References

[1] S. Banach et S. Saks, *Sur convergence forte dans les champs L^p*, Studia Math. 2 (1930), pp. 51–57.
[2] M. M. Day, *Normed Linear Spaces*, Berlin 1958.
[3] S. Kakutani, *Weak convergence in uniformly convex spaces*, Tôhoku Math. J. 45 (1938), pp. 188–193.
[4] V. Klee, *Summability in $l(p_1, p_2, \ldots)$ spaces*, Studia Math. 25 (1965), pp. 277–280.
[5] T. Nishiura and D. Waterman, *Reflexivity and summability*, Studia Math. 23 (1963), pp. 53–57.
[6] S. Sakai, Review of [5], Math. Reviews 27 (1964), p. 974.
[7] J. Schreier, *Ein Gegenbeispiel zur Theorie der schwachen Konvergenz*, Studia Math. 2 (1930), pp. 58–62.
[8] D. Waterman, T. Ito, F. Barber, and J. Ratti, *Reflexivity and summability: the Nakano $l(p_i)$ spaces*, Studia Math. 33 (1969), pp. 141–146.

SYRACUSE UNIVERSITY

Received February 5, 1971 (306)

STUDIA MATHEMATICA T. XLII. (1972)

About the space $\cap\, l_p$, $p > 0$.

by

L. WAELBROECK (Brussels)

Abstract. We give a few properties of the space of sequences (a_n) such that $\sum |a_n|^p$ converges for all $p > 0$. The fact that the algebra of continuous linear transformations of this space has a natural, locally psudo-convex, locally multiplicatively convex, Fréchet topology is rather unexpected. This space also provides a negative answer to a question of W. Żelazko.

l_{+0} is the space of sequences $(a_n)_{n\in N}$ such that $\sum |a_n|^p = \nu_p(a)$ is finite for all positive p, with the Fréchet locally pseudo-convex topology determined by the pseudo norms ν_p. The reader may find the following observations about this space amusing.

The elements of l_{+0} are the sequences whose decreasing rearrangements belong to the space s of rapidly decreasing sequence. If we equip s with the usual topology determined by the norms $\sup n^k |a_n|$, the identity mapping $s \to l_{+0}$ is continuous. Permutations of N induce on l_{+0} an equicontinuous family of linear transformations. A translation invariant topology $\mathscr{T}$, on l_{+0} is weaker than the given one if it induces on s a weaker topology than its usual one, and if permutations of N induce on l_{+0} a $\mathscr{T}_1$-equicontinuous system of transformations at the origin.

These facts are either trivial, well known, or follow from the observation that $|a'_n| < \varepsilon n^{-1/p}$ if (a'_n) is the decreasing rearrangement of (a_n) and $\sum |a_n|^p < \varepsilon^p$.

Let $T\colon l_{+0} \to l_{+0}$ be a continuous linear transformation. Let B_p be the set of sequences $(a_n) \in l_{+0}$ such that $\nu_p\big((a_n)\big) \leqslant 1$. Then B_p is closed, absolutely p-convex, and a neighbourhood of the origin. $\overset{-1}{T}(B_p)$ is then also a closed, absolutely p-convex neighbourhood of the origin in l_{+0}. Being a neighbourhood of the origin, it contains $\varepsilon B_{p'}$ for some $\varepsilon > 0$, $p' > 0$. Further, the closed, absolutely p-convex hull of $B_{p'}$ is B_p when $p' < p$, so that $\overset{-1}{T} B_p \supseteq \varepsilon B_p$.

In other words, T extends to a continuous linear transformation of l_p, for all p, $0 < p \leqslant 1$. We can define $\tilde{\nu}_p(T)$ by

$$\tilde{\nu}_p(T) = \sup\{\nu_p(Tx) \mid \nu_p(x) \leqslant 1\}.$$

$\tilde{\nu}_1$, is a submultiplicative p-norm on $\mathscr{L}(l_{+0})$. With these p-norms, the algebra of continuous linear transformations of l_{+0} is in a natural way a locally pseudo-convex, locally multiplicatively convex Fréchet algebra.

The following is an observation of S. Rolewicz. W. Żelazko has proved (unpublished) that E is a normed space if E is locally convex and if there is a topology on $\mathscr{L}(E)$, the algebra of continuous linear operators on E which makes substitution $(u, e) \to u(e)$, $\mathscr{L}(E) \times E \to E$ continuous. Żelazko's result does not extend to the locally pseudo-convex case, nor even to the locally p-convex case. The space l_{+0} described above is a locally pseudo-convex counter-example. And the considerations above apply clearly to the space $l_{p+0} = \cap_{p'>p} l_p$ with its obvious Fréchet topology. An algebra topology is defined in this way on $\mathscr{L}(l_{p+0})$. Substitution is again a continuous operation $\mathscr{L}(l_{p+0}) \times l_{p+0} \to l_{p+0}$. But l_{p+0} is locally p-convex and not locally bounded.

The last result is trivial. We have the inclusion $l_{+0} \subseteq l_1$, the identity $i: l_{+0} \to l_1$ is continuous. A linear mapping $T: l_{+0} \to l_{+0}$ is continuous if $i \circ T: l_{+0} \to l_1$ is. This is clear, the graph of T is closed in $l_{+0} \times l_1$ and a fortiori in $l_{+0} \times l_{+0}$.

Received February 10, 1971 (298)

STUDIA MATHEMATICA T. XLII. (1972)

Decompositions of non-contractive operator-valued representations of Banach algebras

by

F. H. SZAFRANIEC (Kraków)

Abstract. The present paper deals with some decompositions of non-contractive operator-valued representations of Banach algebras. These decompositions are closely related to the abstract F. and M. Riesz property. An examination of the Boolean character of this property is basic for our purposes. This, when combined with the Sz.-Nagy–Dixmier theorem concerning similarity of certain Boolean algebras of projections shows that the representation in question is similar to a suitable orthogonally decomposed representation.

Let T be the Hilbert space representation of a function algebra A. There are results of Sarason [14] and of Mlak [7], [8] that to every Gleason part of A or intersection of peak sets of A there corresponds a projection which commutes with T. This projection is orthogonal for contractive T. In this case a full decomposition of T with respect to the totality of all Gleason parts or to the Bishop decomposition of A is available.

In both cases an essential role is played by the F. and M. Riesz property. The point is that this property in an abstract form [13] gives rise to a homomorphism of a certain Boolean algebra of projections in the dual space onto a Boolean algebra of projections commuting with T. It seems that this is one of the real reasons why such decompositions as in [7], [8], [14] are available.

Althought our theory concerns representations of general non-commutative algebras, the examples of applications we give in the present paper are commutative. Non-commutative cases will be treated elsewhere.

1. Let B be a (not necessarily commutative) Banach algebra with the unit 1. The norm of $u \in B$ is denoted by $\|u\|$. B^* is the dual of B. For $u \in B$ and $\mu \in B^*$ we write $\langle \mu, u \rangle$ for $\mu(u)$. I stands for the identity operator in B^*.

Let A be a closed subalgebra of B and assume $1 \in A$. If $\langle \mu, u \rangle = 0$ (for all $u \in A$) for $\mu \in B^*$ then we write $\mu \perp A$. For $v \in B$ and $\mu \in B^*$ we define $v\mu$ and μv as the elements of B^* given by the formulae: $\langle v\mu, u \rangle = \langle \mu, vu \rangle$, $\langle \mu v, u \rangle = \langle \mu, uv \rangle$, $u \in B$.