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On topologically nilpotent algebras

by

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Abstract. A B_0-algebra A is called topologically nilpotent if, for each continuous pseudonorm $|\ |$ in A, there exists a neighbourhood of zero V such that

$$\lim_n \sup_{x_i \in V} \sqrt[n]{|x_1 x_2 \dots x_n|} = 0.$$

Our main result states that a commutative B_0-algebra A is topologically nilpotent if and only if there exists a power series having radius of convergence 0, operating in A. The second theorem states that a commutative B_0-algebra is topologically nilpotent if and only if, it is isomorphic with the projective limit of a sequence of topologically nilpotent Banach algebras.

§ 1. Introduction. This paper is devoted to the characterization of topologically nilpotent B_0-algebras. The class of topologically nilpotent Banach algebras was introduced by P. G. Dixon (in a letter to Professor W. Żelazko).

It is easy to see that all topologically nilpotent B_0-algebras must be necessarily multiplicatively-convex.

The first main theorem states that a commutative B_0-algebra A is topologically nilpotent if and only if there exists a power series having radius of convergence 0, operating in A (this result is also new for Banach algebras).

The second theorem states that a commutative B_0-algebra is topologically nilpotent if and only if, it is isomorphic with the projective limit of a sequence of topologically nilpotent Banach algebras.

§ 2. Basic facts and examples.

2.1. DEFINITION. A B_0-algebra A is called *topologically nilpotent* if, for each continuous pseudonorm $|\ |$ in A, there exists a neighbourhood of zero V such that

$$\lim_n \sup_{x_i \in V} \sqrt[n]{|x_1 x_2 \dots x_n|} = 0.$$

2.2. THEOREM. *If A is a topologically nilpotent B_0-algebra then A is m-convex.*

Proof. For any continuous pseudonorm $|\ |_i$ let V_i be a neighbourhood of zero such that

$$\sup_{y_j \in V_i} \sqrt[n]{|y_1 y_2 \dots y_n|_i} = \varepsilon_{i,n} \to 0. \tag{2.2.1}$$

Clearly we may assume that

$$V_i = \{y\colon\ |y|_{i+1} < 1\}.$$

Proof will be based upon the Lemma 13.10 [5]. Thus it will be sufficient to show that there exists a matrix C_{in} of positive reals $i, n = 1, 2, \dots,$ such that

$$|x_1 x_2 \dots x_n|_i \leqslant C_{in} |x_1|_{i+1} \dots |x_n|_{i+1} \tag{2.2.2}$$

where $x_1, \dots, x_n \in A$ and $p_i = \sup \sqrt[n]{C_{in}} < \infty$. Let $x_1, \dots, x_n \in A$. Suppose that $|x_k|_{i+1} = 0$ for an integer k, $1 \leqslant k \leqslant n$. From (2.2.1) it is easy to see that $|x_1 \dots x_n|_i = 0$, thus (2.2.2) holds. If $|x_k|_{i+1} \neq 0$, $k = 1, \dots, n$, we can put

$$y_k = \frac{x_k}{2|x_k|_{i+1}}, \quad \text{so } y_k \in V_i \quad \text{for } k = 1, \dots, n,$$

we have

$$\varepsilon_{in}^n \geqslant |y_1 y_2 \dots y_n|_i = \frac{|x_1 x_2 \dots x_n|_i}{2^n |x_1|_{i+1} \dots |x_n|_{i+1}},$$

so

$$|x_1 x_2 \dots x_n|_i = (2\varepsilon_{in})^n |x_1|_{i+1} \cdot \dots \cdot |x_n|_{i+1},$$

so (2.2.2) holds.

2.3. Example. $A = C(0,1)$ consists of continuous functions on the interval $[0,1]$ with convolution multiplication

$$x \times y(\tau) = \int_0^\tau x(t) \cdot y(\tau - t)\, dt,$$

and supremum norm.

It is clearly a commutative, radical Banach algebra and from the fact, that for $x_1, x_2, \dots, x_n \in A$, $\|x_i\| \leqslant 1$, we have

$$\|x_1 \times x_2 \times \dots \times x_n\| \leqslant \big\| \|x_1\| e \times \dots \times \|x_n\| e \big\|$$
$$\leqslant \|e \times \dots \times e\| \leqslant \frac{1}{(n-1)!},$$

where $e = e(t) \equiv 1$, it follows that A is topologically nilpotent. It easy to see that $A_0 = \{x \in A\colon x(0) = 0\}$ is a subalgebra of A, and A_0 is also topologically nilpotent.

Put

$$u_n(x) = \begin{cases} n\pi/2 \sin n\pi x & \text{for } 0 \leqslant x \leqslant 1/n, \\ 0 & \text{for } 1/n \leqslant x \leqslant 1. \end{cases}$$

One may verify $u_n(x)$ is an approximate identity in A_0. This sequence is unbounded, which also follows from the following:

2.4. Proposition. *If A is a B_0-algebra with a bounded approximate identity, then A is not topologically nilpotent.*

Proof. For a given continuous pseudonorm $\|$ and a neighbourhood V of zero, we choose an $x_1 \in V$ such that $|x_1| = a > 0$. If $\{e_\lambda\}_{\lambda \in \Lambda}$ is a bounded approximate identity, then for a certain constant $C > 0$ we have:

$$\{e_\lambda\}_{\lambda \in \Lambda} \subset C \cdot V. \tag{2.4.1}$$

There exists a λ_2 such that:

$$|x_1 e_{\lambda_2} - x_1| \leqslant a/4.$$

We put $x_2 = e_{\lambda_2}$, and pick a $\lambda_3 > \lambda_2$ such that

$$|x_1 x_2 e_{\lambda_3} - x_1 x_2| \leqslant a/8,$$

and so on. We have

$$|x_1 x_2 \dots x_{n-1} x_n - x_1 x_2 \dots x_{n-1}| \leqslant \frac{a}{2^{n+1}},$$

so

$$|x_1 - x_1 \dots x_n| \leqslant a/2,$$

and

$$|x_1 x_2 \dots x_n| \geqslant a/2. \tag{2.4.2}$$

In virtue of 2.4.1 and 2.4.2 we have

$$\sup_{y_i \in V} \sqrt[n]{|y_1 y_2 \dots y_n|} \geqslant \frac{1}{c} \cdot \sqrt[n]{|x_1 x_2 \dots x_n|} \geqslant \frac{1}{c} \sqrt[n]{\frac{a}{2}};$$

hence A is not topologically nilpotent. ∎

Obviously a topologically nilpotent algebra must be necessarily radical. The converse statement, however is not true, as is shown by the following:

2.5. Example. $A = L(0,1)$ consists of summable functions on the interval $[0,1]$ with convolution multiplication and norm

$$\|x\| = \int_0^1 |x(t)|\, dt.$$

It is a commutative, radical Banach algebra (see [2] A.2.11), but it is easy to see that A has a bounded approximate identity (cf. [4]), so A is not a topologically nilpotent algebra.

Therefore a necessary (but not sufficient) condition for a B_0-algebra to be topologically nilpotent is that the algebra in question be radical and have no a bounded approximate identity.

§ 3. Characterization of topologically nilpotent algebras. We assume A to be a commutative, complex B_0-algebra. Our essential theorem will characterize topologically nilpotent B_0-algebras, and its proof will be based upon the following lemma which is due to W. Żelazko, ([5], Proposition 13.12). For sake of completness we reproduce here the proof.

3.1. Lemma. *Let A be a commutative B_0-algebra and U a convex subset of A. Put $V = \operatorname{conv}\big(U\cup(-U)\big)$ and*

$$Q^n_{\|\,\|,P} = \sqrt[n]{\sup_{x_i\in P}\|x_1\ldots x_n\|}, \qquad \tilde{Q}^n_{\|\,\|,P} = \sqrt[n]{\sup_{x\in P}\|x^n\|};$$

then

$$\tilde{Q}^n_{\|\,\|,U} \leqslant \tilde{Q}^n_{\|\,\|,V} \leqslant Q^n_{\|\,\|,V} = Q^n_{\|\,\|,U} \leqslant M\tilde{Q}^n_{\|\,\|,U}, \tag{3.1.1}$$

where M is a constant >0.

Proof. It is clear that $\tilde{Q}^n_{\|\,\|,U} \leqslant \tilde{Q}^n_{\|\,\|,V} \leqslant Q^n_{\|\,\|,V}$. It remains to prove that $Q^n_{\|\,\|,V} = Q^n_{\|\,\|,U}$ and $Q^n_{\|\,\|,U} \leqslant M\tilde{Q}^n_{\|\,\|,U}$. To prove the first equality let us observe that

$$\sup_{x_i\in V}\|x_1\ldots x_n\| = \sup_{x_i\in U\cup(-U)}\|x_1\ldots x_n\|,$$

so it remains to show that if W is any set, then

$$\sup_{x_i\in W}\|x_1\ldots x_n\| = \sup_{x_i\in \operatorname{conv} W}\|x_1\ldots x_n\|.$$

Let $x_1,\ldots,x_n\in\operatorname{conv}W$; so

$$x_i = \sum_{m=1}^{M_i}a_{i,m}\bar{x}_{i,m}, \quad \bar{x}_{i,m}\in W, \quad 0\leqslant a_{i,m}\leqslant 1, \quad \text{and} \quad \sum_{m=1}^{M_i}a_{i,m} = 1.$$

So we have

$$\begin{gathered} x_1\ldots x_n = \sum_{m_1}\ldots\sum_{m_n}a_{1,m_1}\ldots a_{n,m_n}\bar{x}_{1,m_1}\ldots\bar{x}_{n,m_n}, \\ \sum_{m}a_{1,m_1}\ldots a_{n,m_n} = 1, \end{gathered} \tag{3.1.2}$$

the terms being between 0 and 1. So

$$\begin{aligned} \|x_1\ldots x_n\| &\leqslant \sum_m a_{1,m_1}\ldots a_{n,m_n}\|\bar{x}_{1,m_1}\ldots\bar{x}_{n,m_n}\| \\ &\leqslant \sum_m a_{1,m_1}\ldots a_{n,m_n}\cdot\sup_{x_i\in W}\|x_1\ldots x_n\| = \sup_{x_i\in W}\|x_1\ldots x_n\|. \end{aligned}$$

So

$$\sup_{x_i\in\operatorname{conv}W}\|x_1\ldots x_n\| \leqslant \sup_{x_i\in W}\|x_1\ldots x_n\|;$$

but having $W\subset\operatorname{conv}W$ we obtain the equality. To prove $Q^n_{\|\,\|,U}\leqslant M\tilde{Q}^n_{\|\,\|,U}$ we shall consider the generalization of formula $xy = \frac{1}{2}[(x+y)^2-x^2-y^2]$, namely

$$x_1\ldots x_n = \frac{(-1)^n}{n!}\sum_{k=1}^{n}(-1)^k W^n_k(x_1\ldots x_n),$$

where

$$W^n_k = \sum_{1\leqslant i_1<i_2<\ldots<i_n\leqslant n}(x_{i_1}+\ldots+x_{i_k})^n.$$

Let us observe, that

$$\|x_1\ldots x_n\| \leqslant \frac{1}{n!}\sum_{k=1}^{n}\|W^n_k(x_1\ldots x_n)\|,$$

and since U is convex

$$\sup_{x_i\in U}\|W^n_k(x_1\ldots x_n)\| \leqslant \sum_{i_p<i_{p+1}}\sup_{x_i\in U}\|(x_{i_1}+\ldots+x_{i_k})^n\| \leqslant \binom{n}{k}n^n\sup_{x\in U}\|x^n\|,$$

so

$$\sup_{x_i\in U}\|x_1\ldots x_n\| \leqslant \frac{(2n)^n}{n!}\sup_{x\in U}\|x^n\| \leqslant M^n\sup_{x\in U}\|x^n\|,$$

and $Q^n_{\|\,\|,U}\leqslant M\tilde{Q}^n_{\|\,\|,U}$.

3.2. Theorem. *Let A be a commutative, complex B_0-algebra, then the following conditions are equivalent:*

(i) *A is topologically nilpotent;*

(ii) *For each continuous pseudonorm $|\ |$ in A there exists a neighbourhood of zero V such that:*

$$\lim_n\sup_{x\in V}\sqrt[n]{|x^n|} = 0.$$

(iii) *There exists a power series with complex coefficients $\Sigma a_n\lambda^n$, having radius of convergence 0, operating in A i.e. $\Sigma a_n x^n$ is convergent for each $x\in A$.*

Proof. (i) $\to$ (iii). Let $(|\ |_i)$ be a countable family of homogeneous pseudonorms giving a topology in A. In the presence of Theorem 2.2 we may assume that

$$|xy|_i \leqslant |x|_i|y|_i. \tag{3.2.1}$$

Let V_i be a neighbourhood of zero such that

$$\lim_n\sup_{x_i\in V_i}\sqrt[n]{|x_1x_2\ldots x_n|_i} = 0. \tag{3.2.2}$$

Clearly, we may assume that

$$V_i\subset\{x\colon |x|_i<1/i\}. \tag{3.2.3}$$

Consider the sequence defined by the relation:

$$a_n = (\max_{i\leqslant n}\sup_{x\in V_i}|x^n|_i)^{-1/2}.$$

Choose a positive integer N in such a way that for $n > N$ we have:

(3.2.4) $$\sup_{x\in V_i}\sqrt[n]{|x^n|_i}\leqslant\varepsilon^2 \quad\text{for } i\leqslant 1/\varepsilon^2.$$

We get from 3.2.1, 3.2.3

(3.2.5) $$\sup_{x\in V_i}\sqrt{|x^n|_i}\leqslant 1/i\leqslant\varepsilon^2 \quad\text{for } i\geqslant 1/\varepsilon^2.$$

In virtue of 3.2.4 and 3.2.5 we have:

$$(\sqrt[n]{a_n})^{-1} = \sqrt{\max_{i\leqslant n}\sup_{x\in V_i}\sqrt[n]{|x^n|_i}}\leqslant\varepsilon \quad\text{for } n>N,$$

so $\sum a_n\lambda^n$ has a radius of convergence 0.

It remains to be shown that $\sum a_n\lambda^n$ is operating in A. For given i we choose a positive $t_i\neq 0$ in such a way that

$$xt_i\in V_i.$$

We have the following estimation:

$$\sqrt[n]{|a_nx^n|_j} = \frac{\sqrt[n]{|x^n|_j}}{\sqrt{\max\limits_{i\leqslant n}\sup\limits_{y\in V_i}\sqrt[n]{|y^n|_i}}} \leqslant \frac{1}{t_j}\,\frac{\sup\limits_{y\in V_j}\sqrt[n]{|y^n|_j}}{\sqrt{\max\limits_{i\leqslant n}\sup\limits_{y\in V_i}\sqrt[n]{|y^n|_i}}}$$

$$\leqslant\frac{1}{t_j}\sqrt{\sup_{y\in V_j}\sqrt[n]{|y^n|_j}} \quad\text{for } n>i.$$

In the presence of 3.2.2, $\sum a_n\lambda^n$ is operating in A, which proves (iii).

(iii) → (ii). Suppose that $\sum a_n\lambda^n$ is a power series having radius of convergence 0, operating in A (clearly we may assume that $a_n\neq 0$ for each n).

Setting

(3.2.6) $$A_n^i = \{x\in A:\ |a_kx^k|_i\leqslant 1;\quad \text{for } k\geqslant n\};$$

we have that every A_n^i is closed and for fixed i

$$\bigcup_{n=1}^{\infty}A_n^i = A;$$

so there is an $n(i)$ and a convex open set U_i such that

$$U_i\subset \operatorname{Int}A^i_{n(i)}.$$

Let $V_i = \operatorname{conv}(U_i\cup(-U_i))$. We get from 3.1.1, and 3.2.6 that

$$\sup_{x\in V_i}\sqrt[k]{|a_kx^k|_i}\leqslant M \quad\text{for each } k\geqslant n(i);$$

so

$$\sup_{x\in V_i}\sqrt[n]{|x^n|_i}\leqslant\frac{M}{\sqrt[n]{|a_n|}}\to 0,$$

and we have proved (ii).

(ii) → (i) follows immediately from Lemma 3.1. ∎

The following problem is open:

3.3. Problem. *Is the first conclusion of Theorem* 3.2. ((ii) → (i)) *true for non-commutative* B_0*-algebras.*

§ 4. Some properties of topologically nilpotent algebras. The property of being topologically nilpotent algebra is invariant under the following operations on B_0-algebras:

(a) Let A be a topologically nilpotent B_0-algebra and I a closed ideal ideal in A, then A/I is topologically nilpotent.

(b) Every subalgebra of a topologically nilpotent B_0-algebra is topologically nilpotent.

(c) The Cartesian product of a countable family of topologically nilpotent B_0-algebras is topologically nilpotent.

The above properties follow immediately from the definition.

(d) The projective limit of a countable family of topologically nilpotent B_0-algebras is topologically nilpotent.

This follows from (b) and (c).

(e) The complete Tensor product with the projective topology of a topologically nilpotent B_0-algebra and an m-convex B_0-algebra is topologically nilpotent.

Proof. Let A be a topologically nilpotent B_0-algebra and let B be an m-convex, B_0-algebra.

Consider a continuous pseudonorm p in A and a continuous submultiplicative pseudonorm q in B. The topology in $A\tilde{\otimes}B$ may be given by means of the family of pseudonorms form $p\otimes q$ (cf. [3]).

Chooose a continuous pseudonorm p' in A such that

(4.1) $$\lim_n\sup_{p'(y_i)<1}\sqrt[n]{p(y_1\dots y_n)} = 0.$$

In order to complete the proof it will be sufficient to show that:

$$\lim\sup_{p'\otimes q(z)<1}\sqrt[n]{p\otimes q(z^n)} = 0.$$

If $z \in A \tilde{\otimes} B$ and $p' \otimes q(z) < 1$; then

$$z = \sum_{i=1}^{\infty} \lambda_i x_i \otimes y_i, \quad \text{where } x_i \in A,\ y_i \in B,$$

and

(4.2) $\sum |\lambda_i| \leqslant 1,\ p'(x_i) \leqslant 1,\ q(y_i) \leqslant 1$ (cf. [3] Theorem III 6.4).

From (4.1) and (4.2) we have the following estimation:

$$\begin{aligned} p \otimes q(z^n) &= p \otimes q\Big(\sum_{i_1 \ldots i_n} \lambda_{i_1} \ldots \lambda_{i_n} x_{i_1} \ldots x_{i_n} \otimes y_{i_1} \ldots y_{i_n}\Big) \\ &\leqslant \sum_{i_1 \ldots i_n} |\lambda_{i_1}| \ldots |\lambda_{i_n}| \cdot p(x_{i_1} \ldots x_{i_n}) \cdot q(y_{i_1}) \ldots q(y_{i_n}) \\ &\leqslant \sum_{i_1 \ldots i_n} |\lambda_{i_1}| \ldots |\lambda_{i_n}| \cdot \sup_{p'(y_i)<1} p(y_1 \ldots y_n) \leqslant \sup_{p'(y_i)<1} p(y_1 \ldots y_n). \end{aligned}$$

It follows that

$$\sup_{p' \otimes q(z) < 1} \sqrt[n]{p \otimes q(z^n)} \leqslant \sup_{p'(y_i) \leqslant 1} \sqrt[n]{p(y_1 \ldots y_n)} \to 0.$$

and the disered result follows.

§ 5. Connections between topologically nilpotent Banach and B_0-algebras. We have shown that the projective limit of a countable family of topologically nilpotent Banach algebras is topologically nilpotent. Now we will show that the converse theorem also holds.

5.1. LEMMA. *Let U and V be convex neighbourhoods of zero in a commutative topological algebra A such that*

(5.1.1) $$U^2 \subset U, \quad V \subset \tfrac{1}{4} U, \quad V^n \subset \varepsilon_n U;$$

where $\varepsilon_n \geqslant 0$ is a sequence covergent to zero.

Let

$$W = \operatorname{conv}(V + VU).$$

Then

(5.1.2) $$W^n \subset 2^n \varepsilon_{n-1} W,$$

(5.1.3) $$W \subset U$$

and

(5.1.4) $$W^2 \subset W.$$

Proof. Consider the following inclusions:

$$\operatorname{conv}(V^n) = \operatorname{conv}(V^{n-1} V) \subset \operatorname{conv}(\varepsilon_{n-1} UV) = \varepsilon_{n-1} \operatorname{conv}(UV),$$
$$\operatorname{conv}(V^n U^k) \subset \varepsilon_{n-1} \operatorname{conv}(VU^{k+1}) \subset \varepsilon_{n-1} \operatorname{conv}(VU).$$

From the above inclusions and (5.1.1) we have:

(5.1.5) $$\begin{aligned} (V + VU)^n &\subset \operatorname{conv}(V^n) + \binom{n}{1} \operatorname{conv}(V^n U) + \ldots + \binom{n}{n} \operatorname{conv}(V^n U^n) \\ &\subset \varepsilon_{n-1} \operatorname{conv}(UV) + \binom{n}{1} \varepsilon_{n-1} \operatorname{conv}(UV) + \ldots + \binom{n}{n} \varepsilon_{n-1} \operatorname{conv}(UV) \\ &\subset 2^n \varepsilon_{n-1} \operatorname{conv}(UV) \subset 2^n \varepsilon_{n-1} W. \end{aligned}$$

In the same way as in (3.1.2) we get

$$W^n = [\operatorname{conv}(V + VU)]^n \subset \operatorname{conv}[(V + VU)^n],$$

so

$$W^n \subset 2^n \varepsilon_{n-1} W.$$

Moreover

$$W \subset \operatorname{conv}(\tfrac{1}{4} U + \tfrac{1}{4} U^2) \subset \operatorname{conv}(\tfrac{1}{4} U + \tfrac{1}{4} U) \subset U,$$
$$\begin{aligned} W^2 &\subset \operatorname{conv}(V + VU)^2 \subset \operatorname{conv}(V^2) + 2\operatorname{conv}(V^2 U) + \operatorname{conv}(V^2 U^2) \\ &\subset \operatorname{conv}(VU) \subset W, \end{aligned}$$

which proves the lemma. ■

5.2. THEOREM. *Let A be a commutative topologically nilpotent B_0-algebra. Then A is the projective limit of a countable family of topologically nilpotent Banach algebras.*

Proof. Let $(|\ |_i)$ be a countable family of submultiplicative pseudonorms giving the topology in A.

For a fixed i choose an idempotent, convex neighbourhood of zero V such that

(5.2.1) $$\sup_{x_j \in V} \sqrt[n]{|x_1 \ldots x_n|_i} = \sqrt[n]{\varepsilon_n} \to 0.$$

Put

(5.2.2) $$U = \{x:\ |x|_i < 1\}.$$

Clearly we may assume that

(5.2.3) $$V \subset \tfrac{1}{4} U.$$

We get from (5.2.1) and (5.2.2)

(5.2.4) $$V^n \subset \varepsilon_n U.$$

Let $W = \operatorname{conv}(U + UV)$ and denote the Minkowski functional of W by $\|\ \|_i$. Clearly $\|\ \|_i$ is a submultiplicative pseudonorm in A. From (5.1.3) it follows that the new system $(\|\ \|_i)$ is equivalent to the old one.

In virtue of (5.1.2)

$$\sup_{\|x\|_i \leqslant 1} \sqrt[n]{\|x^n\|_i} \leqslant 2 \sqrt[n]{\varepsilon_{n-1}} \to 0;$$

so each $\tilde{A}_i = \widetilde{A/\ker\| \|_i}$ is a topologically nilpotent Banach algebra. From ([5], Theorem 10.10) it follows that A is the projective limit of $\tilde{A}_i$. ■

5.3. COROLLARY. *An m-convex B_0-algebra A is topologically nilpotent if and only if there exists a system of pseudonorms giving the topology in A such that every $\tilde{A}_i$ is a topologically nilpotent Banach algebra.*

5.4. Remark. For a given system of pseudonorms in A, $\tilde{A}_i$ need not be topologically nilpotent. Indeed, take the Cartesian product $\prod_{i=1}^{\infty} \tilde{A}_i$, where $\tilde{A}_i = C(0, 1)$ from Example 2.3.

Put

$$|x|_i = \|x_1\| + \ldots + \|x_i\| + \int_0^1 |x_{i+1}(t)|\, dt \quad \text{ for } x = (x_1, x_2, \ldots) \in A.$$

One may verify, that for each i, $\widetilde{A/\ker| \ |_i}$ is not a topologically nilpotent algebra, but A is a topologically nilpotent algebra.

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Diagonal nuclear operators

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Abstract. Let E and F be Banach spaces with total biorthogonal sequences (x_n, f_n) and (y_n, g_n) respectively. An operator $T\colon E \to F$ is called diagonal if $g_i(Tx_j) = 0$ for $i \neq j$. The diagonal of a linear operator T is the scalar sequence $(g_i(Tx_i))$. A sequence space representation $\mathscr{S}(E, F)$ for the diagonals of the nuclear operators is given and a necessary and sufficient condition is obtained for $\mathscr{S}(E, F)$ to be the diagonal nuclear operators. In particular this is the case when (x_n, f_n) is an unconditional shrinking basis for E and (y_n, g_n) is an unconditional basis for F. As another application of this result, it is shown that if the coordinate vectors form an unconditional basis for the BK-space E then the vectors from E give precisely the diagonal nuclear operators from l_1 into E.

1. Introduction. Let E and F be Banach spaces with total biorthogonal sequences (x_n, f_n) and (y_n, g_n) respectively. If T is a linear operator from E to F then by the diagonal of T we mean the sequence $\delta(T) = (g_i(Tx_i))_{i=1}^{\infty}$. An operator T from E to F is called diagonal if $g_i(Tx_j) = 0$ for $i \neq j$. The purpose of this paper is to determine the diagonal nuclear operators between certain Banach spaces. In Section 3 we present a simple proof that the diagonal nuclear operators on a space with an unconditional basis are l_1 and we obtain a sequence space representation for the diagonals of the nuclear operators in the case where (x_n, f_n) and (y_n, g_n) are complete biorthogonal sequences. This sequence space $\mathscr{S}(E, F)$ is a generalization of the series space studied by Ruckle in [4]. In Section 4 we show that if E' or F has the approximation property then a necessary and sufficient condition for $\mathscr{S}(E, F)$ to be the diagonal nuclear operators is that the diagonal of every continuous linear operator from E' to F' be well defined as a linear operator from E' to F'. In particular if E has an unconditional shrinking basis and F has an unconditional basis then the diagonal nuclear operators are determined.

After completing this work, the authors became aware of the results of Ruckle in [5]. There is overlap between Ruckle's work and the results that appear in our preliminary section.

2. Notation and terminology. If (x_n, f_n) is a total biorthogonal sequence for the Banach space E then E can be identified with the linear