Now let $\nu_n^k = \mu_n^1 \dots \mu_n^k$, $\nu_n^0 = 1$, $n, k = 1, 2, \dots$ Then for any $k \geqslant 1$, we have $\zeta^k = (\zeta_n^k)_n \in l_2$ with

$$P_{k+1}\Big(x - \sum_{i=1}^{M} \xi_i b_i\Big) = \sum_{n=1}^{\infty} \zeta_n^k e^n$$

so

$$P_1\Big(x - \sum_{i=1}^{M} \xi_i b_i\Big) = A_1 \dots A_k P_{k+1}\Big(x - \sum_{i=1}^{M} \xi_i b_i\Big) = \sum_{n=1}^{\infty} \nu_n^k \zeta_n^k e^n.$$

Equating coefficients we conclude that

$$\zeta_n^k = 0 \text{ for } n \leqslant M \quad \text{ and } \quad \zeta_n^k = \frac{\xi_n}{\nu_n^k} \text{ for } n > M.$$

In particular, applying this for $M = N$ we conclude that

$$\Big(\frac{\xi_n}{\nu_n^k}\Big)_{n=1}^{\infty} \in l_2.$$

And for arbitrary $M \geqslant N$,

$$\Big\| P_k\Big(x - \sum_{i=1}^{M} \xi_i b_i\Big)\Big\| = \Big\| \sum_{n=M+1}^{\infty} \frac{\xi_n}{\nu_n^k} e^n \Big\| = \Big(\sum_{n=M+1}^{\infty} \Big|\frac{\xi_n}{\nu_n^k}\Big|^2\Big)^{\frac{1}{2}}.$$

The last term goes to 0 as M goes to ∞ and this implies

$$\lim_{M\to\infty} \Big\| P_k\Big(x - \sum_{i=1}^{M} \xi_i b_i\Big)\Big\| = 0 \quad \text{so that} \quad x = \sum_{i=1}^{\infty} \xi_i b_i.$$

Remark 5. The case described above is a very primitive example. To go further, it would be very interesting to see what happens if N varies with respect to k and moreover if this approach could be used to approximate an arbitrary lower triangular matrix. Finally one could investigate the connection between upper and lower triangular matrices.

References

[1] Ed Dubinsky, *Equivalent nuclear systems*, Stud. Math., 38 (1970), pp. 373–379.
[2] — *A new definition of nuclear systems with applications to bases in nuclear spaces* Studia Math., 41 (1972) pp. 149–161.
[3] W. B. Johnson, *Markuschevich bases and duality theory*, Trans. Amer. Math. Soc., 149 (1970), pp. 1–7.
[4] A. Pietsch, *Über die Erzeugung von (F)-Räumen durch selbstadjungierte Operatoren*, Math. Ann., 164 (1966), pp. 219–224.
[5] I. Singer, *Bases in Banach spaces*, Heidelberg 1970.

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On operator-valued solutions of d'Alembert's functional equation, II

by

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Abstract. In the paper several negative examples are given, connected with the problem of representation of a cosine operator function $\mathscr{C}(t)$ in the form $\mathscr{C}(t) = \frac{1}{2}\mathscr{G}(t) + \frac{1}{2}\mathscr{G}(-t)$, where $\mathscr{G}(t)$ is an one parameter group of operators.

Introduction and results. Let X be a real or complex topological vector space and let $\mathscr{L}_s(X)$ be the space of all linear continuous operators of X into itself with the topology of simple convergence. A continuous mapping $\mathscr{C}$ of $(-\infty, \infty)$ into $\mathscr{L}_s(X)$ is called the cosine operator function if it satisfies the d'Alembert functional equation

$$\mathscr{C}(t+s) + \mathscr{C}(t-s) = 2\mathscr{C}(t)\mathscr{C}(s), \quad -\infty < s, t < \infty$$

and if, moreover,

$$\mathscr{C}(0) = 1.$$

We shall say that $\mathscr{C}$ has an exponential representation if there is a one-parameter continuous group $\{\mathscr{G}(t)\colon -\infty < t < \infty\} \subset \mathscr{L}_s(X)$ such that

$$\mathscr{C}(t) = \tfrac{1}{2}\mathscr{G}(t) + \tfrac{1}{2}\mathscr{G}(-t), \quad -\infty < t < \infty.$$

It was proved in [5] that if X is a complex Banach space and an $\mathscr{L}(X)$-valued cosine function is bounded on $(-\infty, \infty)$ and continuous in the sence of the norm in $\mathscr{L}(X)$, then this cosine function has an exponential representation. Without the assumption of continuity in the sense of norm in $\mathscr{L}(X)$ a similar theorem is not true. Namely, as shown in [5], if X is the space of all complex impair continuous functions on $(-\infty, \infty)$ having period 2π, or if X is the space of all complex impair functions almost periodic in the sense of Bohr, and if

$$\big(\mathscr{C}(t)x\big)(s) = \tfrac{1}{2}x(s+t) + \tfrac{1}{2}x(s-t), \quad x \in X, \ -\infty < s, t < \infty,$$

then $\mathscr{C}$ has no exponential representation.

In the present paper some other examples of this type will be presented and the results may be summarized as follows. Consider following complex functional spaces on $(-\infty, \infty)$:

$C_u(-\infty, \infty) = \{$all bounded uniformly continuous functions$\}$,
$C[-\infty, \infty] = \{$all continuous functions with finite limits at $-\infty$ and at $\infty\}$,
$C_0(-\infty, \infty) = \{$all continuous functions with limits zero at $-\infty$ and at $\infty\}$,
$C_{2\pi} = \{$all continuous functions having period $2\pi\}$,
$AP =$ the space of almost periodic functions in the sense of Bohr,
$L^p(-\infty, \infty)$, $1 \leqslant p \leqslant \infty$,
$L^p_{2\pi} = \{$all functions locally integrable with power p and having period $2\pi\}$, $1 \leqslant p \leqslant \infty$,
$M(-\infty, \infty) =$ the space of all bounded Borel measures,
$M_{2\pi} =$ the space of all finite Borel measures having period 2π.

We shall regard $M(-\infty, \infty)$, $M_{2\pi}$, $L^\infty(-\infty, \infty)$ and $L^\infty_{2\pi}$ as adjoint spaces of $C_0(-\infty, \infty)$, $C_{2\pi}$, $L^1(-\infty, \infty)$ and $L^1_{2\pi}$ with the $*$-weak topology. The remaining of the above spaces will be regarded as Banach spaces with the norm topology.

Let X be any of the spaces listed above, let $\{T(t)\colon -\infty < t < \infty\} \subset \mathscr{L}(X)$ be the one-parameter group of left translations and let X_{imp} denote the subspace of all impair elements of X. We shall say that X has property (E) if the $\mathscr{L}(X_{\text{imp}})$-valued cosine function

$$\mathscr{C}_0(t) = \tfrac{1}{2}[T(t)+T(-t)]|X_{\text{imp}}$$

has an exponential representation.

THEOREM. *All spaces L^p and $L^p_{2\pi}$ with $1 < p < \infty$ have property* (E). *The spaces $C_u(-\infty, \infty)$, $C[-\infty, \infty]$, $C_0(-\infty, \infty)$, $C_{2\pi}$, AP, $L^1(-\infty, \infty)$, $L^1_{2\pi}$, $M(-\infty, \infty)$, $M_{2\pi}$, $L^\infty(-\infty, \infty)$ and $L^\infty_{2\pi}$ do not have property* (E).

The proof will be given in several sections, devoted to various spaces.

The author expresses his warmest thanks to professor C. Ryll-Nardzewski, who suggested that the problem of the existence of an exponential representation for the cosine function $\mathscr{C}_0(t)$ may be interesting not only for the space $C_{2\pi,\text{imp}}$, considered in [5], but also for other spaces of impair functions. Also to professor C. Ryll-Nardzewski the author owes many technical hints and some important parts of the proofs.

1. The spaces $L^p_{2\pi}$ and $L^p(-\infty, \infty)$, $1 < p < \infty$, have property (E). Let $1 < p < \infty$ and $X = L^p_{2\pi}$ or $X = L^p(-\infty, \infty)$. Let

$$(Rx)(s) = x(-s), \quad x \in X, \ -\infty < s < \infty.$$

Then there is a projector $P \in \mathscr{L}(X)$ such that

(a) $$T(t)P = PT(t), \quad -\infty < t < \infty$$

and

(b) $$T(t)(RP+PR-R) \quad \text{does not depend of sign}\, t.$$

Indeed, if $X = L^p_{2\pi}$, then, according to a theorem of M. Riesz (of. [4], Chapter 9), there is a projector $P \in \mathscr{L}(L^p_{2\pi})$ such that $Px_n = x_n$ for $n = 0, 1, \ldots$ and $Px_n = 0$ for $n = -1, -2, \ldots$, where $x_n(s) = e^{ins}$. It is easy to see that this projector satisfies (a) and (b). If $X = L^p(-\infty, \infty)$ then, according to another theorem of M. Riesz (cf. [1], Chapter XI, § 7, Theorem 8), the Hilbert transformation H belongs to $\mathscr{L}(X)$. As known, $H^2 = -1$, $HR+RH = 0$ and $HT(t) = T(t)H$. Therefore $P = \frac{1}{2}+iH/2$ is a projector satisfying (a) and (b).

Now we may proceed simultaneously for $X = L^p_{2\pi}$ and $X = L^p(-\infty, \infty)$. Let $P \in \mathscr{L}(X)$ be a projector satisfying (a) and (b) and put

$$G(t) = T(t)P+T(-t)(1-P).$$

By (a) we see that $\{G(t)\colon -\infty < t < \infty\} \subset \mathscr{L}_s(X)$ is a continuous one-parameter group. By (b) we have

$$RG(t) = T(-t)RP+T(t)R(1-P) = T(-t)(1-P)R+T(t)PR = G(t)R,$$

which, in view of the equality $X_{\text{imp}} = \{x\colon x \in X, Rx = -x\}$, implies that

$$G(t)X_{\text{imp}} \subset X_{\text{imp}}, \quad -\infty < t < \infty.$$

Therefore, if $\mathscr{G}(t) = G(t)|X_{\text{imp}}$ then $\{\mathscr{G}(t)\colon -\infty < t < \infty\} \subset \mathscr{L}(X_{\text{imp}})$ is a continuous one-parameter group such that

$$\mathscr{C}_0(t) = \tfrac{1}{2}\mathscr{G}(t)+\tfrac{1}{2}\mathscr{G}(-t).$$

2. The spaces $C_{2\pi}$, AP, $L^1_{2\pi}$, $L^\infty_{2\pi}$ and $M_{2\pi}$ do not have property (E). For $C_{2\pi}$ and AP this was proved in [5]. Following the argumentation used there for $C_{2\pi}$, we shall give the complete proof for $C_{2\pi}$, $L^\infty_{2\pi}$, $L^1_{2\pi}$ and $M_{2\pi}$. Let X denote any of these four spaces. For every $n = 1, 2, \ldots$ let $x_n(s) = \sin ns$; let

$$P_n x = \left(\frac{2}{\pi}\int_0^\pi x_n(s)x(s)\,ds\right)x_n, \quad x \in X_{\text{imp}}$$

in the case when $X = C_{2\pi}$, $L^\infty_{2\pi}$ or $L^1_{2\pi}$ and let

$$P_n x = \left(\frac{2}{\pi}\int_0^\pi x_n(s)x(ds)\right)x_n, \quad x \in M_{2\pi,\text{imp}},$$

when $X = M_{2\pi}$. Then $P_n \in \mathscr{L}(X_{\text{imp}})$ is a projector and, since

(a) $$\mathscr{C}_0(t)x_n = (\cos nt)x_n,$$

we have

$$\text{(b)} \qquad P_n = \frac{2}{\pi}\int_0^{\pi} \cos nt\, \mathscr{C}_0(t)\,dt.$$

Suppose on the contrary to the assertion that there is a continuous one-parameter group $\{\mathscr{G}(t)\colon -\infty < t < \infty\} \subset \mathscr{L}(X_{\rm imp})$ such that $\mathscr{C}_0(t) = \frac{1}{2}\mathscr{G}(t)+\frac{1}{2}\mathscr{G}(-t)$. Then, by (b), $P_n\mathscr{G}(t) = \mathscr{G}(t)P_n$ and consequently, by (a), for every $n = 1, 2, \ldots$ we have

$$\text{(c)} \qquad \mathscr{G}(t)P_n = e^{\varepsilon_n int}P_n, \qquad -\infty < t < \infty,$$

where $\varepsilon_n = 1$ or $\varepsilon_n = -1$ do not depend on t. Assume additionally that $\varepsilon_0 = 0$ and $\varepsilon_n = -\varepsilon_{-n}$ for $n = -1, -2, \ldots$ and consider the impair periodic distribution S with Fourier series

$$S \sim \sum_{n=-\infty}^{\infty} \varepsilon_n e^{ins}.$$

According to a theorem of Helson [3], if the sequence of Fourier coefficients of a measure on $[0, 2\pi)$ consists of finitely many distinct values only, then this sequence may be made periodic by a change of a finite number of its elements. The sequence $\{\varepsilon_n\}$ takes only the values $-1, 0$ and 1, but since $|\varepsilon_n| = 1$ for all $m \neq 0$ and $\{\varepsilon_n\}$ is impair, it cannot be made periodic by such a change. Therefore S is not a measure.

Now we shall obtain a contradiction by showing that S is a measure. We shall use here the argumentation given by professor C. Ryll-Nardzewski. We have

$$\frac{2}{\pi}\int_0^{\pi} e^{\varepsilon_n int}\sin n(s+t)\,dt = -i\varepsilon_n e^{-\varepsilon_n ins} = (x_n - S*x_n)(s)$$

and, by (c)

$$(T(t)\mathscr{G}(t)x_n)(s) = e^{\varepsilon_n int}\sin n(s+t),$$

so that

$$S*x = x - \frac{2}{\pi}\int_0^{\pi} T(t)\mathscr{G}(t)x\,dt$$

for every impair trigonometric polynomial x. Hence, by an application of the Banach–Steinhaus theorem, we infer that there is a constant K such that

$$\text{(d)} \qquad \|S*x\|_X \leqslant K\,\|x\|_X$$

for every impair trigonometric polynomial x. If $X = C_{2\pi}$ or $X = L^{\infty}_{2\pi}$ then, S being impair, we have

$$|\langle S, x\rangle| = |\langle S, x_{\rm imp}\rangle| = |-(S*x_{\rm imp})(0)| \leqslant K\|x\|_X$$

for every trigonometric polynomial x with the impair part $x_{\rm imp}$ and this implies that S is a measure.

If $X = L^1_{2\pi}$ or $X = M_{2\pi}$, then let y_n, $n = 1, 2, \ldots$ be an approximative unit in the convolution algebra $L^1_{2\pi}$, such that y_n are pair trigonometric polynomials and $\|y_n\| \leqslant 2$ for $n = 1, 2, \ldots$ Then, for every fixed $t \in (-\infty, \infty)$

$$S_{t,n} = \left(T\left(\frac{t}{2}\right)S\right) * \left(T\left(\frac{t}{2}\right)y_n - T\left(-\frac{t}{2}\right)y_n\right) \to T(t)S - S$$

as $n \to \infty$, in the sense of distributional convergence. On the other hand, applying (d) to the impair trigonometric polynomial $x = T\left(\frac{t}{2}\right)y_n - T\left(-\frac{t}{2}\right)y_n$, we see that

$$\|S_{t,n}\|_{L^1_{2\pi}} \leqslant \text{const}$$

for every $n = 1, 2, \ldots$ and $t \in (-\infty, \infty)$. It follows that

$$\text{(e)} \qquad T(t)S - S \in M_{2\pi} \quad \text{and} \quad \|T(t)S - S\|_{M_{2\pi}} \leqslant \text{const}$$

for every $t \in (-\infty, \infty)$. Since S is impair, we have $\int_0^{2\pi} T(t)S\,dt = 0$ and thus

$$S = \frac{1}{2\pi}\int_0^{2\pi} (S - T(t)S)\,dt$$

in the sense of distributional convergence of the Riemann approximating sums. On the other hand, by (e), these approximating sums form a bounded set in $M_{2\pi}$. It follows that $S \in M_{2\pi}$.

3. Some lemmas on cosine operator functions. In this section X always denotes a sequentially complete real or complex linear locally convex space and our reasonings will base on a boundedness principle formulated in Theorem 7.4.4 of the book of Edwards [2]. Throughout this section $\mathscr{C}$ denotes a $\mathscr{L}_s(X)$-valued continuous cosine function. According to Sova [6], the infinitesimal generator of $\mathscr{C}$ is the linear operator A defined by the conditions

$$\mathscr{D}(A) = \left\{x\colon x \in X,\ \lim_{t\to 0}\frac{2}{t^2}(\mathscr{C}(t)x - x) \text{ exists in } X\right\},$$

$$Ax = \lim_{t\to 0}\frac{2}{t^2}(\mathscr{C}(t)x - x) \quad \text{for } x \in \mathscr{D}(A).$$

LEMMA 1. *The operator A is sequentially closed and its domain $\mathscr{D}(A)$ is sequentially dense in X. If $x \in \mathscr{D}(A)$, then $\mathscr{C}(t)x$ is an X-valued function of t, twice continuously differentiable on $(-\infty, \infty)$ and such that $\mathscr{C}(t)x \in \mathscr{D}(A)$ and $\frac{d^2}{dt^2}\mathscr{C}(t)x = A\mathscr{C}(t)x = \mathscr{C}(t)Ax$ for every $t \in (-\infty, \infty)$.*

Proof. For every X-valued function f continuous on $(-\infty, \infty)$ and every $h > 0$ we have

$$\frac{d}{dt}\int_{-h}^{0}\int_{0}^{h} f(t+u+v)\,du\,dv = \Big(\int_{t}^{t+h} - \int_{t-h}^{t}\Big) f(v)\,dv$$
$$= \Big(\int_{h}^{t+h} + \int_{-h}^{t-h} - 2\int_{0}^{t}\Big) f(v)\,dv + \Big(\int_{0}^{h} - \int_{-h}^{0}\Big) f(v)\,dv$$
$$= \int_{0}^{t} \big(f(v+h)+f(v-h)-2f(v)\big)\,dv +$$
$$+ \int_{0}^{h} \big(f(v)-f(-v)\big)\,dv$$

and therefore

$$\int_{-h}^{0}\int_{0}^{h} \big(f(t+u+v)-f(u+v)\big)\,du\,dv$$
$$= t\int_{0}^{h} \big(f(v)-f(-v)\big)\,dv + \int_{0}^{t}\int_{0}^{u} \big(f(v+h)+f(v-h)-2f(v)\big)\,dv\,du.$$

We shall apply the former equality to $f(t) = \mathscr{C}(t)x$. Since, by the d'Alembert equation, $\mathscr{C}(t) = \mathscr{C}(-t)$ and

$$\text{(a)}\qquad \frac{1}{h^2}\big(\mathscr{C}(t+h)+\mathscr{C}(t-h)-2\mathscr{C}(t)\big) = \mathscr{C}(t)\frac{2}{h^2}\big(\mathscr{C}(h)-1\big)$$
$$= \frac{2}{h^2}\big(\mathscr{C}(h)-1\big)\mathscr{C}(t),$$

we obtain

$$\text{(b)}\qquad \frac{1}{h^2}\int_{-h}^{0}\int_{0}^{h} \big(\mathscr{C}(t+u+v)x - \mathscr{C}(u+v)x\big)\,du\,dv$$
$$= \int_{0}^{t}\int_{0}^{u} \mathscr{C}(v)\frac{2}{h^2}\big(\mathscr{C}(h)x-x\big)\,dv\,du = \frac{2}{h^2}\big(\mathscr{C}(h)-1\big)\int_{0}^{t}\int_{0}^{u} \mathscr{C}(v)x\,dv\,du.$$

Passing in the first and last term of (b) to the limit as $h \to 0$, we infer that $\int_0^t\int_0^u \mathscr{C}(v)x\,dv\,du \in \mathscr{D}(A)$ for every $t \in (-\infty, \infty)$ and $x \in X$. Because $\lim_{t\to 0}\frac{2}{t^2}\int_0^t\int_0^u \mathscr{C}(v)x\,dv\,du = x$ for every $x \in X$, it follows that $\mathscr{D}(A)$ is sequentially dense in X.

Now we show that

$$\text{(c)}\qquad \mathscr{C}(t)x - x = \int_{0}^{t}\int_{0}^{u} \mathscr{C}(v)Ax\,dv\,du \qquad \text{for every } x \in \mathscr{D}(A) \text{ and } t \in (-\infty, \infty).$$

This is obvious for $t = 0$ and since the argumentation for $t < 0$ is similar to that for $t > 0$, let us assume that $t > 0$ is fixed. Let $x \in \mathscr{D}(A)$ also be fixed. Then, by the continuity of $\mathscr{C}$ and by the definition of A, it follows that $B = \Big\{\frac{2}{h^2}\big(\mathscr{C}(h)x-x\big)\colon 0 < |h| \leqslant 1\Big\}$ is a bounded subset of X and, moreover, for any $y \in X$ the set $\{\mathscr{C}(v)y\colon 0 \leqslant v \leqslant t\}$ is a bounded subset of X. Therefore, according to the boundedness principle for locally convex sequentially complete spaces, formulated in Theorem 7.4.4 of the book of Edwards [2], the set $\bigcup\{\mathscr{C}(v)B\colon 0 \leqslant v \leqslant t\} = \Big\{\mathscr{C}(v)\frac{2}{h^2}\big(\mathscr{C}(h)x-x\big)\colon 0 \leqslant v \leqslant t,\ 0 < |h| \leqslant 1\Big\}$ also is a bounded subset of X. Moreover, we have $\lim_{h\to 0}\mathscr{C}(v)\frac{2}{h^2}\big(\mathscr{C}(h)x-x\big) = \mathscr{C}(v)Ax$ for every $v \in [0, t]$ and therefore, by the Lebesgue bounded convergence theorem, we see that $\lim_{h\to 0}\int_0^t\int_0^u \mathscr{C}(v)\frac{2}{h^2}\big(\mathscr{C}(h)x-x\big)\,dv\,du = \int_0^t\int_0^u \mathscr{C}(v)Ax\,dv\,du$. Thus (c) follows by passing in the first and second term of (b) to the limit as $h \to 0$.

By a similar argumentation, based on the Lebesgue bounded convergence theorem, we obtain from (c) that if a pair $(x, y) \in X \times X$ lies in the sequential closure of the graph of A, then $\frac{2}{t^2}\big(\mathscr{C}(t)x-x\big) = \frac{2}{t^2}\int_0^t\int_0^u \mathscr{C}(v)y\,dv\,du$ for $t \neq 0$. Passing in this equality to the limit as $t \to 0$, we obtain that $x \in \mathscr{D}(A)$ and $Ax = y$, which proves that A is sequentially closed.

If $x \in \mathscr{D}(A)$, then it follows from (a) that $\mathscr{C}(t)x \in \mathscr{D}(A)$ and $A\mathscr{C}(t)x = \mathscr{C}(t)Ax$, and from (c) it follows that $\frac{d^2}{dt^2}\mathscr{C}(t)x = \mathscr{C}(t)Ax$ for every $t \in (-\infty, \infty)$. This completes the proof.

LEMMA 2. *If K is a sequentially closed operator with domain $\mathscr{D}(K)$ sequentially closed in X and range in X, such that $K \subset A$ and $\mathscr{C}(t)\mathscr{D}(K) \subset \mathscr{D}(K)$ for every $t \in (-\infty, \infty)$, then $K = A$.*

Proof. We have to prove that $\mathscr{D}(A) \subset \mathscr{D}(K)$. Since $K \subset A$ and $\mathscr{C}(t)\mathscr{D}(K) \subset \mathscr{D}(K)$, so, by Lemma 1, for every $x \in \mathscr{D}(K)$ and $t \in (-\infty, \infty)$ we have $\frac{d^2}{dt^2}\mathscr{C}(t)x = K\mathscr{C}(t)x = \mathscr{C}(t)Kx$ and consequently $\mathscr{C}(t)x - x = \int_0^t\int_0^u K\mathscr{C}(v)x\,dv\,du$. Since in the last equality $\mathscr{C}(v)x$ and $K\mathscr{C}(v)x$ are continuous functions of v and since K is sequentially closed, we may transport K before the integrals. We then obtain that

$$(*) \qquad \int_0^t\int_0^u \mathscr{C}(v)x\,dv\,du \in \mathscr{D}(K) \quad \text{and} \quad K\int_0^t\int_0^u \mathscr{C}(v)x\,dv\,du = \mathscr{C}(t)x - x$$

for every $x \in \mathscr{D}(K)$ and $t \in (-\infty, \infty)$. Let now $x \in X \setminus \mathscr{D}(K)$. Because $\mathscr{D}(K)$ is sequentially dense in X, so there is a sequence $\{x_n\}$, $n = 1, 2, \ldots$ such that $x_n \in \mathscr{D}(K)$ and $\lim_{n\to\infty} x_n = x$. Then $\lim_{n\to\infty} K\int_0^t\int_0^u \mathscr{C}(v)x_n\,dv\,du = \lim_{n\to\infty}(\mathscr{C}(t)x_n - x_n) = \mathscr{C}(t)x - x$ and $\lim_{n\to\infty}\int_0^t\int_0^u \mathscr{C}(v)x_n\,dv\,du = \int_0^t\int_0^u \mathscr{C}(v)x\,dv\,du$, by a boundedness principle for locally convex sequentially complete spaces from the book of Edwards and by the Lebesque bounded convergence theorem. Hence by the sequential closedness of K, it follows that $(*)$ holds true for every $x \in X$ and $t \in (-\infty, \infty)$. Now it is easy to finish the proof. Indeed, if $x \in \mathscr{D}(A)$ and $x_t = \frac{2}{t^2}\int_0^t\int_0^u \mathscr{C}(v)x\,dv\,du$, then $\lim_{t\to 0} x_t = x$ and, by $(*)$, $x_t \in \mathscr{D}(K)$ and $\lim_{t\to 0} Kx_t = \lim \frac{2}{t^2}(\mathscr{C}(t)x - x) = Ax$, which, K being sequentially closed, implies that $x \in \mathscr{D}(K)$.

Lemma 3. *If $\mathscr{C}(t) = \frac{1}{2}G(t) + \frac{1}{2}G(-t)$, where $\{G(t)\colon -\infty < t < \infty\} \subset \mathscr{L}_s(X)$ is an one-parameter continuous group with infinitesimal generator B, then $B^2 = A$.*

Proof. Let us recall that

$$\mathscr{D}(B) = \left\{x\colon x \in X,\ \lim_{t\to 0}\frac{1}{t}(G(t)x - x) \quad \text{exists in } X\right\},$$

$$Bx = \lim_{t\to 0}\frac{1}{t}(G(t)x - x) \quad \text{for} \quad x \in \mathscr{D}(B).$$

Then for every $n = 1, 2, \ldots$ we have

(i) $\quad G(t)\mathscr{D}(B^n) \subset \mathscr{D}(B^n)$ and $\quad \frac{d^n}{dt^n}G(t)x = G(t)B^n x = B^n G(t)x$

for every $t \in (-\infty, \infty)$ and $x \in \mathscr{D}(B^n)$. Moreover,

(ii) all the operators B^n, $n = 1, 2, \ldots$, are sequentially closed

and

(iii) $\bigcap_{n=1}^{\infty} \mathscr{D}(B^n)$ is a sequentially dense subset of X.

Having (i), (ii) and (iii), the equality $B^2 = A$ follows immediately by an application of Lemma 2 to $K = B^2$. We may prove (iii) in the same fashion as in the book of Yosida [7], in the case of equicontinuous semi-groups. However, if we want to prove (i) or (ii) without the assumption of equicontinuity, we have to use a new argument.

Ad (i). We shall proceed by induction in n. If we assume that $\mathscr{D}(B^0) = X$ and $B^0 = 1$ then (i) is trivial for $n = 0$. Suppose now that (ii) is true for a certain $n \geqslant 0$ and let $x \in \mathscr{D}(B^{n+1})$. Then, for every $h > 0$ we have

$$\frac{1}{h}\int_0^h \left(\frac{d^n}{du^n}G(t+u)x - \frac{d^n}{du^n}G(u)x\right)du = \int_0^t G(u)\frac{1}{h}(G(h)-1)B^n x\,du$$

and, passing to the limit as $h \to 0$, we obtain that, for every $t \in (-\infty, \infty)$,

$$(*) \qquad \frac{d^n}{du^n}G(u)x\Big|_{u=0}^{u=t} = \int_0^t G(u)B^{n+1}x\,du.$$

Indeed, the only non-trivial point in this limit passage is that

$$\lim_{h\to 0}\int_0^t G(u)\frac{1}{h}(G(h)-1)B^n x\,du = \int_0^t G(u)B^{n+1}x\,du,$$

and this may be proved by an application of the boundedness principle from the book of Edwards and the Lebesque bounded convergence theorem, similarly as this was done in the deduction of (c) in the proof of our Lemma 1. From $(*)$ it follows by a differentiation that if $x \in \mathscr{D}(B^{n+1})$, then $\frac{d^{n+1}}{dt^{n+1}}G(t)x = G(t)B^{n+1}x$ for every $t \in (-\infty, \infty)$. On the other hand, if $x \in \mathscr{D}(B^{n+1})$ then for $y = B^n x \in \mathscr{D}(B)$ we have

$$G(t)B^{n+1}x = G(t)By = \lim_{h\to 0} G(t)\frac{1}{h}(G(h)y - y) = \lim_{h\to 0}\frac{1}{h}(G(h)-1)G(t)y$$

so that $B^n G(t)x = G(t)B^n x = G(t)y \in \mathscr{D}(B)$ and $G(t)B^{n+1}x = BG(t)y = B^{n+1}G(t)x$. Thus (i) is proved.

Ad (ii). It follows from (i) that, for $x \in \mathscr{D}(B^n)$,

$$G(t)x = x + tBx + \ldots + \frac{t^{n-1}}{(n-1)!}B^{n-1}x + \int_0^t \frac{(t-u)^{n-1}}{(n-1)!}G(u)B^n x\,du.$$

Applying this formula and proceeding by induction in n, it is easy to prove that for every $n = 1, 2, \ldots$ the operator B^n is sequentially closed. Indeed, if $\mathcal{D}(B^0) = X$, $B^0 = 1$, then B^0 is closed. Let now $n \geq 1$ and suppose that $B, B^2, \ldots$ and B^{n-1} all are closed. Let the pair $(x, y) \in X \times X$ lie in the sequential closure of the graph of B^n. Then there is a sequence $\{x_k\}$, $k = 1, 2, \ldots$ such that $x_k \in \mathcal{D}(B^n)$, $\lim_{k\to\infty} x_k = x$ and $\lim_{k\to\infty} B^n x_k = y$. By the boundedness principle from the book of Edwards and by the Lebesque bounded convergence theorem, we have

$$\lim_{k\to\infty} \int_0^t \frac{(t-u)^{n-1}}{(n-1)!} G(u) B^n x_k du = \int_0^t \frac{(t-u)^{n-1}}{(n-1)!} G(u) y\, du$$

for every $t \in (-\infty, \infty)$. Since obviously $\lim_{k\to\infty} G(t)x_k = G(t)x$, it follows that $\lim_{k\to\infty}\left(x_k + tBx_k + \ldots + \frac{t^{n-1}}{(n-1)!} B^{n-1}x_k\right)$ exists for every t. Therefore $\lim_{k\to\infty} B^m x_k$ exists for every $m = 1, 2, \ldots, n-1$ and, B^m being closed, $x \in \mathcal{D}(B^m)$ and $\lim_{k\to\infty} B^m x_k = B^m x$. Therefore $x \in \mathcal{D}(B^{n-1})$ and

$$G(t)x = x + tBx + \ldots + \frac{t^{n-1}}{(n-1)!} B^{n-1}x + \int_0^t \frac{(t-u)^{n-1}}{(n-1)!} G(u) y\, du$$

for every $t \in (-\infty, \infty)$. Differentiating this $n-1$ times and using (i), we obtain that $\frac{1}{t}\big(G(t)-1\big)B^{n-1}x = \frac{1}{t}\int_0^t G(u) y\, du$ for every $t \neq 0$. Finally passing in the former equality to the limit as $t \to 0$, we infer that $B^{n-1}x \in \mathcal{D}(B)$ and $B^n x = y$, so that B^n is sequentially closed.

LEMMA 4. *Let $\mathcal{K}$ denote the field of scalars of the linear structure of X and define the $\mathcal{L}(X \times \mathcal{K})$ — valued cosine function $\tilde{\mathscr{C}}$ by the formula*

$$\tilde{\mathscr{C}}(t)(x, \lambda) = \big(\mathscr{C}(t)x, \lambda\big), \quad x \in X,\ \lambda \in \mathcal{K},\ -\infty < t < \infty.$$

Suppose that $\tilde{\mathscr{C}}$ has an exponential representation and that A is invertible. Then $\mathscr{C}$ also has an exponential representation.

The invertibility of A is essential in this lemma. Indeed, let $X = C \times C$ be the two-dimensional complex space and let $\mathscr{C}(t) = \begin{pmatrix} 1 & \frac{1}{2}t^2 \\ 0 & 1 \end{pmatrix}$. Then $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has no square root and therefore $\mathscr{C}$ has no exponential representation. On the other hand, in this case

$$\tilde{\mathscr{C}}(t) = \tfrac{1}{2}\exp\left(t\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right) + \tfrac{1}{2}\exp\left(-t\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right).$$

Proof of Lemma 4. We shall write elements of $X \times \mathcal{K}$ in the form of columns $\begin{pmatrix} x \\ \lambda \end{pmatrix}$, where $x \in X$ and $\lambda \in \mathcal{K}$. Let $\{\tilde{\mathscr{G}}(t)\colon -\infty < t < \infty\} \subset \mathcal{L}_s(X \times \mathcal{K})$ be a continuous one-parameter group such that

$$\tfrac{1}{2}\tilde{\mathscr{G}}(t) + \tfrac{1}{2}\tilde{\mathscr{G}}(-t) = \tilde{\mathscr{C}}(t)$$

and let $\tilde{\mathscr{B}}$ be its infinitesimal generator. Then, by Lemma 3, $\tilde{\mathscr{B}}^2$ is the infinitesimal generator of $\tilde{\mathscr{C}}$ and so $\mathcal{D}(\tilde{\mathscr{B}}^2) = \mathcal{D}(A) \times \mathcal{K}$. It follows that $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathcal{D}(\tilde{\mathscr{B}}^2) \subset \mathcal{D}(\tilde{\mathscr{B}})$ and thus

$$\mathcal{D}(\tilde{\mathscr{B}}) = L \times \mathcal{K},$$

where L is a dense linear subset of X. Therefore we may represent $\tilde{\mathscr{B}}$ in the form of a matrix

$$\tilde{\mathscr{B}} = \begin{pmatrix} B & x_0 \\ l & \lambda_0 \end{pmatrix},$$

where B is a linear operator defined on L with values in X, $x_0 \in X$, $\lambda_0 \in \mathcal{K}$ and l is a linear form on L. If $\tilde{x} = \begin{pmatrix} x \\ \lambda \end{pmatrix} \in \mathcal{D}(\tilde{\mathscr{B}}) = L \times \mathcal{K}$ then, according to the general rule of multiplication of matrices,

$$\tilde{\mathscr{B}}\tilde{x} = \begin{pmatrix} B & x_0 \\ l & \lambda_0 \end{pmatrix}\begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} Bx + \lambda x_0 \\ lx + \lambda\lambda_0 \end{pmatrix}.$$

Since $\tilde{\mathscr{B}}^2$ is the infinitesimal generator of $\tilde{\mathscr{C}}$, we have

$$\tilde{\mathscr{B}}^2 = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \quad \text{on } \mathcal{D}(\tilde{\mathscr{B}}^2) = \mathcal{D}(A) \times \mathcal{K}$$

which implies that

(i) $$B\mathcal{D}(A) \subset L$$

and

(ii) $$\begin{pmatrix} B^2 + lx_0, & Bx_0 + \lambda_0 x_0 \\ lB + \lambda_0 l, & lx_0 + \lambda_0^2 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \quad \text{on } \mathcal{D}(A) \times \mathcal{K}.$$

Since $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathcal{D}(A^2) \times \mathcal{K} = \mathcal{D}(\tilde{\mathscr{B}}^4) \subset \mathcal{D}(\tilde{\mathscr{B}}^3)$, we have $\begin{pmatrix} x_0 \\ \lambda_0 \end{pmatrix} = \tilde{\mathscr{B}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathcal{D}(\tilde{\mathscr{B}}^2) = \mathcal{D}(A) \times \mathcal{K}$ and consequently

(iii) $$x_0 \in \mathcal{D}(A).$$

It follows from (i)–(iii) that

$$Ax_0 = B^2x_0+(lx_0)x_0 = B^2x_0-\lambda_0^2x_0 = (B-\lambda_0)(Bx_0+\lambda_0x_0) = 0$$

and so $x_0 = 0$, since A is invertible. Now, as we already know that $x_0 = 0$, we can see from (ii) that also $\lambda_0 = 0$. Thus

$$\tilde{\mathscr{B}} = \begin{pmatrix} B & 0 \\ l & 0 \end{pmatrix} \quad \text{on } \mathscr{D}(\tilde{\mathscr{B}}) = L\times\mathscr{K},$$

so that $\frac{d}{dt}\tilde{\mathscr{G}}(t)\begin{pmatrix}0\\ \lambda\end{pmatrix} = \tilde{\mathscr{G}}(t)\tilde{\mathscr{B}}\begin{pmatrix}0\\ \lambda\end{pmatrix} = 0$ and consequently

$$\text{(iv)} \qquad \tilde{\mathscr{G}}(t)\begin{pmatrix}x\\ \lambda\end{pmatrix} = \tilde{\mathscr{G}}(t)\begin{pmatrix}x\\ 0\end{pmatrix}+\begin{pmatrix}0\\ \lambda\end{pmatrix}$$

for every $x\in X$, $\lambda\in\mathscr{K}$ and $t\in(-\infty,\infty)$.

Let now P denote the natural projection of $X\times\mathscr{K}$ onto X and let $\mathscr{I}$ denote the natural imbedding of X into $X\times\mathscr{K}$, i.e.

$$P\begin{pmatrix}x\\ \lambda\end{pmatrix} = x, \qquad \mathscr{I}x = \begin{pmatrix}x\\ 0\end{pmatrix} \quad \text{for } x\in X.$$

Put

$$\mathscr{G}(t) = P\tilde{\mathscr{G}}(t)\mathscr{I}.$$

Then $\mathscr{G}(0) = 1$ in $\mathscr{L}(X)$ and since, by (iv), $P\tilde{\mathscr{G}}(t) = P\tilde{\mathscr{G}}(t)\mathscr{I}P$, we have

$$\mathscr{G}(t)\mathscr{G}(s) = P\tilde{\mathscr{G}}(t)\mathscr{I}P\tilde{\mathscr{G}}(t)\mathscr{I} = P\tilde{\mathscr{G}}(t)\tilde{\mathscr{G}}(s)\mathscr{I} = P\tilde{\mathscr{G}}(t+s)\mathscr{I} = \mathscr{G}(t+s).$$

Therefore $\{\mathscr{G}(t)\colon -\infty<t<\infty\}\subset\mathscr{L}_s(X)$ is a continuous one-parameter group. Moreover, $\frac{1}{2}\mathscr{G}(t)+\frac{1}{2}\mathscr{G}(-t) = \frac{1}{2}P\big(\tilde{\mathscr{G}}(t)+\tilde{\mathscr{G}}(-t)\big)\mathscr{I} = P\tilde{\mathscr{C}}(t)\mathscr{I} = \mathscr{C}(t)$, which gives an exponential representation for $\mathscr{C}$.

4. The spaces $L^1(-\infty,\infty)$ and $M(-\infty,\infty)$ do not have property (E). Since our reasonings, except of some details, are the same for $L^1(-\infty,\infty)$ and for $M(-\infty,\infty)$, we admit in this section the convention that X denotes $L^1(-\infty,\infty)$ or $M(-\infty,\infty)$. Any of our statements concerning X should be understood as a statement true simultaneously for $X = L^1(-\infty,\infty)$ and for $X = M(-\infty,\infty)$. Let us recall that we consider $L^1(-\infty,\infty)$ under its norm topology and that we consider $M(-\infty,\infty)$ with the topology of weak convergence of measures, i.e. $M(-\infty,\infty)$ is regarded as the adjoint space of $C_0(-\infty,\infty)$ with the $*$-weak topology.

Let H denote the set of all finite linear combinations of Hermite functions $\varphi_n(s) = \frac{d^n}{ds^n}e^{-s^2}$, $n = 0,1,\ldots$ Of course, $H\subset L^1(-\infty,\infty)\subset M(-\infty,\infty)$ in the obvious sense and it is known that H is dense in $L^1(-\infty,\infty)$ in the sense of norm topology (this in particular follows from the statement (I) of our next section) and that $L^1(-\infty,\infty)$ is sequentially dense in $M(-\infty,\infty)$ in the sense of weak convergence of measures. Therefore

(i) $\qquad H_{\mathrm{imp}}$ is sequentially dense in X_{imp},

where H_{imp} denotes the set of all finite linear combinations of the functions φ_n, $n = 1,3,5,\ldots$

The infinitesimal generator A_0 of the $\mathscr{L}(X_{\mathrm{imp}})$-valued cosine function

$$\mathscr{C}_0(t) = \tfrac{1}{2}[T(t)+T(-t)]|X_{\mathrm{imp}}$$

is defined by the equality

$$A_0x = \lim_{h\to 0}\frac{1}{h^2}\big(T(h)x+T(-h)x-2x\big),$$

its domain $\mathscr{D}(A_0)$ being the set of all those elements $x\in X_{\mathrm{imp}}$, for which this limit exists in the sense of topology admitted in X.

Let $\mathscr{D}$ denote the space of all complex-valued infinitely differentiable functions on $(-\infty,\infty)$ with compact supports and let $\mathscr{D}'$ be the corresponding space of distributions. In the obvious sense we have $X\subset\mathscr{D}'$. For any $x\in\mathscr{D}'$ let x'' denote its second distributional derivative. We shall show that

(ii) $\quad \mathscr{D}(A_0) = \{x\colon x\in X_{\mathrm{imp}} \text{ and } x''\in X_{\mathrm{imp}}\}$, $A_0x = x''$ for $x\in\mathscr{D}(A_0)$.

For the proof of (ii), for every real $h\neq 0$ put

$$\delta_h(s) = \max\left(\frac{1}{|h|}-\left|\frac{s}{h^2}\right|,\ 0\right), \qquad -\infty<s<\infty.$$

Then it is easy to verify that

$$\frac{1}{h^2}\big(T(h)\varphi+T(-h)\varphi-2\varphi\big) = \delta_h*\varphi''$$

for every $\varphi\in\mathscr{D}$ and real $h\neq 0$. Now suppose that $x\in X_{\mathrm{imp}}$ and $x''\in X_{\mathrm{imp}}$. Then $T(t)x''$ is an X-valued function of t, continuous in $(-\infty,\infty)$. Therefore, for every real $h\neq 0$ the integral $\int_{-\infty}^{\infty}\delta_h(t)T(t)x''\,dt$ has a sense and, as is easy to prove,

$$(\alpha) \qquad \lim_{h\to 0}\int_{-\infty}^{\infty}\delta_h(t)T(t)x''\,dt = x''$$

in the sense of topology amitted in X. Furthermore, for every $\varphi \in \mathscr{D}$ and real $h \neq 0$, we have

$$\Big\langle \int_{-\infty}^{\infty} \delta_h(t)T(t)x''\,dt, \varphi\Big\rangle = \langle x'', \delta_h * \varphi\rangle = \langle x, \delta_h * \varphi''\rangle$$

$$= \Big\langle x, \frac{1}{h^2}\big(T(h)\varphi + T(-h)\varphi - 2\varphi\big)\Big\rangle = \Big\langle \frac{1}{h^2}\big(T(-h)x + T(h)x - 2x\big), \varphi\Big\rangle$$

so that

$$(\beta)\qquad \int_{-\infty}^{\infty} \delta_h(t)T(t)x''\,dt = \frac{1}{h^2}\big(T(-h)x + T(h)x - 2x\big).$$

It follows from (α) and (β) that, if $x \in X_{\mathrm{imp}}$ and $x'' \in X_{\mathrm{imp}}$, then $x \in \mathscr{D}(A_0)$ and $A_0 x = x''$. On the other hand if $x \in \mathscr{D}(A_0)$, then for every $\varphi \in \mathscr{D}$ we have

$$\langle A_0 x, \varphi\rangle = \lim_{h\to 0} \frac{1}{h^2}\langle\big(T(h)x + T(-h)x - 2x\big), \varphi\rangle$$

$$= \lim_{h\to 0}\Big\langle x, \frac{1}{h^2}\big(T(h)\varphi + T(-h)\varphi - 2\varphi\big)\Big\rangle = \langle x, \varphi''\rangle,$$

so that $A_0 x$ is equal to the second distributional derivative of x. The assertion (ii) is proved.

It follows immediately from (ii) that

$$\text{(iii)}\qquad \varphi_1 \in \bigcap_{n=1}^{\infty} \mathscr{D}(A_0^n) \quad \text{and} \quad \varphi_{2n+1} = A_0^n \varphi_1 \quad \text{for} \quad n = 1, 2, \ldots$$

After this preparation we shall prove the assertion stated in the title of this section. The proof will be by proceeding ad absurdum. We assume that there is a one-parameter continuous group $\{\mathscr{G}(t)\colon -\infty < t < \infty\} \subset \mathscr{L}_s(X_{\mathrm{imp}})$ such that

$$\tfrac{1}{2}\mathscr{G}(t) + \tfrac{1}{2}\mathscr{G}(-t) = \mathscr{C}_0(t).$$

Under this assumption we shall prove some lemmas, which will lead us to a contradiction.

For any $x \in X$ let $\mathscr{F}x$ be its Fourier transform, i.e.

$$(\mathscr{F}x)(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isu} x(s)\,ds, \quad -\infty < u < \infty, \quad \text{if } x \in L^1(-\infty, \infty)$$

and

$$(\mathscr{F}x)(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isu} x(ds), \quad -\infty < u < \infty, \quad \text{if } x \in M(-\infty, \infty).$$

Then $\mathscr{F}x \in C_u(-\infty, \infty)$. In particular,

$$(\mathscr{F}\varphi_1)(u) = \frac{iu}{\sqrt{2}} e^{-\frac{1}{4}u^2}.$$

If $x \in \mathscr{D}(A_0^n)$, then, by (ii),

$$(\mathscr{F}A_0^n x)(u) = (-u^2)^n (\mathscr{F}x)(u).$$

For any real t and u put

$$g_t(u) = \begin{cases} \dfrac{\sqrt{2}}{iu} e^{\frac{1}{4}u^2}\big(\mathscr{F}\mathscr{G}(t)\varphi_1\big)(u) & \text{if } u \neq 0, \\ 1 & \text{if } u = 0. \end{cases}$$

Then, for fixed t, $g_t(u)$ is a pair function of u, continuous in $(-\infty, \infty)$ except, perhaps, of the point $u = 0$.

Lemma A. *For every $t \in (-\infty, \infty)$ and $x \in X_{\mathrm{imp}}$ we have*

$$\mathscr{F}\mathscr{G}(t)x = g_t \mathscr{F}x.$$

Proof. From Lemma 3 of Section 3 and from the property (i) of one-parameter groups stated in the proof of this lemma, it follows that

$$\mathscr{G}(t)\mathscr{D}(A_0^n) \subset \mathscr{D}(A_0^n) \quad \text{and} \quad \mathscr{G}(t)A_0^n x = A_0^n \mathscr{G}(t)x$$

for every $t \in (-\infty, \infty)$, $x \in \mathscr{D}(A_0^n)$ and $n = 1, 2, \ldots$ Therefore, by (iii), for every $n = 1, 2, \ldots$ and $t \in (-\infty, \infty)$ we have $\mathscr{G}(t)\varphi_{2n+1} = A_0^n \mathscr{G}(t)\varphi_1$ and $\big(\mathscr{F}\mathscr{G}(t)\varphi_{2n+1}\big)(u) = (-u^2)^n\big(\mathscr{F}\mathscr{G}(t)\varphi_1\big)(u) = g_t(u)(-u^2)^n(\mathscr{F}\varphi_1)(u) = g_t(u)(\mathscr{F}\varphi_{2n+1})(u)$, so that our lemma is true for every $x \in H_{\mathrm{imp}}$. Let now $x \in X_{\mathrm{imp}} \setminus H_{\mathrm{imp}}$. Then, by (i), there is a sequence $x_1, x_2, \ldots$ of elements of H_{imp} converging to x. For any $t \in (-\infty, \infty)$ we have $\lim_{n\to\infty} \mathscr{G}(t)x_n = \mathscr{G}(t)x$.

If $x \in L^1(-\infty, \infty)$; then it follows that, for every $t \in (-\infty, \infty)$,

$$(\alpha)\qquad \lim_{n\to\infty} \mathscr{F}\mathscr{G}(t)x_n = \mathscr{F}\mathscr{G}(t)x \quad \text{and} \quad \lim_{n\to\infty} \mathscr{F}x_n = \mathscr{F}x$$

in the sense of uniform convergence on $(-\infty, \infty)$ and so, for every $u \in (-\infty, \infty)$,

$$(\beta)\qquad \big(\mathscr{F}\mathscr{G}(t)x\big)(u) = \lim_{n\to\infty}\big(\mathscr{F}\mathscr{G}(t)x_n\big)(u) = \lim_{n\to\infty} g_t(u)(\mathscr{F}x_n)(u) = g_t(u)(\mathscr{F}x)(u).$$

Therefore in the case of $X = L^1(-\infty, \infty)$ our lemma is proved. If $x \in M_{\mathrm{imp}}(-\infty, \infty) \setminus H_{\mathrm{imp}}$ then we cannot assert that the convergence in

(α) and (β) is pointwise, but then (α) still is true in the sense of $*$-weak convergence in $L^\infty(-\infty, \infty)$.

Therefore the proof will be complete, if we show that $g_t \in L^\infty(-\infty, \infty)$, because this implies that (β) holds in the sense of $*$-weak convergence in $L^\infty(-\infty, \infty)$. In order to prove the boundedness of g_t, for any $k > 0$ put

$$\psi_k(s) = \frac{1}{k}\varphi_1\left(\frac{s}{k}\right), \qquad -\infty < s < \infty.$$

Then $\|\psi_k\| = \|\varphi_1\|$, in the sense of norm in $L^1(-\infty, \infty)$. By the closed graph theorem, the operator $\mathscr{G}(t) \in \mathscr{L}(M_{\mathrm{imp}}(-\infty, \infty))$ is continuous also with respect to the norm topology in $M_{\mathrm{imp}}(-\infty, \infty)$. Consequently $\{\mathscr{G}(t)\psi_k\colon k > 0\}$ is a bounded subset of $M(-\infty, \infty)$ and therefore $\{\mathscr{F}\mathscr{G}\psi_k\colon k > 0\}$ is a bounded subset of $C_u(-\infty, \infty)$. But $\psi_k \in L^1_{\mathrm{imp}}(-\infty, \infty)$ and so, as we have stated above, $\mathscr{F}\mathscr{G}(t)\psi_k = g_t\mathscr{F}\psi_k$. Since $(\mathscr{F}\psi_k)(u) = (\mathscr{F}\varphi_1)(ku) = \dfrac{iku}{\sqrt{2}}e^{-\frac{k^2}{4}u^2}$, it follows that $\sup\left\{\left|g_t(u)\dfrac{iku}{\sqrt{2}}e^{-\frac{k^2}{4}u^2}\right| : k > 0,\ -\infty < u < \infty\right\} < \infty$ and consequently, putting $k = \dfrac{1}{|u|}$, we see that g_t is bounded.

Lemma B. *We have $g_t(u) = e^{it|u|}$ for every real u and t or $g_t(u) = e^{-it|u|}$ for every real u and t.*

Proof. For every $x \in X_{\mathrm{imp}}$ we have $g_0\mathscr{F}x = \mathscr{F}\mathscr{G}(0)x = \mathscr{F}x$ and $g_{t+s}\mathscr{F}x = \mathscr{F}\mathscr{G}(t+s)x = \mathscr{F}\mathscr{G}(t)\mathscr{G}(s)x = g_t\mathscr{F}\mathscr{G}(s)x = g_tg_s\mathscr{F}x$, so that. Since $g_t(0) \equiv 1$, we have

(a) $$g_0(u) = 1, \qquad -\infty < u < \infty,$$

and

(b) $$g_{t+s}(u) = g_t(u)g_s(u), \qquad -\infty < s, t, u < \infty.$$

Moreover,

(c) for every fixed $u \in (-\infty, \infty)$ $g_t(u)$ is a function of t measurable on $(-\infty, \infty)$.

Indeed, let $u \neq 0$ be fixed and let $x \in X_{\mathrm{imp}}$ be such that $(\mathscr{F}x)(u) = 1$. Then $g_t(u) = g_t(u)(\mathscr{F}x)(u) = (\mathscr{F}\mathscr{G}(t)x)(u)$. In the case of $X = L^1(-\infty, \infty)$, $\mathscr{F}\mathscr{G}(t)x$ is a continuous $C_0(-\infty, \infty)$-valued function of t and therefore $g_t(u)$ is continuous in t. If $X = M(-\infty, \infty)$ then $\mathscr{F}\mathscr{G}(t)x \in C_u(-\infty, \infty)$ depends on t continuously in the sense of the $*$-weak topology in $L^\infty(-\infty, \infty)$ and so $f_n(t) = n\int\limits_u^{u+1/n}(\mathscr{F}\mathscr{G}(t)x)(v)\,dv$, $n = 1, 2, \ldots$ is a sequence of continuous functions of t, converging pointwise to $g_t(u)$.

It follows from (a), (b) and (c) that for every $u \in (-\infty, \infty)$ there is a complex number $k(u)$ such that

$$g_t(u) = e^{k(u)t}$$

for every $t \in (-\infty, \infty)$. Since for every $x \in X_{\mathrm{imp}}$ we have

$$\begin{aligned}\tfrac{1}{2}(e^{k(u)t} + e^{-k(u)t})(\mathscr{F}x) &= \tfrac{1}{2}\mathscr{F}(\mathscr{G}(t) + \mathscr{G}(-t))x(u)\\ &= (\mathscr{F}\mathscr{C}_0(t)x)(u) = \cos tu(\mathscr{F}x)(u)\end{aligned}$$

and since $e^{k(0)t} = g_t(0) = 1$, it follows that $\frac{1}{2}e^{k(u)t} + \frac{1}{2}e^{-k(u)t} = \cos tu$ for every real u and t and consequently for every real u we have $k(u) = iu$ or $k(u) = -iu$. But we already know that, for every fixed t, $g_t(u) = e^{k(u)t}$ is a pair function of u, continuous everywhere except, perhaps, of the point $u = 0$. It follows that $k(u) = i|u|$ for every u or $k(u) = -i|u|$ for every u, which completes the proof.

Lemma C. *For every $t \in (-\infty, \infty)$ and every $x \in \mathscr{D}_{\mathrm{imp}}$ we have*

$$\mathscr{G}(t)x = \mathscr{C}_0(t)x \pm \tfrac{1}{2}iH(T(t) - T(-t))x,$$

where H is the Hilbert transformation, i.e.

$$(Hx)(s) = \frac{1}{\pi}\,\mathrm{V.\,P.}\int\limits_{-\infty}^{\infty}\frac{x(u)}{s-u}\,du, \qquad -\infty < s < \infty.$$

Proof. As known, $(\mathscr{F}Hx)(u) = i\,\mathrm{sign}\,u(\mathscr{F}x)(u)$ and therefore

$$\begin{aligned}&\big(\mathscr{F}[\mathscr{C}_0(t)x \pm \tfrac{1}{2}iH(T(t) - T(-t))x]\big)(u) =\\ &\quad = \big(\cos tu \mp \tfrac{1}{2}(e^{itu} - e^{-itu})\,\mathrm{sign}\,u\big)(\mathscr{F}x)(u) = e^{\mp it|u|}(\mathscr{F}x)(u).\end{aligned}$$

Now consider the following statement:

(S) *For every $t \in (-\infty, \infty)$ there is $c_t \in (0, \infty)$ such that*

$$\|H(T(t) - T(-t))x\|_{L^1(-\infty,\infty)} \leqslant c_t\|x\|_{L^1(-\infty,\infty)}$$

for every $x \in \mathscr{D}_{\mathrm{imp}}$.

Lemma C implies (S). This is obvious if $X = L^1(-\infty, \infty)$. If $X = M(-\infty, \infty)$ then the operator $(\mathscr{C}_0(t) - \mathscr{G}(t)) \in \mathscr{L}(M_{\mathrm{imp}}(-\infty, \infty))$, continuous in the $*$-weak topology, is, by the closed graph theorem, also continuous with respect to the norm topology in $M_{\mathrm{imp}}(-\infty, \infty)$. On the other hand if $x \in \mathscr{D}_{\mathrm{imp}}$, then $H(T(t) - T(-t))x \in C^\infty$, so that also in this case (S) follows from Lemma 3.

Now we shall show that (S) is not true. Let $0 < a < b$. For any function x defined on (a, b) let

$$(Kx)(s) = x(a+b-s).$$

Let $R_{a,b}$ be the operator of restriction to (a,b) of functions defined on $(-\infty,\infty)$. Let E and F be operators of extension of functions from (a,b) onto $(-\infty,\infty)$ defined by the formulae

$$(Ex)(s)=\begin{cases}x(s) & \text{if } s\in(a,b),\\ 0 & \text{otherwise}\end{cases}\qquad (Fx)(s)=(Ex)(s)-(Ex)(-s)$$

Consider the operators $A\colon \mathscr{D}(a,b)\to C^\infty(a,b)$ and $B\in\mathscr{L}\big(L^1(a,b)\big)$ defined as follows:

$$A=R_{a,b}HE,\ \text{i.e.}\ (Ax)(s)=\frac{1}{\pi}\,\mathrm{V.\,P.}\int\limits_a^b\frac{x(u)}{s-u}\,du,\qquad x\in\mathscr{D}(a,b),\quad s\in(a,b),$$

$$(Bx)(s)=\frac{1}{\pi}\int\limits_a^b\left[\frac{1}{a+b+s-u}-\frac{1}{a+b+s+u}-\frac{1}{s-a-b-u}\right]x(u)\,du,$$
$$x\in L^1(a,b),\quad s\in(a,b).$$

Then, as is easy to verify, $AKx+Bx=R_{a,b}H\big(T(a+b)-T(-a-b)\big)Fx$ or, which is the same, in view of $K^2=1$,

$$Ax=R_{a,b}H\big(T(a+b)-T(-a-b)\big)FKx-BKx$$

for every $x\in\mathscr{D}(a,b)$. From this formula we see that (S) implies the following:

(S′) $\quad\|Ax\|_{L^1(a,b)}\leqslant c\|x\|_{L^1(a,b)},\ c=\text{const},$ for every $x\in\mathscr{D}(a,b)$.

However, (S′) is not true. Indeed, for every $\varepsilon\in\left(0,\dfrac{b-a}{3}\right)$ let $x_\varepsilon\in\mathscr{D}(a,b)$ have the following properties:

$$\operatorname{supp}x_\varepsilon\subset(a,a+3\varepsilon),\qquad 0\leqslant x_\varepsilon(u)\leqslant\frac{1}{\varepsilon},\quad x_\varepsilon(u)=\frac{1}{\varepsilon}\ \text{for}\ u\in[a+\varepsilon,a+2\varepsilon].$$

Then

$$\|x_\varepsilon\|_{L^1(a,b)}\leqslant 3$$

and, on the other hand, if $s\in(a+3\varepsilon,b)$ then

$$(Ax_\varepsilon)(s)=\frac{1}{\pi}\int\limits_a^{a+3\varepsilon}\frac{x(u)}{s-u}\,du\geqslant\frac{1}{\pi\varepsilon}\int\limits_{a+\varepsilon}^{a+2\varepsilon}\frac{du}{s-u}$$
$$=\frac{1}{\pi\varepsilon}\log(s-a-\varepsilon)-\frac{1}{\pi\varepsilon}\log(s-a-2\varepsilon)\geqslant\frac{1}{\pi(s-a-\varepsilon)},$$

so that

$$\|Ax_\varepsilon\|_{L^1(a,b)}\geqslant\frac{1}{\pi}\int\limits_{a+3\varepsilon}^{b}\frac{ds}{s-a-\varepsilon}=\frac{1}{\pi}\log\frac{b-a-\varepsilon}{2\varepsilon}.$$

5. The spaces $C_0(-\infty,\infty)$, $C[-\infty,\infty]$; $C_u(-\infty,\infty)$ and $L^\infty(-\infty,\infty)$ do not have property (E). For $C_0(-\infty,\infty)$ and $C[-\infty,\infty]$ this follows easily from the results of the preceding section and from Lemma 4 of Section 3. Indeed, the adjoint space of $C_{0,\mathrm{imp}}(-\infty,\infty)$ with the $*$-weak topology may be represented as $M_{\mathrm{imp}}(-\infty,\infty)$ with the topology of weak convergence of measures, in the sense that any $m\in M_{\mathrm{imp}}(-\infty,\infty)$ defines a continuous linear form on $C_{0,\mathrm{imp}}(-\infty,\infty)$ according to the formula

$$m(x)=\int\limits_{-\infty}^{\infty}x(s)m(ds),\qquad x\in C_{0,\mathrm{imp}}(-\infty,\infty).$$

In this representation, for any $t\in(-\infty,\infty)$ the operator adjoint to $\mathscr{C}_0(t)\in\mathscr{L}\big(C_{0,\mathrm{imp}}(-\infty,\infty)\big)$ equals again $\mathscr{C}_0(t)$, but now viewed upon as an element of $\mathscr{L}\big(M_{\mathrm{imp}}(-\infty,\infty)\big)$. Therefore, if a one-parameter group should give an exponential representation for $\mathscr{C}_0$ in $\mathscr{L}\big(C_{0,\mathrm{imp}}(-\infty,\infty)\big)$, then the corresponding group of adjoint operators would give an exponential representation for $\mathscr{C}_0$ in $\mathscr{L}\big(M_{\mathrm{imp}}(-\infty,\infty)\big)$, contrary to the result of the preceeding section.

The adjoint space of $C_{\mathrm{imp}}[-\infty,\infty]$ with the $*$-weak topology may be represented as the direct sum $M_{\mathrm{imp}}(-\infty,\infty)+\boldsymbol{C}$, where $\boldsymbol{C}$ is the field of complex numbers and $M_{\mathrm{imp}}(-\infty,\infty)$ is equipped with the topology of weak convergence of measures, in the sense that any element $m+\lambda\in M_{\mathrm{imp}}(-\infty,\infty)+\boldsymbol{C}$ defines a continuous linear form on $C_{\mathrm{imp}}[-\infty,\infty]$ according to the formula

$$(m+\lambda)(x)=\int\limits_{-\infty}^{\infty}x(s)m(ds)+\lambda\lim_{s\to\infty}x(s),\quad x\in C_{\mathrm{imp}}[-\infty,\infty].$$

In this representation, we have for operators $\mathscr{C}_0^*(t)$ adjoint to $\mathscr{C}_0(t)$,

$$[\mathscr{C}_0^*(t)(m+\lambda)](x)=\int\limits_{-\infty}^{\infty}\big(\mathscr{C}_0(t)x\big)(s)m(ds)+\lambda\lim_{s\to\infty}\mathscr{C}_0(t)x\big)(s)$$
$$=\int\limits_{-\infty}^{\infty}x(s)\big(\mathscr{C}_0(t)m\big)(ds)+\lambda\lim_{s\to\infty}x(s),$$

so that

$$(*)\qquad \mathscr{C}_0^*(t)(m+\lambda)=\mathscr{C}_0(t)m+\lambda$$

for every $m+\lambda\in M_{\mathrm{imp}}(-\infty,\infty)+\boldsymbol{C}$ and $t\in(-\infty,\infty)$, where $\mathscr{C}_0(t)$ is treated as an element of $\mathscr{L}\big(M_{\mathrm{imp}}(-\infty,\infty)\big)$. Moreover, according to the statement (ii) of Section 4, the infinitesimal generator A_0 of $\mathscr{C}_0$ in $\mathscr{L}\big(M_{\mathrm{imp}}(-\infty,\infty)\big)$ is the operator of the second derivative in the sense of distributions, defined on the set $\mathscr{D}(A_0)=\{m\colon m \text{ and } m''\in M_{\mathrm{imp}}(-\infty,\infty)\}$.

If $m\in\mathscr{D}(A_0)$ and $m''=A_0m=0$ then the density of m with respect to the Lebessue measure is a linear function. But since m is a bounded measure, this is possible only if $m = 0$. Therefore A_0 is an invertible operator.

Now suppose that a one-parameter group gives an exponential representation for $\mathscr{C}_0(t)$ in $\mathscr{L}(C_{\mathrm{imp}}[-\infty, \infty])$. Then the corresponding group of adjoint operators would give an exponential representation for $\mathscr{C}_0^*(t)$ in $\mathscr{L}(M_{\mathrm{imp}}(-\infty, \infty)+C)$. But then, since the operators $\mathscr{C}_0^*(t)$ have the form (*) and since A_0 is invertible, by Lemma 4 of Section 3, $\mathscr{C}_0(t)$ would have an exponential representation in $\mathscr{L}(M_{\mathrm{imp}}(-\infty, \infty)$, which is impossible, as we already know from Section 4.

The above method of proofs was suggested to the author by professor C. Ryll-Nardzewski, whose suggestion was also that $M(-\infty, \infty)$ with the *-weak topology may be treated similarly to $L^1(-\infty, \infty)$. A direct approach to $C_0(-\infty, \infty)$ (without a use of adjoint operators), which was a former idea of the author is more complicated. However this direct proof works without any further additional complications also for $C[-\infty, \infty]$, $C_u(-\infty, \infty)$ and $L^\infty(-\infty, \infty)$. We shall present it for the last two spaces.

Proof of the fact that $C_u(-\infty, \infty)$ and $L^\infty(-\infty, \infty)$ do not have property (E). We admit the convention that X always denotes $C_u(-\infty, \infty)$ or $L^\infty(-\infty, \infty)$ and that any statement concerning X should be considered as a statement true for $X = C_u(-\infty, \infty)$ and for $X = L^\infty(-\infty, \infty)$ simultaneously. Let us recall that $C_u(-\infty, \infty)$ is considered with the norm topology, while $L^\infty(-\infty, \infty)$ is regarded as the adjoint space of $L^1(-\infty, \infty)$ with the *-weak topology. The Fourier transforms of elements of X are temper distributions, i.e. they are elements of the space $\mathscr{S}'$ of L. Schwartz.

Let H denote the set of all finite linear combinations of Hermite functions $\varphi_n(s) = \dfrac{d^n}{ds^n} e^{-s^2}$, $n = 0, 1, \ldots$ We shall need the fact that

(I) *H is a dense subset of the space $\mathscr{S}$.*

Here $\mathscr{S}$ is the space of L. Schwartz of infinitely differentiable rapidly decreasing functions.

Let $S \in \mathscr{S}'$ and $\langle S, x\rangle = 0$ for every $x \in H$. Then (I) will be proved if we show that $S = 0$. Let $\psi_n(u) = u^n e^{-u^2}$. Then $\psi_n \in H$ and for every real s the series $\sum\limits_{n=0}^{\infty} \dfrac{(-is)^n}{n!}\psi_n$ converges in the sense of the topology of $\mathscr{S}$, so that

$$\text{(a)} \qquad \langle S, e^{-isu}e^{-u^2}\rangle = \sum_{n=0}^{\infty} \frac{(-is)^n}{n!}\langle S, \psi_n\rangle = 0,$$

where S acts onto $e^{-isu}e^{-u^2}$ as a function of u. Let now $\varphi \in \mathscr{D}(-\infty, \infty)$, $\operatorname{supp}\varphi \subset [a, b]$. If $a = s_{0,n} < s_{1,n} < \ldots < s_{m_n,n} = b$, $n = 1, 2, \ldots$, is a normal sequence of partitions of the interval $[a, b]$, then the Riemann sums

$$\text{(b)} \qquad \sigma_n(u) = \sum_{k=1}^{m_n} \varphi(s_{k,n}) e^{-is_{k,n}u}(s_{k,n} - s_{k-1,n}), \quad n = 1, 2, \ldots,$$

form a sequence of functions of u, such that, for every fixed $l = 0, 1, \ldots$, the sequence $\dfrac{d^l\sigma_n(u)}{du^l}$, $n = 1, 2, \ldots$, is bounded and converges to $\dfrac{d^l(\mathscr{F}\varphi)(u)}{du^l}$ almost uniformly in u on $(-\infty, \infty)$. Therefore $\tau_n(u) = e^{-u^2}\sigma_n(u)$, $n = 1, 2, \ldots$, is a sequence of functions of u, converging to $e^{-u^2}(\mathscr{F}\varphi)(u)$ in the sense of the topology of $\mathscr{S}$. Moreover, by (a) and (b), $\langle S, \tau_n\rangle = 0$ and $\langle S, e^{-u^2}\mathscr{F}\varphi\rangle = \lim\limits_{n\to\infty}\langle S, \tau_n\rangle = 0$. Since $e^{-u^2}S \in \mathscr{S}'$ and $\mathscr{F}(\mathscr{D}(-\infty, \infty))$ is dense in $\mathscr{S}$, it follows that $e^{-u^2}S = 0$ as an element of $\mathscr{S}'$ and therefore also as an element of $\mathscr{D}'(-\infty, \infty)$. It follows that $S = 0$ as an element of $\mathscr{D}'(-\infty, \infty)$. But $S \in \mathscr{S}'$ and $\mathscr{D}(-\infty, \infty)$ is dense in $\mathscr{S}$, and therefore $S = 0$ also as an element of $\mathscr{S}'$. The statement (I) is proved.

Similarly as in Section 4, it may be proved that if A_0 is the infinitesimal generator of $\mathscr{L}(X_{\mathrm{imp}})$-valued cosine function

$$\mathscr{C}_0(t) = \tfrac{1}{2}[T(t) + T(-t)]|X_{\mathrm{imp}}$$

then

$$\mathscr{D}(A_0) = \{x\colon x \in X_{\mathrm{imp}},\ x'' \in X_{\mathrm{imp}}\},$$
$$A_0 x = x'' \quad \text{for} \quad x \in \mathscr{D}(A_0),$$

where x'' denotes the second distributional derivative of x.

After this preparation suppose that $\{\mathscr{G}(t)\colon -\infty < t < \infty\} \subset \mathscr{L}_s(X_{\mathrm{imp}})$ is a one-parameter continuous group such that

$$\tfrac{1}{2}\mathscr{G}(t) + \tfrac{1}{2}\mathscr{G}(-t) = \mathscr{C}_0(t)$$

for every $t \in (-\infty, \infty)$. An investigation of the structure of this group will lead us to a contradiction.

For any real t let the distribution $g_t \in \mathscr{D}'(0, \infty)$ be defined by the equality

$$(\mathscr{F}\varphi_1)|_{(0,\infty)}\, g_t = (\mathscr{F}\mathscr{G}(t)\varphi_1)|_{(0,\infty)}.$$

This is a correct definition because $(\mathscr{F}\varphi_1)(u) = \dfrac{iu}{\sqrt{2}} e^{-\frac{1}{4}u^2}$ is positive and infinitely differentiable in $(0, \infty)$. Both terms in this equality are elements of $\mathscr{D}'(0, \infty)$.

LEMMA A$_1$. *For every $x \in \mathscr{S}_{\mathrm{imp}}$ and every $t \in (-\infty, \infty)$ we have*

$$(\mathscr{F}x)|_{(0,\infty)}\, g_t = (\mathscr{F}\mathscr{G}(t)x)|_{(0,\infty)}.$$

Proof. By an argument to that similar used at the beginning of the proof of Lemma A in Section 4, it follows that Lemma A_1 is true for every $x \in H_{\mathrm{imp}}$. Let now $x \in \mathscr{S}_{\mathrm{imp}} \setminus H_{\mathrm{imp}}$. Then, by (I), there is a sequence $\{x_n\}$, $n = 1, 2, \ldots$, of elements of H_{imp}, converging to x in the sense of topology of the space $\mathscr{S}$ and then, as is easy to see, $(\mathscr{F}x)|_{(0,\infty)} g_t = \lim_{n\to\infty} (\mathscr{F}x_n)|_{(0,\infty)} g_t = \lim_{n\to\infty} \big(\mathscr{F}\mathscr{G}(t)x_n\big)|_{(0,\infty)} = \big(\mathscr{F}\mathscr{G}(t)x\big)|_{(0,\infty)}$ in the sense of convergence in $\mathscr{D}'(0, \infty)$.

LEMMA A_2. *For every $t \in (-\infty, \infty)$ the distribution $g_t \in \mathscr{D}'(0, \infty)$ is a function continuous in $(0, \infty)$.*

Proof. It is sufficient to show that for every $\varphi \in \mathscr{D}(0, \infty)$ the distribution $g_{t,\varphi}$, which by definition equals φg_t on $(0, \infty)$ and equals zero on $(-\infty, \infty) \setminus \operatorname{supp}\varphi$, is the Fourier transform of a bounded measure on $(-\infty, \infty)$. For a fixed $\varphi \in \mathscr{D}(0, \infty)$ let $h \in \mathscr{S}_{\mathrm{imp}}$ be such that $(\mathscr{F}h)(u) = 1$ for every $u \in \operatorname{supp}\varphi$. Let $\psi \in \mathscr{D}(-\infty, \infty)$ be equal φ on $(0, \infty)$ and zero on $(-\infty, 0]$. For any $x \in \mathscr{S}$ let x_+ and x_- denote respectively the pair and the impair part of x. Then

$$\begin{aligned}
|\langle \mathscr{F}^{-1} g_{t,\varphi}, x\rangle| &= |\langle g_{t,\varphi}, \mathscr{F}x_+ - \mathscr{F}x_-\rangle| = |\langle g_{t,\varphi}, \mathscr{F}h\mathscr{F}x_+ - \mathscr{F}x_-\rangle| \\
&= |\langle g_{t,\varphi}, \mathscr{F}(h * x_+ - x_-)\rangle| = |\langle \big(\mathscr{F}(h * x_+ - x_-)\big)|_{(0,\infty)} g_t, \varphi\rangle| \\
&= |\langle \mathscr{G}(t)(h * x_+ - x_-), \mathscr{F}\psi\rangle| \\
&\leqslant \|\mathscr{G}(t)(h * x_+ - x_-)\|_{X_{\mathrm{imp}}} \|\mathscr{F}\psi\|_{L^1(-\infty,\infty)} \\
&\leqslant \|\mathscr{F}\psi\|_{L^1(-\infty,\infty)} \|\mathscr{G}(t)\|_{\mathscr{L}(X_{\mathrm{imp}})} (\|h\|_{L^1(-\infty,\infty)} + 1) \sup_{-\infty<s<\infty} |x(s)|,
\end{aligned}$$

which proves that $\mathscr{F}^{-1}_{g_{t,\varphi}}$ is a bounded measure on $(-\infty, \infty)$. We must only make clear that, in the case of $X = L^\infty(-\infty, \infty)$, $\| \ \|_{X_{\mathrm{imp}}}$ should be understood as ess sup and $\| \ \|_{\mathscr{L}(X_{\mathrm{imp}})}$ — as the corresponding norm for operators. Since $\mathscr{G}(t)$ is continuous in the $*$-weak topology, it is also continuous in the norm topology, so that $\|\mathscr{G}(t)\|_{\mathscr{L}(X_{\mathrm{imp}})} < \infty$.

On account of Lemma A_2, it is convenient to extend g_t to a pair function on $(-\infty, \infty)$, whose value at 0 is 1. Henceforth by g_t we shall mean such an extended function. Let Z_{imp} denote the set of all impair functions continuous in $(-\infty, \infty)$ with compact supports not containing zero. Let $Y_{\mathrm{imp}} = \{\mathscr{F}z\colon\ z \in Z_{\mathrm{imp}}\}$. Then clearly $Y_{\mathrm{imp}} \subset C_{0,\mathrm{imp}}(-\infty, \infty)$.

LEMMA A_3. *$\mathscr{G}(t) Y_{\mathrm{imp}} \subset Y$ and $\mathscr{F}\mathscr{G}(t)x = g_t \mathscr{F}x$ for every $t \in (-\infty, \infty)$ and $x \in Y_{\mathrm{imp}}$.*

Proof. If $x \in Y_{\mathrm{imp}}$ then also $\mathscr{F}^{-1}(g_t \mathscr{F}x) \in Y_{\mathrm{imp}}$, and therefore in Lemma A_3 the inclusion follows from the equality. Let $x \in Y_{\mathrm{imp}}$. Then there are positive numbers a and b, $b > a$, and a sequence $\{z_n\}$, $n = 1, 2, \ldots$, of elements of $\mathscr{D}_{\mathrm{imp}}(-\infty, \infty)$ such that $z_n \to \mathscr{F}^{-1}x = -\mathscr{F}x$ uniformly on $(-\infty, \infty)$ as $n \to \infty$ and $\operatorname{supp} z_n \subset [-b, -a] \cup [a, b]$. Put $x_n = \mathscr{F}z_n$. Then $x_n \in \mathscr{S}_{\mathrm{imp}}$ and $x_n \to x$ uniformly on $(-\infty, \infty)$ as $n \to \infty$, which implies that $\mathscr{G}(t)x_n \to \mathscr{G}(t)x$ in the sense of the topology in X and, consequently, $\mathscr{F}\mathscr{G}(t)x_n \to \mathscr{F}\mathscr{G}(t)x$ in the sense of convergence in $\mathscr{S}'$. On the other hand, the sequence $g_t \mathscr{F}x_n = -g_t z_n$, $n = 1, 2, \ldots$, of functions belonging to Z_{imp} converges uniformly on $(-\infty, \infty)$ to $g_t \mathscr{F}x \in Z_{\mathrm{imp}}$. Since, by Lemma A_1, for every $n = 1, 2, \ldots$ we have $\mathscr{F}\mathscr{G}(t)x_n = g_t \mathscr{F}x_n$ on $(-\infty, 0) \cup (0, \infty)$, it follows that $\mathscr{F}\mathscr{G}(t)x = g_t \mathscr{F}x$ on $(-\infty, 0) \cup (0, \infty)$. Consequently the difference between the distribution $\mathscr{F}\mathscr{G}(t)x \in \mathscr{S}'_{\mathrm{imp}}$ and the function $g_t \mathscr{F}x \in Z_{\mathrm{imp}}$ is an impair distribution with the one-point support at zero. Therefore $\mathscr{F}\mathscr{G}(t)x - g_t \mathscr{F}x = \sum_{k=0}^{m} C_k \delta^{(2k+1)}$, so that the function $\mathscr{G}(t)x - \mathscr{F}^{-1}(g_t \mathscr{F}x) \in X_{\mathrm{imp}}$, bounded on $(-\infty, \infty)$, equals to an impair polynomial. This is possible only if this polynomial vanishes identically, i.e. only if $\mathscr{G}(t)x = \mathscr{F}^{-1}(g_t \mathscr{F}x)$.

Our further reasonings follow very closely the reasonigs of Section 4. Lemma B may be transported without any change to the present section. The only difference in the proof is that instead of Lemma A now we use Lemma A_3 and that the proof of the fact that

(c) for every fixed $u \in (-\infty, \infty)$ $g_t(u)$ is a function of t measurable on $(-\infty, \infty)$

must be a little modified. Now we prove (c) as follows. Given a fixed $u \neq 0$, we take a function $x \in \mathscr{S}_{\mathrm{imp}}$ such that $(\mathscr{F}x)(u) = 1$ and we take a δ-like sequence $\{y_n\}$, $n = 1, 2, \ldots$, of non-negative functions in $\mathscr{S}$. Then

$$f_n(t) = \big(y_n * \mathscr{F}\mathscr{G}(t)x\big)(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ius} [\mathscr{F}^{-1} y_n](s) [\mathscr{G}(t)x](s)\, ds,$$

$n = 1, 2, \ldots$, are continuous functions of t and for every fixed t we have

$$\lim_{n\to\infty} f_n(t) = \big(\mathscr{F}\mathscr{G}(t)x\big)(u) = g_t(u)(\mathscr{F}x)(u) = g_t(u).$$

Lemma C together with its proof may be transported to the present section without any change. By means of an argument similar to that used in Section 4, this lemma implies the following statement:

(S_1) *For every $t \in (-\infty, \infty)$ there is $c_t \in (0, \infty)$ such that*

$$\sup_{-\infty<s<\infty} |\{H\big(T(t) - T(-t)\big)x\}(s)| \leqslant c_t \sup_{-\infty<s<\infty} |x(s)|$$

for every $x \in \mathscr{D}_{\mathrm{imp}}(-\infty, \infty)$.

Further, for arbitrarily fixed positive a and $b, b > a$, by a similar reasoning as in Section 4, (S_1) implies the following statement

$$(S_1') \quad \sup_{s\in[a,b]} \left| \mathrm{V.P.} \int_a^b \frac{x(u)}{s-u}\, du \right| \leqslant \mathrm{const} \cdot \sup_{u\in[a,b]} |x(u)| \quad \text{for every} \quad x \in \mathscr{D}(a, b).$$

Now the whole indirect proof is completed by showing that (S_1') is not true. Indeed, if $\varepsilon \in \left(0, \frac{b-a}{2}\right)$ and $x_\varepsilon \in \mathscr{D}(a, b)$ is such that $0 \leqslant |x_\varepsilon(s)| \leqslant 1$ for $s \in (a, b)$, and that $x_\varepsilon(s) = 1$ for $s \in [a+\varepsilon, b-\varepsilon]$, then

$$\int_a^b \frac{x(u)}{b-u}\, du \geqslant \int_{a+\varepsilon}^{b-\varepsilon} \frac{du}{b-u} = \log\left(\frac{b-a}{\varepsilon} - 1\right).$$

References

[1] N. Dunford and J. T. Schwartz, *Linear Operators*, Part II, *Spectral Theory and Selfadjoint Operators in Hilbert Space*, New York 1963.
[2] R. E. Edwards, *Functional Analysis, Theory and Applications*, New York 1965.
[3] H. Helson, *Note on harmonic functions*, Proc. Amer. Math. Soc. 4 (1953), pp. 686–691.
[4] K. Hoffman, *Banach Spaces of Analytic Functions*, New York 1962
[5] J. Kisyński, *On operator valued solutions of d'Alemberts functional equation*, I, Colloq. Math. to appear.
[6] M. Sova, *Cosine operator functions*, Dissertationes Math. (Rozprawy Mat.) 49 (1966), pp. 1–47.
[7] K. Yosida, *Functional Analysis*, Berlin-Göttingen-Heidelberg 1965.

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On uniform symmetrization of analytic matrix functions

by

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Abstract. Let A be real-analytic function of ξ on open set $M \subset R^n$, which values $A(\xi)$ are $m \times m$ matrices with purely diagonal and real canonical Jordan form. If the characteristic roots of $A(\xi)$ are restricted to change their multiplicities only in a suitable, very simple manner, then for every $\xi \in M$ we construct a hermitean positive $m \times m$ matrix $H(\xi)$, such that $H(\xi)A(\xi)$ is hermitean and that $\|H(\xi)\|$ and $\|H^{-1}(\xi)\|$ are locally bounded functions of ξ.

1. The result. Let A be a function defined on a set M, which values are $m\times m$ complex matrices. We shall say that A is *uniformly symmetrizable on M* if the following condition is satisfied:

(S) there is a constant $c \geqslant 1$, such that for every $\xi \in M$ there is a hermitean $m\times m$ matrix $H(\xi)$, such that $c^{-1} \leqslant H(\xi) \leqslant c$ and that $H(\xi)A(\xi)$ is hermitean.

According to Kreiss [2], [3], the uniform symmetrizability of A on M is equivalent to either of the following conditions:

(D) there is a constant $d \geqslant 1$, such that for every $\xi \in M$ there is an on singular $m\times m$ matrix $T(\xi)$, such that $\|T(\xi)\| \leqslant d$, $\|T^{-1}(\xi)\| \leqslant d$ and that $T^{-1}(\xi)A(\xi)T(\xi)$ is purely diagonal and real;

(E) $\sup\{\|\exp(itA(\xi))\|: t \in (-\infty, \infty),\ \xi \in M\} < \infty$;

(R) $\sup\{\|(E - isE - itA(\xi))^{-1}\|: s, t \in (-\infty, \infty),\ \xi \in M\} < \infty$, where E denotes the unit $m\times m$ matrix.

The theorem, which we state below may be treated as a contribution to the following problem. Let A be a matrix-valued function on a set M and suppose that $A(\xi)$ is symmetrizable for every fixed $\xi \in M$. Under which additional conditions A is uniformly symmetrizable on M? Our additional conditions have the form of restrictions on the behaviour of characteristic roots of $A(\xi)$ near the points of branching. We consider only the simplest case, when two roots come together.

THEOREM. *Let $M \subset R^n$ be open and let A be an analytic function on M, which values are $m\times m$ complex matrices. Suppose that for every $\xi \in M$ the matrix $A(\xi)$ has purely diagonal and real canonical Jordan form. Moreover,*