1}(\varphi) = A^k(\varphi) \subset A^k(l_2)$ so $b^{1,n} = U(e^n) \in \bigcap_k A^k(l_2)$ and $b_n \in \hat{E}$. If $\sum_{i=1}^n t_i b_i = 0$ then $\sum_{i=1}^n t_i U e^i = \sum_{i=1}^n t_i b^{1,i} = 0$ and since U is injective on φ, $\sum_{i=1}^n t_i e^i = 0$ so $t_1 = \ldots = t_n = 0$ and (b_n) is linearly independent. Finally, for each k, the subspace generated by $(b^{k,n})_n$ is $A^{-k+1}U(\varphi) = \varphi$ so (b_n) is total in $\hat{E}$.

Thus we may finish the proof by applying Proposition C to show that (b_n) is not a Schauder basis. We shall show that the norm criteria is not satisfied for $k = 1$. Let $j \geqslant 0$ and consider (all maps restricted to $A^{-(j+1)}U(\varphi) = \varphi$),

$$A^{-1} U \Pi_p U^{-1} A^{j+1} = UD^{-1}\Pi_p D U^{-1} A^i = U \Pi_p U^{-1} A^j.$$

Let $U\Pi_p U^{-1} = (\mu_{mn})$ and we compute $\mu_{p,n}$ for $n = 1, 2, \ldots, p+1$. Now in $\Pi_p U^{-1}$, the first p columns are the same as in U^{-1} and the $(p+1)$st column is the same as in U^{-1} except the $(p+1)$st row which is 0 instead of 1. Taking the inner product of each of these columns with the pth row of U, we obtain,

$$\mu_{p,n} = \begin{cases} 0 & n < p, \\ 1 & n = p, \\ -\dfrac{\mu_p}{\mu_p - \mu_{p+1}} & n = p+1. \end{cases}$$

We have already partially computed the $(p+1)$st column of $A^j = (a_{mn}^j)$ in the proof of Proposition 3, but we also need to evaluate $a_{p,p+1}^j$. We claim

$$a_{p,p+1}^j = -\mu_p \sum_{n=0}^{j-1} \mu_p^n \mu_{p+1}^{j-n-1}, \qquad j \geqslant 1$$

and $a_{p,p+1}^0 = 0$. This is clear for $j = 1$ so we suppose it holds for j. Then using the information in the proof of Proposition 3, we obtain

$$\begin{aligned} a_{p,p+1}^{j+1} &= \sum_{i=1}^{\infty} a_{pi} a_{i,p+1}^j = \mu_p a_{p,p+1}^j - \mu_p \mu_{p+1}^j \\ &= -\mu_p \Big(\mu_p \sum_{n=0}^{j-1} \mu_p^n \mu_{p+1}^{j-n-1} + \mu_{p+1}^j\Big) = -\mu_p \sum_{n=0}^{j} \mu_p^n \mu_{p+1}^{j-n-1}. \end{aligned}$$

Hence we have,

$$a_{m,p+1}^j = \begin{cases} -\mu_p \sum_{n=0}^{j-1} \mu_p^n \mu_{p+1}^{j-n-1}, & m = p,\ j \geqslant 1, \\ 0, & m = p,\ j = 0, \\ \mu_{p+1}^j, & m = p+1,\ j \geqslant 0, \\ 0, & m > p+1,\ j \geqslant 0. \end{cases}$$

Therefore we conclude that the element in the pth row and $(p+1)$st column of $A^{-1} U \Pi_p U^{-1} A^{j+1}$ is given by

$$-\mu_p \sum_{n=0}^{j-1} \mu_p^n \mu_{p+1}^{j-n-1} - \frac{\mu_p \mu_{p+1}^j}{\mu_p - \mu_{p+1}}, \qquad j \geqslant 1,$$

$$-\frac{\mu_p \mu_{p+1}^j}{\mu_p - \mu_{p+1}}, \qquad j = 0.$$

Hence it follows that

$$\|A^{-1} U \Pi_p U^{-1} A^{j+1}\| \geqslant \|A^{-1} U \Pi_p U^{-1} A^{j+1}(e^{p+1})\| \geqslant \frac{\mu_p \mu_{p+1}^j}{\mu_p - \mu_{p+1}},$$

and by hypothesis, this last sequence is unbounded with respect to p for each $j \geqslant 0$ so the result follows from Proposition C.

Remark 4. It is important to note that there exist $\mu - \nu$ matrices which satisfy the hypotheses of both Proposition 3 and 6. Indeed, if we take μ such that

$$\mu_n = \begin{cases} \dfrac{1}{2^m}, & n = 2^{2^m} \\ \dfrac{1}{2^m} + \dfrac{1}{2^{m^2}}, & n = 2^{2^m} - 1, \end{cases} \tag{1}$$

then (see Remark 1), the conditions of Proposition 3 are satisfied. Moreover, if we take any $j \geqslant 0$ and $p = 2^{2^m} - 1$, then

$$\frac{\mu_p \mu_{p+1}^j}{\mu_p - \mu_{p+1}} = \frac{\left(\dfrac{1}{2^m} + \dfrac{1}{2^{m^2}}\right)\dfrac{1}{2^{jm}}}{\dfrac{1}{2^{m^2}}} = 2^{m^2 - m(j+1)} + \frac{1}{2^{jm}}$$

which is clearly unbounded so Proposition 6 is satisfied. Thus all of the methods that we know of fail to produce a basis in this case and we are led to the following

CONJECTURE. *If A is a $\mu-\lambda$ matrix with $\mu=\lambda$, $0<\mu_n\neq\mu_m$ for all $n\neq m$ and such that μ satisfies* (1), *then A generates a nuclear system whose associated space is a nuclear Fréchet space which does not have a Schauder basis.*

3. Lower triangular matrices. In this section we consider a very simple case of lower triangular matrices which give nuclear systems and construct a basis by direct computation.

Let N be a fixed positive integer and for each $k=1,2,\ldots$ let $a^{1,k},\ldots,a^{Nk}$ be elements of l_2 such that $a_j^{ik}=0$ for $j\leqslant i$ and let $\mu^k\in l_1$, $\mu_j^k\neq 0$ for all j,k. Then we can define $A_k\colon l_2\to l_2$ by

$$A_k x=\sum_{i=1}^{N}x_i a^{i,k}+\sum_{j=1}^{\infty}x_j\mu_j^k e^j.$$

As a matrix we can describe A_k by noting that A_k is a diagonal matrix except for its ith column ($1\leqslant i\leqslant N$) which is the sequence $\mu_i^k e^i+a^{ik}$.

PROPOSITION 7. *(A_k) is a nuclear system. For each n, the map $f_n=\Pi_n P_1$ is in the dual of $\hat{E}$.*

Proof. Each A_k is the sum of a nuclear diagonal map and a map with finite dimensional range so it is nuclear. If A_k^* is the adjoint of A_k, then we can see by inspection that $A_k^*(\varphi)=\varphi$ so that A_k^* has dense range so A_k is injective. Next, it is clear that $e^i=\dfrac{1}{\mu_i^k}A_k(e^i)$ for $i>N$. For $1\leqslant n\leqslant N$, let $\varepsilon>0$ and define $x=(x_i)\in l_2$ by

$$x_i=\begin{cases}0, & i<n,\\[4pt] \dfrac{1}{\mu_n^k}, & i=n,\\[4pt] -\dfrac{1}{\mu_i^k}(a_i^{1,k}x_1+\ldots+a_i^{i-1,k}x_{i-1}), & n<i\leqslant N,\\[4pt] -\dfrac{1}{\mu_i^k}(a_i^{1,k}x_1+\ldots+a_i^{N}x_N), & N<i\leqslant M,\\[4pt] 0, & i>M,\end{cases}$$

where $M\geqslant N$ is an integer chosen such that

$$\sum_{j=n}^{N}\Big(\sum_{i=M+1}^{\infty}|a_i^{j,k}x_j|^2\Big)^{\frac{1}{2}}\leqslant\varepsilon.$$

Then

$$\|A_k(x)-e^n\|=\Big\|\sum_{i=M+1}^{\infty}(a_i^{1,k}x_1+\ldots+a_i^{N,k}x_N)e^i\Big\|\leqslant\varepsilon,$$

so A_k has dense range. Thus (A_k) is a nuclear system.

The second statement is obvious since Π_n, P_1 are continuous.

PROPOSITION 8. *The associated space of (A_k) has a Schauder basis.*

Proof. For $i>N$ it follows that $e^i\in\bigcap_k A_1\ldots A_k(l_2)$ so we can define $b_i=P_1^{-1}(e^i)\in\hat{E}$. Now $P_1(\hat{E})$ is a dense subspace of l_2 so if we define $\Pi\colon l_2\to R^N$ by $\Pi(x)=(x_1,\ldots,x_N)$ then $\Pi P_1(\hat{E})=R^N$. Hence there exists $b_1,\ldots,b_N$ in $\hat{E}$ such that $\Pi P_1(b_i)=e^i\in R^N$, $i=1,\ldots,N$.

We claim that (b_i) is a Schauder basis for $\hat{E}$. First suppose that $x\in\hat{E}$ and $x=\sum\xi_i t_i$. Then by Proposition 7, for each n, $f_n(x)=\sum\xi_i f_n(b_i)$. Thus we obtain,

$$\xi_n=\begin{cases}f_n(x) & n\leqslant N,\\ f_n(x)-\sum_{i=1}^{N}f_i(x)f_n(b_i) & n>N.\end{cases}$$

This shows that the representation is unique and we need only show that for $x\in\hat{E}$, the series $\sum\xi_i b_i$ converges to x, where (ξ_i) is given by the above relations.

We consider for $n=1,2,\ldots$

$$\Pi_n P_1\Big(x-\sum_{i=1}^{N}\xi_i b_i\Big)=f_n(x)-\sum_{i=1}^{N}\xi_i f_n(b_i)=f_n(x)-\sum_{i=1}^{N}f_i(x)f_n(b_i)$$
$$=\begin{cases}0, & n\leqslant N,\\ \xi_n, & n>N\end{cases}$$

and since $P_1\big(x-\sum_{i=1}^{N}\xi_i b_i\big)\in l_2$ we have $\xi\in l_2$ and

$$P_1\Big(x-\sum_{i=1}^{N}\xi_i b_i\Big)=\sum_{n=N+1}^{\infty}\xi_n e^n.$$

Hence for $M\geqslant N$,

$$P_1\Big(x-\sum_{i=1}^{M}\xi_i b_i\Big)=P_1\Big(x-\sum_{i=1}^{N}\xi_i b_i\Big)-\sum_{i=N+1}^{M}\xi_i P_1(b_i)$$
$$=\sum_{n=N+1}^{\infty}\xi_n e^n-\sum_{i=N+1}^{M}\xi_i e^i=\sum_{n=M+1}^{\infty}\xi_n e^n.$$

Now let $\nu_n^k = \mu_n^1 \dots \mu_n^k$, $\nu_n^0 = 1$, $n, k = 1, 2, \dots$ Then for any $k \geqslant 1$, we have $\zeta^k = (\zeta_n^k)_n \in l_2$ with

$$P_{k+1}\Big(x - \sum_{i=1}^{M} \xi_i b_i\Big) = \sum_{n=1}^{\infty} \zeta_n^k e^n$$

so

$$P_1\Big(x - \sum_{i=1}^{M} \xi_i b_i\Big) = A_1 \dots A_k P_{k+1}\Big(x - \sum_{i=1}^{M} \xi_i b_i\Big) = \sum_{n=1}^{\infty} \nu_n^k \zeta_n^k e^n.$$

Equating coefficients we conclude that

$$\zeta_n^k = 0 \text{ for } n \leqslant M \quad \text{and} \quad \zeta_n^k = \frac{\xi_n}{\nu_n^k} \text{ for } n > M.$$

In particular, applying this for $M = N$ we conclude that

$$\Big(\frac{\xi_n}{\nu_n^k}\Big)_{n=1}^{\infty} \in l_2.$$

And for arbitrary $M \geqslant N$,

$$\Big\| P_k\Big(x - \sum_{i=1}^{M} \xi_i b_i\Big)\Big\| = \Big\| \sum_{n=M+1}^{\infty} \frac{\xi_n}{\nu_n^k} e^n \Big\| = \Big(\sum_{n=M+1}^{\infty} \Big|\frac{\xi_n}{\nu_n^k}\Big|^2\Big)^{\frac{1}{2}}.$$

The last term goes to 0 as M goes to ∞ and this implies

$$\lim_{M\to\infty} \Big\| P_k\Big(x - \sum_{i=1}^{M} \xi_i b_i\Big)\Big\| = 0 \quad \text{so that} \quad x = \sum_{i=1}^{\infty} \xi_i b_i.$$

Remark 5. The case described above is a very primitive example. To go further, it would be very interesting to see what happens if N varies with respect to k and moreover if this approach could be used to approximate an arbitrary lower triangular matrix. Finally one could investigate the connection between upper and lower triangular matrices.

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On operator-valued solutions of d'Alembert's functional equation, II

by

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Abstract. In the paper several negative examples are given, connected with the problem of representation of a cosine operator function $\mathscr{C}(t)$ in the form $\mathscr{C}(t) = \frac{1}{2}\mathscr{G}(t) + \frac{1}{2}\mathscr{G}(-t)$, where $\mathscr{G}(t)$ is an one parameter group of operators.

Introduction and results. Let X be a real or complex topological vector space and let $\mathscr{L}_s(X)$ be the space of all linear continuous operators of X into itself with the topology of simple convergence. A continuous mapping $\mathscr{C}$ of $(-\infty, \infty)$ into $\mathscr{L}_s(X)$ is called the cosine operator function if it satisfies the d'Alembert functional equation

$$\mathscr{C}(t+s) + \mathscr{C}(t-s) = 2\mathscr{C}(t)\mathscr{C}(s), \qquad -\infty < s, t < \infty$$

and if, moreover,

$$\mathscr{C}(0) = 1.$$

We shall say that $\mathscr{C}$ has an exponential representation if there is a one-parameter continuous group $\{\mathscr{G}(t): -\infty < t < \infty\} \subset \mathscr{L}_s(X)$ such that

$$\mathscr{C}(t) = \tfrac{1}{2}\mathscr{G}(t) + \tfrac{1}{2}\mathscr{G}(-t), \qquad -\infty < t < \infty.$$

It was proved in [5] that if X is a complex Banach space and an $\mathscr{L}(X)$-valued cosine function is bounded on $(-\infty, \infty)$ and continuous in the sence of the norm in $\mathscr{L}(X)$, then this cosine function has an exponential representation. Without the assumption of continuity in the sense of norm in $\mathscr{L}(X)$ a similar theorem is not true. Namely, as shown in [5], if X is the space of all complex impair continuous functions on $(-\infty, \infty)$ having period 2π, or if X is the space of all complex impair functions almost periodic in the sense of Bohr, and if

$$(\mathscr{C}(t)x)(s) = \tfrac{1}{2}x(s+t) + \tfrac{1}{2}x(s-t), \qquad x \in X, \ -\infty < s, t < \infty,$$

then $\mathscr{C}$ has no exponential representation.

In the present paper some other examples of this type will be presented and the results may be summarized as follows. Consider following complex functional spaces on $(-\infty, \infty)$: