STUDIA MATHEMATICA, T. XLIII. (1972)

Free semigroups and unitary group representations

by

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Abstract. This paper is concerned with locally compact groups that contain closed, or uniformly discrete, free semigroups on two generators. These groups are discussed in terms of transformation groups, their unitary presentations, and the question of symmetry of their group algebras.

The occurrence of free semigroups on two generators as subsemigroups of discrete groups was first considered by Frey [5]. He proved that all subsemigroups of a discrete, amenable group are amenable if and only if the group does not contain a free semigroup on two generators.

Appel and Djorup [1] gave the first example of a non-free group generated by a free semigroup on two generators. Hochster [8] later constructed an amenable group with such a semigroup.

Free subsemigroups have been shown to be an issue when considering the question of symmetry of the group algebra of a locally compact group. In [9], this author proved that $l^1(G)$ is not symmetric if G is a discrete group containing a free semigroup on two generators. This result was extended in [10] to locally compact groups containing a semigroup on two generators that, in addition to being free, satisfying a stringent topological condition (see § 5).

In this paper we are concerned principally with topological groups that contain either a closed, or a uniformly discrete, free semigroup on two generators.

In § 1, the notion of non-asymptotic disjoint ideal semigroup (NADIS) is defined, and used to characterize groups containing uniformly discrete free semigroups.

In § 2, groups containing a NADIS are characterized in terms of transformation groups. Using this characterization, several examples are presented.

In § 3, it is shown that a connected locally compact group containing an open NADIS is homomorphic to a matrix group containing a NADIS.

* This research was partially supported by National Science Foundation Grants GP-12027 and GU-3171.

Groups containing closed free semigroups are discussed in § 4.

In § 5, an example is presented of a locally compact group that contains a closed free semigroup on two generators and whose group algebra is symmetric.

In § 6, the unitary representations of groups containing free semigroups is discussed. In particular, the locally compact groups that contain a NADIS are characterized in term of their unitary representations.

§ 1. Throughout this section H and G will denote topological groups. By a subsemigroup of G we mean a nonempty subset of G that is closed under the multiplication of G. If S is a semigroup, a nonempty subset I of S is a right ideal if for each a in I and s in S, $as \in I$.

If a and b are elements of G, $[a, b]$ will denote the subsemigroup generated by a and b. If $A \subset G$, $\langle A\rangle$ will denote the (not necessarily closed) subgroup generated by A; $\langle g\rangle$ denotes $\langle\{g\}\rangle$. e will always denote the group identity.

Definition 1.1. A subsemigroup S of G is *uniformly discrete* if there is a neighborhood U of e in G such that $sU \cap s'U = \emptyset$ if $s \neq s'$ for all s, s' in S.

Definition 1.2. A subsemigroup S of G is said to be a *non-asymptotic, disjoint ideal semigroup* (*NADIS*) if S contains disjoint right ideals I and J such that e is not in the closure of $I^{-1}J$. (NADIS may also be read in the plural if the context so warrants.)

A subsemigroup S of G has disjoint right ideals I and J if, and only if, I^{-1} and J^{-1} are disjoint left ideals of S^{-1}. Hence, the existence of a NADIS in G is not retricted by the use of "right" in definition 1.2.

Theorem 1.3. *G contains a NADIS if, and only if, there exist a and b in G such that $[a, b]$ is a uniformly discrete free semigroup.*

Proof. Assume that S is a NADIS in G with disjoint right ideals I and J. Let $a \in I$ and $b \in J$.

Suppose that $x_1x_2\ldots x_n = y_1y_2\ldots y_m$, where $x_i, y_j \in \{a, b\}$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Unless $n = m$ and $x_i = y_i$ for $1 \leqslant i \leqslant n$, there is a minimum k such that $x_k \neq y_k$. If $k > n$ or if $k > m$ we have $e \in I \cup J$. This is clearly impossible. Thus, $x_kx_{k+1}\ldots x_n = y_ky_{k+1}\ldots y_m$, and $x_k \neq y_k$. But then $x_kx_{k+1}\ldots x_n \in I$, say, while $y_ky_{k+1}\ldots y_m \in J$. This contradiction implies that $n = m$ and $x_i = y_i$ for $1 \leqslant i \leqslant n$. Therefore, $[a, b]$ is free.

Let U be a neighborhood of e in G such that $U \cap I^{-1}J = \emptyset$. Let V be a symmetric neighborhood of e such that $V^2 \subset U$. If $s, t \in [a, b]$ such that $s \neq t$ then $s^{-1}t \in I^{-1}J$ or $t^{-1}s \in I^{-1}J$. Assume the former. Now, if $sV \cap tV \neq \emptyset$, $s^{-1}t \in VV^{-1} \subset U$. But U was chosen so that $U \cap I^{-1}J = \emptyset$. Hence $sV \cap tV = \emptyset$ if $s, t \in [a, b]$ and $s \neq t$. Therefore, $[a, b]$ is a uniformly discrete free semigroup.

Conversely, if $a, b \in G$ such that $S = [a, b]$ is a uniformly discrete free semigroup then $I = aS$ and $J = bS$ are disjoint right ideals of S. Furthemore, if U is a neighborhood of e in G such that $sU \cap tU = \emptyset$ for $s, t \in S$, $s \neq t$, then $I^{-1}J \cap U = \emptyset$. Hence S is a NADIS in G.

Theorem 1.4. *If H has a NADIS and $H \subset G$ then G has a NADIS.*

Proof. Merely observe that if S is a NADIS in H with disjoint right ideals I and J and if U is a neighborhood of e in G then $I^{-1}J \cap U = I^{-1}J \cap (U \cap H) = \emptyset$, for some U. Hence, S is a NADIS in G.

Theorem 1.5. *G contains a NADIS if there is a continuous homomorphism of G onto a group H containing a NADIS.*

Proof. Let S be a NADIS in H and let π be a continuous homomorphism of G onto H. If I and J are disjoint right ideals of S then $\pi^{-1}(I)$ and $\pi^{-1}(J)$ are disjoint right ideals of $\pi^{-1}(S)$. Furthermore, if $I^{-1}J \cap U = \emptyset$, where U is a neighborhood of e in H, then $\pi^{-1}(I^{-1})\pi^{-1}(J) \cap \pi^{-1}(U) = \emptyset$ and $\pi^{-1}(U)$ is a neighborhood of e in G.

Theorem 1.6. *If G contains an open NADIS and if H is a subgroup of the center of G, G/H contains a NADIS.*

Proof. Let I and J be disjoint right ideals of the open NADIS S, let $a \in \mathrm{int}(I)$ and $b \in \mathrm{int}(J)$. We will show that $[aH, bH]$ is a uniformly discrete free subsemigroup of G/H.

Let U be a neighborhood of e in G such that $aU \subset I$ and $bU \subset J$. Let V be a neighborhood of e in G such that $V^2 \subset U$. For $g \in G$ and $A \subset G$, let $\bar{g} = gH \in G/H$ and $\bar{A} = \{aH|\ a \in A\} \subset G/H$. Now, if $w, x \in [a, b]$ and $w \neq x$, there exist $y, z \in [a, b]$ such that $w^{-1}x = y^{-1}z$ and y is in one of the ideals I or J while z is in the other. If $\bar{w}\bar{V} \cap \bar{x}\bar{V} \neq \emptyset$ then there exist v_1, v_2 in V and h_1, h_2 in H such that $wv_1h_1 = xv_2h_2$. Hence $x^{-1}wv_1v_2^{-1} = h_2h_1^{-1}$.

Therefore, there is a $u \in U$ such that $y^{-1}zu = x^{-1}wv_1v^{-1} \in H \subset Z(G)$, the center of G. Consequently, $y(y^{-1}zu) = zu = (y^{-1}zu)y$. Thus $yzu = zuy$. But, since $zu \in S$, yzu is in either the ideal I or J and zuy is in the other ideal. Therefore, for arbitrary w and x in $[a, b]$, $w \neq x$, $\bar{w}\bar{V} \cap \bar{x}\bar{V} = \emptyset$, and $[aH, bH]$ is a uniformly discrete free semigroup in G/H.

§ 2. The following theorems give a method of determining groups which contain NADIS.

Theorem 2.1. *If G contains an open semigroup with disjoint right ideals then G contains a NADIS.*

Proof. Let S be the open subsemigroup and I and J disjoint right ideals of S. If $a \in I$ and $b \in J$ then aS and bS are open disjoint right ideals of S. Let $c \in aS$, $d \in bS$ and U a neighborhood of e in G such that $cU \subset aS$ and $dU \subset bS$. Let $s, t \in [c, d]$ and $s \neq t$. Then $s = wx_1s'$ and $t = wx_2t'$ where $w \in G$, $\{x_1, x_2\} = \{c, d\}$ and $s', t' \in [c, d]$. Now,

$$sU \cap tU = w(x_1s'U \cap x_2t'U).$$

But $s'U \cup t'U \subset S$. Thus, $x_1 s'U$ is in one of the disjoint ideals, either aS or bS, and $x_2 t'U$ is in the other. Thus $sU \cap tU = \emptyset$. Therefore $[c, d]$ is a uniformly discrete free subsemigroup of G and so, by Theorem 1.3, G contains a NADIS.

COROLLARY 2.2. *G contains a NADIS if there is a subsemigroup S of G with disjoint right ideals such that* $\operatorname{int}(S) \neq \emptyset$ *in* $\langle S \rangle$.

Proof. If I and J are disjoint right ideals of S, $a \in I$ and $b \in J$ then $\operatorname{int}(aS) \cup \operatorname{int}(bS)$ is an open subsemigroup of $\langle S \rangle$ with disjoint right ideals. By Theorem 2.1, $\langle S \rangle$ contains a NADIS, and by Theorem 1.4, G contains a NADIS.

A topological space X is called a *G-space* if there is a homomorphism ϱ of G into the group of homeomorphisms of X, $G(X)$, such that for each x in X, $g \to \varrho(g) \cdot x$ is continuous.

THEOREM 2.3. *G contains a NADIS if, and only if, there is a G-space X containing disjoint open subsets X_1 and X_2 such that $S_i = \{g \in G \mid \varrho(g): X_1 \cup X_2 \to X_i\} \neq \emptyset$ for $i = 1, 2$.*

Proof. Assume G contains a NADIS. Let $[a, b]$ be a uniformly discrete free subsemigroup of G. Let $X = G$, $X_1 = a[a, b]U$ and $X_2 = b[a, b]U$ where U is an open neighborhood of e chosen so that $sU \cap tU = \emptyset$ if $s, t \in [a, b]$, $s \neq t$. If $\varrho(g)$ is left multiplication by g the conditions of the theorem are clearly satisfied.

Conversely, let X, X_i and S_i, $i = 1, 2$, be as in the theorem. Then S_1 and S_2 are disjoint ideals of the semigroup $S = \{g \in G \mid \varrho(g): X_1 \cup X_2 \to X_1 \cup X_2\}$. We will show that e is not in the closure of $S_1^{-1} S_2$.

Suppose there exist nets $\{a_n\} \subset S_1$ and $\{b_n\} \subset S_2$ such that $a_n^{-1} b_n \to e$. Then, for each $x \in X$, $\varrho(a_n)^{-1} \varrho(b_n) \cdot x \to x$. If $x \in X_1$ then, since X_1 is a neighborhood of x, there is an m such that $\varrho(a_n)^{-1} \varrho(b_n) \cdot x \in X_1$ for all $n > m$. But then, $\varrho(b_n) \cdot x \in \varrho(a_n) \cdot X_1$ for all $n > m$. This is impossible since $\varrho(b_n) \cdot x \in X_2$ while $\varrho(a_n) \cdot X_1 \subset X_1$. Therefore e is not in the closure of $S_1^{-1} S_2$, and so S is a NADIS.

We can now give some easy examples of groups containing NADIS.

EXAMPLE 2.4. Let $G = SL(n, R)$, $n \geq 2$. Let $X = R^n$,

$$X_1 = \{(x_i) \in R^n \mid 0 < x_i < x_{i+1}, \quad 1 \leq i \leq n-1\}$$

and

$$X_2 = \{(x_i) \in R^n \mid 0 < x_{i+1} < x_i, \quad 1 \leq i \leq n-1\}.$$

One easily sees that X_1 and X_2 are open disjoint subsets of X and that, considering G a subgroup of $G(X)$, $S_i = \{g \in G \mid g(X_1 \cup X_2) \subset X_i\} \neq \emptyset$. Hence, by Theorem 2.3 $SL(n, R)$ contains a NADIS. Since $GL(n, R)$, $SL(n, C)$ and $GL(n, C)$ each contains $SL(n, R)$, they each contain NADIS.

EXAMPLE 2.5. Let G denote the real affine group, i.e.,

$$G = \{g: R \to R \mid g(x) = ax + b, \ a \neq 0\}$$

with composition as multiplication and with the obvious topology. If $X = R$, $X_1 = (0, 1)$ and $X_2 = (1, 2)$,

$$S_i = \{g \in G \mid g: (X_1 \cup X_2) \to X_i\} \neq \emptyset \quad \text{for } i = 1, 2.$$

Since the continuity requirement is obviously satisfied G contains a NADIS.

We consider one final example.

EXAMPLE 2.6. Let B be a ball in R^n. Let $P_1, P_2 \in \operatorname{int}(B)$ and let U_1, U_2 be disjoint open subsets of B such that $\overline{U}_1, \overline{U}_2 \subset \operatorname{int}(B)$. One can easily construct homeomorphism f_1, f_2 of B that are the identity on the boundary of B and such that $f_i(U_1 \cup U_2) \subset U_i$ for $i = 1, 2$. (If h is a homeomorphism of B that is fixed on the boundary of B and contracts all other points radially toward P, then for some m, $h^m(U_1 \cup U_2) \subset U_1$.)

Let M be an n-manifold and let (U, φ) be a coordinate patch of M. Let B be a ball in $\varphi(U)$ and let $U_i, f_i, i = 1, 2$ be as before. Then, if $V_i = \varphi^{-1}(U_i)$ and $h_i = \varphi^{-1} f_i \varphi$ for $i = 1, 2$, h_i is a homeomorphism on $\varphi^{-1}(B)$ that is the identity on the boundary of $\varphi^{-1}(B)$ and such that $h_i(V_1 \cup V_2) \subset V_i$ for $i = 1, 2$. h_1 and h_2 can be extended to M by making them the identity outside $\varphi^{-1}(B)$. Therefore, the group of homeomorphisms on M, with, say, the compact-open topology, contains NADIS

§ 3. The converse of Theorem 1.5 is trivial. Namely, if G has a NADIS, then using the identity map and G for H, there is a continuous homomorphism of G onto a group containing a NADIS. Of more interest perhaps is the fact that for a connected locally compact group containing an open NADIS, H can be taken to be a matrix group. We formulate this more precisely in

THEOREM 3.1. *Let G be a locally compact connected group with an open NADIS, S. There is a continuous homomorphism, π, of G into $GL(n\ R)$, for some $n \geq 2$, such that $\pi(G)$ contains a NADIS.*

Proof. Let I and J be disjoint right ideals of S and U a neighborhood of e in G such that $I^{-1}J \cap U = \emptyset$. Let V be a neighborhood of e in G such that $V^2 \subset U$.

By the standard approximation theorem by Lie groups (see [11], p. 175) there is a compact normal subgroup H of G, such that $H \subset V$ and G/H is a Lie group. As before, if $A \subset G$, let $\overline{A} = \{aH \mid a \in A\} \subset G/H$. If $\overline{I}^{-1} \overline{J} \cap \overline{V} \neq \emptyset$, there exist $a \in I$, $b \in J$ and $v \in V$ such that $a^{-1}bH = vH$. Hence $a^{-1}b \in vH \subset vV \subset V^2 \subset U$. This contradicts our choice of U. $\overline{S}$ is a NADIS in G/H. Furthermore, since S is open, $\overline{S}$ is open.

Now, if $G' = G/H$ and if $Z(G')$ denotes the center of G', $G'/Z(G')$ is isomorphic to a matrix group (see [11], p. 159). By Theorem 1.6, $G'/Z(G')$ contains a NADIS.

§ 4. If $[a, b]$ is a uniformly discrete subsemigroup of G then $[a, b]$ is closed in G. One is led to ask if every closed free subsemigroup, $[a, b]$, of G is uniformly discrete. The answer is no, as the following example will show.

EXAMPLE 4.1. Let ε denote the 3×3 matrix $[\varepsilon_{ij}]$ where $\varepsilon_{ii} = \delta > 1$ for $i = 1, 2, 3$; and $\varepsilon_{ij} = 0$ if $i \neq j$. Let $SO(3)$ be the real special orthogonal group and set $G = SO(3)\langle\varepsilon\rangle \subset GL(3, R)$. There exist elements α and β in $SO(3)$ such that $\langle\alpha, \beta\rangle$ is free (cf. e.g. [6], p. 9). Let $a = \alpha\varepsilon$ and $b = \beta\varepsilon$. If $s \in [a, b]$ then there is a $\sigma \in [\alpha, \beta]$ and a positive integer n such that $s = \sigma\varepsilon^n$. Suppose that $s, t \in [a, b]$ and that $s = t$. Let $\sigma, \tau \in [\alpha, \beta]$ and m, n be positive integers such that $s = \sigma\varepsilon^m$ and $t = \tau\varepsilon^n$. Since $\det(s) = \delta^{3m}$ and $\det(t) = \delta^{3n}$, $m = n$. Therefore $\sigma = \tau$. Thus, if $\sigma = \alpha^{p_1}\beta^{q_1}\ldots \ldots\alpha^{p_r}\beta^{q_r}$ where $p_i, q_i \geqslant 0$ for $1 \leqslant i \leqslant r$, and $(p_1+q_1)+\ldots+(p_r+q_r) = n$, then

$$s = \alpha^{p_1}\varepsilon^{p_1}\beta^{q_1}\varepsilon^{q_1}\ldots\alpha^{p_r}\varepsilon^{p_r}\beta^{q_r}\varepsilon^{q_r} = a^{p_1}b^{q_1}\ldots a^{p_r}b^{q_r} = t.$$

Consequently, $[a, b]$ is a free semigroup.

Since for any M, $[a, b]$ contains only finitely many s with $\det(s) < M$, $[a, b]$ is closed.

Finally, we show that $[a, b]$ is not uniformly discrete. We can choose sequences of positive integers $\{n_p\}$ and $\{m_p\}$ such that $\alpha^{n_p} \to e$ and $\beta^{m_p} \to e$. (Since $SO(3)$ is compact, $\{\alpha^n |\ n > 0\}$ has a limit point λ. Let $\{q_k\}$ be an increasing sequence such that $\alpha^{q_k} \to \lambda$. Then $n_k = q_k - q_{k-1} > 0$ and $\alpha^{n_k} \to e$.) Let $\mu_p = \alpha^{n_p}\beta^{m_p}$ and $\nu_p = \beta^{m_p}\alpha^{n_p}$. Then $\mu_p \to e$ and $\nu_p \to e$. Finally, set $u_p = a^{n_p}b^{m_p}$ and $v_p = b^{m_p}a^{n_p}$. Then $u_p \neq v_p$ for all p but $u_p^{-1}v_p = \mu_p^{-1}\nu_p \to e$. Therefore, $[a, b]$ is not uniformly discrete.

The following theorem will be crucial in § 6.

THEOREM 4.2. *If $[a, b]$ is a closed, free subsemigroup of G and K is any compact subset of G then $K\cap[a, b]$ is finite.*

Proof. Assume that for some compact subset K of G, $T = K\cap[a, b]$ is infinite. Then, by compactness of K, there is an s in $[a, b]$ and a net $\{s_\alpha\} \subset [a, b]$ such that $s_\alpha \to s$. Suppose that $s = xt$ where $x \in \{a, b\}$ and $t \in [a, b]$. If $\{y\} = \{a, b\} \sim \{x\}$ then $s \notin y[a, b]$. Since $y[a, b]$ is closed in G, there is a neighborhood U of e in G such that $sU\cap[a, b] \subset x[a, b]$. Hence, there is an α_1 such that if $\alpha > \alpha_1$, $s_\alpha \in x[a, b]$. Therefore, for all $\alpha > \alpha_1$, there is a t_α in $[a, b]$ such that $s_\alpha = xt_\alpha$. Hence, since $xt_\alpha \to xt$, $t_\alpha \to t$. Repeating this argument "the length of s" times, we get a net $\{u_\alpha\}$ in $[a, b]$ such that $u_\alpha \to e$. This is impossible since $[a, b]$ is closed in G, and $e \notin [a, b]$.

An immediate corollary to this is

COROLLARY 4.3. *If K is a compact semigroup of G and if $[a, b]$ is a closed, free subsemigroup of G then $K\cap[a, b] = \emptyset$.*

Remark. An attempt to characterize the groups G that contain closed free semigroups, in terms of G-spaces meets with the following difficulty: One can easily show that if a G-space X contains closed disjoint subsets X_1 and X_2 such that

$$S_i = \{g \in G|\ \varrho(g)\colon X_1\cup X_2 \to X_i\} \neq \emptyset \quad \text{for } i = 1, 2,$$

then S_1 and S_2 are closed disjoint right ideals of $S_1\cup S_2$. Although no example is known, it appears that the existence of such a semigroup, $S_1\cup S_2$, in G is not sufficient to guarantee the existence of a closed free semigroup, $[a, b]$, in G.

§ 5. As was mentioned in the introduction, $l^1(G)$ is not symmetric if G contains a free semigroup $[a, b]$. What can be said of $\mathscr{L}^1(G)$ if G is a nondiscrete locally compact group containing a free semigroup $[a, b]$? In order to conclude that $\mathscr{L}^1(G)$ is not symmetric, $[a, b]$ must at least satisfy some topological condition. ($\mathscr{L}^1(SO(3))$ is symmetric but $SO(3)$ contains a free group $\langle\alpha, \beta\rangle$.) In [10], we have shown that the following condition implies nonsymmetry of $\mathscr{L}^1(G)$: G contains a neighborhood U of e and elements a and b such that

$$(aU)^{n_1}(bU)^{m_1}\ldots(aU)^{n_s}(bU)^{m_s}\cap(aU)^{p_1}(bU)^{q_1}\ldots(aU)^{p_r}(bU)^{q_r} = \emptyset$$

if $n_i, p_j > 0$ for $2 \leqslant i \leqslant s$, $2 \leqslant j \leqslant r$; $m_i, q_j > 0$ for $1 \leqslant i \leqslant s-1$, $1 \leqslant j \leqslant r-1$; and $n_1, p_1, m_s, q_r \geqslant 0$: unless $r = s$ and $n_i = p_i, m_i = q_i$ for $1 \leqslant i \leqslant r$. Many groups not satisfying this condition have nonsymmetric group algebras, such as the noncompact semisimple lie groups. Hence, one hopes for a weaker sufficient condition on $[a, b]$. The following theorem gives a lower bound.

THEOREM 5. *There is a locally compact group G containing a closed free semigroup $[a, b]$ such that $\mathscr{L}^1(G)$ is symmetric.*

Proof. Let $G = SO(3)\times\langle\varepsilon\rangle$ where ε is as in Example 4.1. Since $\langle\varepsilon\rangle \subset Z(SO(3))$, G is topologically isomorphic to $SO(3)\langle\varepsilon\rangle$, and hence, contains a closed free semigroup of two generators (see Example 4.1).

By a theorem of Grothendieck, [7], $\mathscr{L}^1(G)$, after a suitable normalization, is isometrically *-isomorphic to the projective tensor product $\mathscr{L}^1(SO(3))\otimes l^1(\langle\varepsilon\rangle)$. Let $R(SO(3))$ be the Banach *-algebra obtained by adjoining the identity to $\mathscr{L}^1(SO(3))$. There is an obvious embedding of $\mathscr{L}^1(SO(3))\otimes\mathscr{L}^1(\langle\varepsilon\rangle)$ onto a closed *-subalgebra of $R(SO(3))\otimes l^1(\langle\varepsilon\rangle)$. Therefore, $\mathscr{L}^1(G)$ is symmetric if $R(SO(3))\otimes l^1(\langle\varepsilon\rangle)$ is symmetric.

Since $SO(3)$ is compact, $\mathscr{L}^1(SO(3))$ is symmetric (cf. v. Dijk [3]) and so also $R(SO(3))$. It is well known that $l^1(\langle\varepsilon\rangle)$ is symmetric. Thus, by a generalization of the Wiener–Gelfand theorem (cf. Bonic [2], Corollary 3.2) $R(SO(3))\otimes l^1(\langle\varepsilon\rangle)$ is symmetric.

This author's principal interest in groups containing NADIS can best be summarized by the following

CONJECTURE. *The group algebra of a locally compact, connected group G is not symmetric if, and only if, G contains a uniformly discrete free semigroup on two generators.*

§ 6. In this section we discuss the unitary representations of groups containing closed, or uniformly discrete, free semigroups on two generators. The principal result is

THEOREM 6.1. *A locally compact group G contains a NADIS if, and only if, there is a unitary representation, $g \to \pi(g)$, of G on a Hilbert space H such that* $S_i = \{g \in G \mid \pi(g)(H_1 \oplus H_2) \subset H_i\} \neq \emptyset$, for $i = 1, 2$, for some pair of non-trivial orthogonal subspaces H_1 and H_2 of H.

Proof. The sufficiency of the condition readily follows from Theorem 2.3.

Conversely, suppose that G contains a NADIS S. Let S_1 and S_2 be right ideals of S and U a neighborhood of e in G such that $S_1^{-1}S_2 \cap U^2 = \emptyset$. Define x in $\mathscr{L}^2(G)$ by $x(g) = x_U(g)$ for λ-almost all g, where x_U denotes the characteristic function of U and λ denotes left Haar measure. Let $g \to \pi(g)$ be the regular left representation of G on $\mathscr{L}^2(G)$, let H_i be the closed subspace of $\mathscr{L}^2(G)$ spanned by $\{\pi(g)x \mid g \in S_i\}$, for $i = 1, 2$. We have only to show that $H_1 \perp H_2$.

For any g in G,

$$\pi(g)x(t) = x(g^{-1}t) = x_U(g^{-1}t) = x_{gU}(t)$$

for λ-almost all t in G. Let $y = \sum_{L=1}^{n} a_i \pi(s_i)x$ be in H_1 and $z = \sum_{j=1}^{m} \beta_j \pi(t_j)x$ be in H_2. (Note that $s_i \in S_1$ for $1 \leqslant i \leqslant n$ and $t_j \in S_2$ for $1 \leqslant j \leqslant m$.) Then

$$\begin{aligned}(y, z) &= \int y(g)\overline{z(g)}\,d\lambda(g)\\ &= \int \Big[\sum_{i=1}^{n} a_i(\pi(s_i)x)(g)\Big]\Big[\sum_{j=1}^{m} \overline{\beta}_j(\pi(t_j)x)(g)\Big]\,d\lambda(g)\\ &= \sum_{L=1}^{n}\sum_{j=1}^{m} a_i\overline{\beta}_j\Big(\int x_{s_iU}(g)x_{t_jU}(g)\,d\lambda(g)\Big).\end{aligned}$$

But for each $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$, $s_iU \cap t_jU = \emptyset$. Therefore

$$\int x_{s_iU}(g)x_{t_iU}(g)\,d\lambda(g) = 0$$

for each $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Consequently, $(y, z) = 0$. Approximating arbitrary elements of H_1 and H_2 by such finite sums one sees that $H_1 \perp H_2$.

From the proof of Theorem 6.1 we immediately get

COROLLARY 6.2. *A locally compact group G contains a NADIS if, and only if, there exist nontrivial orthogonal subspaces H_1 and H_2 of $\mathscr{L}^2(G)$ such that $S_i = \{g \in G \mid \pi(g)(H_1 \oplus H_2) \subset H_i\} = \emptyset$ for $i = 1, 2$ where $g \to \pi(g)$ is the left regular representation of G.*

We will show that if G contains a closed free semigroup on two generators a similar, though weaker, statement can be made about the regular left representation.

Let $[a, b]$ be a closed free subsemigroup of G, and let $\{s_1, s_2, \ldots\}$ be an enumeration of $[a, b]$. Using Theorem 4.2, a sequence of compact neighborhoods of e in G, $\{U_n\}$, can be chosen so that $s_nU_n \cap s_mU_m = \emptyset$ if $n \neq m$. Furthermore, we can assume that $|U_n| \leqslant 4^{-n}$, where $|U_n|$ denotes the left Haar measure of U_n, and that $U_{n+1} \subset U_n$ for all n. If x_n is the element of $\mathscr{L}^2(G)$ which agrees with x_{U_n} λ-almost everywhere, and if $x = \sum_{n=1}^{\infty} x_n$ then $x \in \mathscr{L}^2(G)$ and $\|x\|_2 \leqslant 1$. Let $g \to \pi(g)$ be the left regular representation of G.

LEMMA 6.3. *Let $s_k \in [a, b]$ and let K be a compact subset of G such that $s_kU_1^2 \subset K$. Then $\pi(s_k)x$ is not in the closed linear span of $\{x_K(\pi(s_n)x) \mid n \neq k\}$.*

Proof. Assume that $\pi(s_k)x$ is in the closed linear span of $\{x_K(\pi(s_n)x) \mid n \neq k\}$. For some finite set $\{s_{n_1}, \ldots, s_{n_p}\}$, $x_K(\pi(s_n)x) = 0$ if $n \neq n_i$ for all $1 \leqslant i \leqslant p$. Hence, for some $(a_1, \ldots, a_p) \in C_p$,

$$\pi(s_k)x = \sum_{i=1}^{p} a_i x_K(\pi(s_{n_i})x) = y.$$

If $M = \max\{n_i \mid 1 \leqslant i \leqslant p\}$, then

$$s_{n_i}U_M \cap s_kU_M = \emptyset$$

for $1 \leqslant i \leqslant p$. Therefore, if $U \subset U_M$, then for λ-almost all $t \in s_kU$,

$$\begin{aligned}|y(t)| &= \Big|\sum_{i=1}^{p} a_i x_K(t)(\pi(s_{n_i})x)(t)\Big| = \Big|\sum_{i=1}^{p} a_i x(s_{n_i}^{-1}t)\Big|\\ &= \Big|\sum_{i=1}^{p} a_i\Big(\sum_{m=1}^{\infty} x_{U_m}(s_{n_i}^{-1}t)\Big)\Big| = \Big|\sum_{i=1}^{p} a_i \sum_{m=1}^{M} x_{U_m}(s_{n_i}^{-1}t)\Big| \leqslant M\Big|\sum_{i=1}^{p} a_i\Big|.\end{aligned}$$

Therefore, for all $U \subset U_M$,

$$\operatorname{ess\,sup}\{|y(t)| \mid t \in s_kU_m\} \leqslant M\Big|\sum_{i=1}^{p} a_i\Big|.$$

But for any n, and λ-almost all t in s_kU_n,

$$(\pi(s_k)x)(t) = \sum_{m=1}^{\infty} x_{U_m}(s_k^{-1}t) = n + \sum_{m=n+1}^{\infty} x_{U_m}(s_k^{-1}t) \geqslant n.$$

Choosing $n > M\Big|\sum_{i=1}^{p} a_i\Big|$ we get a contradiction.

Let H_1 and H_2 denote the closed linear spans of $\{\pi(as)x \mid s \in [a, b]\}$ and $\{\pi(bs) \mid s \in [a, b]\}$ respectively.

LEMMA 6.4. $H_1 \cap H_2 = \{0\}$.

Proof. Suppose that $w \in H_1 \cap H_2$, $w \neq 0$. For any compact subset K of G, $x_K w$ is in the closed linear span of $\{x_K(\pi(as)x) \mid s \in [a, b]\}$. But this latter set is finite. Hence

$$x_K w = \sum_{i=1}^{r} a_i x_K(\pi(as_{n_i})x).$$

Let K be a compact subset of G such that $\|x_K w\|_2 > 0$ and such that if

$$x_K w = \sum_{i=1}^{r} a_i x_K(\pi(as_{n_i})x)$$

then $as_{n_i}U_1 \subset K$ for some $1 \leqslant i \leqslant r$. (If no such K exist, then using the sets $K_m = \bigcup_{n=1}^{m} as_n U_1$, we contradict the fact that $0 \neq w \in H_1$.) Renumbering if necessary, we may assume that $as_{n_1} U_1 \subset K$. Thus

$$x_K w = a_1 \pi(as_{n_1}) + \sum_{i=2}^{r} a_i x_K(\pi(as_{n_i})x).$$

Since $w \in H_2$, we also have

$$x_K w = \sum_{j=1}^{t} \beta_j x_K(\pi(bs_{m_j})x).$$

Hence $\pi(as_{n_i})x$ is in the linear span of $\{x_K(\pi(s)x) \mid s \in [a, b], s \neq as_{n_1}\}$. This contradiction of Lemma 6.3 implies that $H_1 \cap H_2 = \{0\}$.

Let H denote the closed linear span of $\{H_1, H_2\}$. If $z \in H$, $z = \lim_n z_n$ where each z_n is of the form $\sum_{i=1}^{m} x_i \pi(s_i)x$. Hence, $\pi(a)z = \lim_n \pi(a)z_n$, and $\pi(a)z_n = \sum_{i=1}^{m} a_i \pi(as_i)x \in H_1$. Thus, $\pi(a)(H) \subset H_1$. Similarly, $\pi(b)(H) \subset H_2$. We have therefore proved

THEOREM 6.5. *Let G be a locally compact group containing a closed free semigroup $[a, b]$. There exist non-trivial closed subspaces H_1 and H_2 of $\mathscr{L}^2(G)$ such that $H_1 \cap H_2 = \{0\}$ and*

$$S_i = \{g \in G \mid \pi(g)\,(\overline{H_1 + H_2}) \subset H_i\} \neq \emptyset \quad \text{for } i = 1, 2,$$

where $g \to \pi(g)$ is the left regular representation of G.

Remark. It is not known if the converse of Theorem 6.5 is true. Given a representation as in the theorem, one can easily conclude that G contains a closed semigroup with disjoint right ideals I and J. As mentioned earlier, this does not appear to be sufficient to imply that G contains a closed free semigroup on two generators.

Let $\Phi(= \Phi(G))$ denote the set of all continuous positive definite functions defined on G that are one at e. $\varphi \in \Phi$ if and only if there is a unitary representation $g \to \pi(g)$ of G on H and an x in H, $\|x\| = 1$, such that $\varphi(g) = (\pi(g)x, x)$ for all g in G. This π is said to be associated with φ. φ in Φ is pure if there is an irreducible representation associated with φ. (For these definitions, see Dixmier [4].)

THEOREM 6.6. *A locally compact group G has a NADIS if and only if there exist a and b in G and a φ in Φ such that $\varphi(s^{-1}b^{-1}at) = 0$ for all $s, t \in [a, b]$.*

Proof. Assume G has a NADIS. Let $[a, b]$ be a uniformly discrete free subsemigroup of G. Let U be a compact neighborhood of e in G such that $sU \cap tU = \emptyset$ if $s, t \in [a, b]$, $s \neq t$. Let x be the $\mathscr{L}^2$-normalized characteristic function of U. If $g \to \pi(g)$ is the left regular representation of G and if $\varphi(g) = (\pi(g)x, x)$ then $\varphi \in \Phi$ and for each s, t in $[a, b]$,

$$\begin{aligned}\varphi(s^{-1}t) &= (\pi(s^{-1}t)x, x) = (\pi(t)x, \pi(s)x) \\ &= \lambda|U|^{-1} \int x_U(t^{-1}g)x_U(s^{-1}g)\,d\lambda(g) \\ &= \lambda|U|^{-1}\lambda|tU \cap sU|.\end{aligned}$$

Hence, if $s, t \in [a, b]$, $s \neq t$, $\varphi(s^{-1}t) = 0$. In particular, $\varphi(s^{-1}b^{-1}at) = 0$ for all s, t in $[a, b]$.

Conversely, if for some φ in Φ and a, b in G, $\varphi(s^{-1}b^{-1}at) = 0$ for all s, t in $[a, b]$ then the ideals $I = a[a, b]$ and $J = b[a, b]$ are clearly disjoint. Since φ is continuous and $\varphi(e) = 1$, e is not in the closure of $I^{-1}J$. Thus, $[a, b]$ is a NADIS.

Given a representation π that satisfies the conditions of Theorem 6.1, one is led to ask if there is an irreducible representation π' that satisfies these conditions. Equivalently, in light of Theorem 6.6, given a φ in $\Phi(G)$ and a, b in G such that $\varphi(s^{-1}b^{-1}at) = 0$ for all s, t in $[a, b]$, is there a pure φ' in $\Phi(G)$ such that $\varphi'(s^{-1}b^{-1}at) = 0$ for all s, t in $[a, b]$. Although the answer is not known, the following example shows that this can be the case, even for very "nice" groups, such as the real affine group, which is type I (see Nelson and Steinspring [13]).

EXAMPLE 6.7. Let G be the real affine group (see Example 2.5). If $g \in G$, there exist α, β in R, $\alpha > 0$, such that $g(x) = \alpha x + \beta$ for each x in R. Let H denote the space of Fourier transforms of $\mathscr{L}^2([0, \infty))$, i.e., H is the space of "functions" square integrable on R which are the limit values of functions that are analytic in the upper half-plane. For each g in G, define $\pi(g)$ on H by $\pi(g)f(x) = f(\alpha^{-1}(x+\beta))$ where $g(x) = \alpha x + \beta$ for all x in R. Naimark [12] has shown that $g \to \pi(g)$ is an irreducible representation of G. Let $f \in H$ such that $\|f\| = 1$ and $f(x) = 0$ for a.a. $x \notin [0, 1]$. Then if $\varphi(g) = (\pi(g)f, f)$ for all g in G, φ is a pure continuous positive definite function on G.

Let $a, b \in G$ such that $a(x) = \frac{1}{8}x + \frac{3}{4}$ and $b(x) = \frac{1}{8}x + \frac{1}{4}$ for all x in R. Then, if $s \in [a, b]$, $as([0,1]) \subset [\frac{3}{4}, \frac{7}{8}]$ and $bs([0,1]) \subset [\frac{1}{4}, \frac{3}{8}]$. Let $s, t \in [a, b]$ and suppose that $s^{-1}b^{-1}at(x) = \alpha x + \beta$ for all x in R. If $at(x) = \alpha_1 x + \alpha_2$ and $bs(x) = \beta_1 x + \beta_2$ then $\alpha_1, \beta_1 < \frac{1}{8}$, $\alpha_2 \in [\frac{3}{4}, \frac{7}{8}]$, $\beta_2 \in [\frac{1}{4}, \frac{3}{8}]$, $\alpha = \alpha_1/\beta_1$, and $\beta = (\alpha_2 - \beta_2)/\beta_1$. Since $\alpha, \beta > 0$, $\alpha^{-1}(x+\beta) \in [\beta/\alpha, (1+\beta)/\alpha]$ for each x in $[0, 1]$. Finally,

$$[\beta/\alpha, (1+\beta)/\alpha] = [(\alpha_2-\beta_2)/\alpha_1, (\alpha_2-\beta_2)/\alpha_1 + \beta_1/\alpha_1] \subset (1, \infty).$$

Hence $f(\alpha^{-1}(x+\beta))\overline{f(x)} = 0$ for a.a. x in R, and

$$\varphi(s^{-1}b^{-1}at) = (\pi(s^{-1}b^{-1}at)f, f) = \int_{-\infty}^{\infty} f(\alpha^{-1}(x+\beta))\overline{f(x)}\,dx = 0.$$

Therefore for arbitrary s, t in $[a, b]$, $\varphi(s^{-1}b^{-1}at) = 0$.

Although there exist type I groups that contain a NADIS, this is never the case for discrete groups.

THEOREM 6.8. *Suppose G is a discrete group and that G contains elements a and b such that $[a, b]$ is free, then G is not type* I.

Proof. The proof follows readily from the fact that a countable discrete group is not type I unless it contains a normal Abelian group of finite index (see Dixmier [4]).

Assume G is type I, then since $C^*(\langle a, b\rangle)$; the C^* algebra of $\langle a, b\rangle$, is a C^* subalgebra of $C^*(G)$, $\langle a, b\rangle$ is type I (see Dixmier [4]). Hence $\langle a, b\rangle$ contains a normal Abelian subgroup H of finite index. Hence, for some positive integers n, m, $a^n H = H$ and $b^m H = H$. But then $a^n, b^m \in H$, which is Abelian, and thus $a^n b^m = b^m a^n$, contradicting the assumption that $[a, b]$ is free.

A locally compact group G is CCR if for each irreducible unitary representation $g \to \pi(g)$ of G, $\pi(x)$ is a compact operator for each x in $\mathscr{L}^1(G)$. One can easily see that the group of Example 4.1, $G = SO(3) \times \langle \varepsilon\rangle$, is CCR. (If $g \to \pi(g)$ is an irreducible unitary representation of G then, since the factors of G are type I, there exist irreducible unitary representations $s \to \pi_1(s)$ of $SO(3)$ and $t \to \pi_2(t)$ of $\langle\varepsilon\rangle$ such that $\pi = \pi_1 \oplus \pi_2$. Since both π_1 and π_2 are finite dimensional, π finite dimensional, and hence G is CCR.) Therefore, a CCR group may contain a closed free subsemigroup on two generators.

References

[1] K. I. Appel and F. M. Djorup, *On the group generated by a free semigroup*, Proc. Amer. Math. Soc. 15 (1964), pp. 838–840.

[2] R. A. Bonic, *Symmetry in group algebras of discrete groups*, Pacific J. Math. 11 (1961), pp. 73–94.

[3] G. van Dijk, *On symmetry of group algebras of motion groups*, Math. Ann. 179 (1969), pp. 219–226.

[4] J. Dixmier, *Les C* algebres et leurs representations*, Paris 1964.

[5] A. H. Frey, *Studies in amenable semigroups*, Thesis, University of Washington, Seattle, Washington 1960.

[6] F. Greenleaf, *Invariant means on topological groups*, New York 1969.

[7] A. Grothendieck, *Produits tensoriels topologique et espaces nucléaires*, Mem. Amer. Math. Soc. 16 (1955).

[8] M. Hochster, *Subsemigroups of amenable groups*, Proc. Amer. Math. Soc. 21 (1969), pp. 363–364.

[9] J. W. Jenkins, *Symmetry and nonsymmetry in the group algebras of discrete groups*, Pacific J. Math. 32 (1970), pp. 131–145.

[10] — *Nonsymmetric group algebras* (to appear).

[11] D. Montgomery and L. Zippin, Topological transformation groups, New York 1955.

[12] M. A. Naimark, *Normed Rings*, Groningen 1960.

[13] E. Nelson and W. F. Steinspring, *Representation of elliptic operators in an enveloping algebra*, Amer. J. Math. 81 (1959), pp. 547–560.

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Received April 18, 1971 (323)