STUDIA MATHEMATICA, T. XLIII. (1972)

Interpolation of sublinear operators on generalized Orlicz and Hardy-Orlicz spaces

by

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Abstract. The Riesz–Thorin interpolation theorem is proved for sublinear operators on certain generalized Orlicz spaces, L^{φ}. The corresponding interpolation theorem for the Hardy–Orlicz spaces, H^{φ}, is also obtained. The interpolation theory above is extended to the case when there are certain factors, resembling the Randon–Nikodym derivative of measures, included. This treatment includes the known work in change of measures in the L^p-theory and generalizes the work to certain Orlicz spaces. Finally, the interpolation of $\{T_z\}$, a family of operators which depend on a smooth parameter, is obtained.

INTRODUCTION

Let L^{p_i}, L^{q_i} $(i = 1, 2)$ be Lebesgue spaces on a measure space. Let $T\colon L^{p_i} \to L^{q_i}$ be a linear operator such that $\|Tf\|_{q_i} \leqslant M_i \|f\|_{p_i}, f \in L^{p_i}$. If $p_t^{-1} = (1-t)p_1^{-1} + tp_2^{-1}$ and $q_t^{-1} = (1-t)q_1^{-1} + tq_2^{-1}$, then by the classical Riesz–Thorin interpolation theorem $T\colon L^{p_t} \to L^{q_t}$ such that

$$\|Tf\|_{q_t} \leqslant M_1^{1-t} M_2^t \|f\|_{p_t}.$$

The importance of this result in analysis (both classical and abstract) is well-known.

In many problems of Fourier analysis, an operator T is defined on the spaces above that is not linear. It is sublinear instead, i.e., it satisfies

(i) $T(f_1+f_2)$ is defined whenever Tf_i are defined,

(ii) $|T(f_1+f_2)| \leqslant |Tf_1| + |Tf_2|$,

(iii) $|T(af)| = |a|\,|Tf|$, for every scalar a.

Calderón and Zygmund [6] were the first to treat the interpolation of sublinear operators.

Numerous other generalizations have been obtained. Stein and Weiss [30] have extended the result when the underlying measures are varied with the spaces, and Stein [29] proved an interpolation theorem for operators T depending on a complex parameter z. Riordan [28], has extended Marcinkiewicz's result to Orlicz spaces, and the Riesz–Thorin theorem was extended to these spaces by Rao in (24).

Recently work on Hardy spaces was also of interest and the interpolation problem was considered there (cf. [5] and [32]). Since there is a close relation between these spaces and the Lebesgue (Orlicz) spaces, the problem is considered to include both types of spaces.

In this paper, the interpolation problem for sublinear operators is considered for certain generalized Orlicz spaces and these results are then used to obtain similar results on Hardy–Orlicz spaces. When specialized to the Lebesgue case, the L^p spaces for $0 < p \leqslant \infty$ are included, and they apply to the H^p spaces, $0 < p \leqslant \infty$, as well. Moreover, the study is always made in the case of sublinear operators. Most of the above mentioned results are subsumed in this study.

In Section 2, the interpolation theorem for sublinear operations on generalized Orlicz spaces, L^φ, is proved and then, using this, the corresponding interpolation theorem for the Hardy–Orlicz spaces, H^φ, is obtained.

In Section 3, the interpolation theory of the preceeding section is extended to the case when there are certain factors, resembling the Radon–Nikodym derivatives of measures, included. This treatment includes the known work on the change of measures in the L^p-theory and generalizes the work to certain Orlicz spaces.

Finally Section 4 contains the interpolation of $[T_z]$, a family of operators which depend on a smooth complex parameter. This generalizes the analytic parameter case of Stein [29]. Also the relation between interpolation with factors and change of measures is discussed here.

Generally the notation used is from [8] and [34]. Also Theorem (Lemma and Corollary) 1.2.3 will mean Theorem (Lemma or Corollary) 3 of subsection 2 in Section 1. Similarly equation (2.1.25) will mean equation 25 of subsection 1 in Section 2.

1. PRELIMINARIES

In this section generalized Orlicz and Hardy spaces are defined and some needed properties of these spaces will be given. Also some results on subharmonic functions are included for use later.

1.1. Generalized Orlicz spaces. The following results are from [18]. Generalized Young's functions, called φ-functions are needed and are given by the following:

DEFINITION 1.1.1. A function $\varphi(\cdot)$ is called a *φ-function* if φ is continuous, defined for $u \geqslant 0$ non-decreasing, vanishing only at $u = 0$, and such that $\varphi(u) \to \infty$ as $u \to \infty$.

Let φ be a φ-function and (Ω, Σ, μ) a measure space. The class of scalar valued functions f measurable on (Ω, Σ, μ) such that $\int_\Omega \varphi(|f|)\,d\mu < \infty$ is denoted by $L^{*\varphi}(\Omega, \Sigma, \mu) = L^{*\varphi}$ if the measure space is understood. The class $L^\varphi(\Omega, \Sigma, \mu)$ is the class of all (equivalence classes of) f such that $\lambda f \in L^{*\varphi}(\Omega, \Sigma, \mu)$ for some $\lambda > 0$. The class $L^{*\varphi}$ is a lattice and the class L^φ is linear. If φ is also convex, then L^φ is called an Orlicz space on (Ω, Σ, μ).

PROPOSITION 1.1.1. *A necessary and sufficient condition for $L^{*\varphi} = L^\varphi$ is that φ satisfy*

$$\varphi(2u) \leqslant k\varphi(u) \quad \textit{for } u \geqslant 0. \tag{1.1.1}$$

For a proof of this proposition, see [31].

The condition (1.1.1) is called the Δ_2-condition. The notation $\varphi \in \Delta_2$ will mean φ satisfies (1.1.1).

Remark. If $\varphi \in \Delta_2$ then it is clear that for every $k > 0$, $\varphi(ku) \leqslant C_k\varphi(u)$ for $u \geqslant 0$, where C_k depends only on k.

DEFINITION 1.1.2. A real valued non-negative function $\|\cdot\|$ on a linear space X is called a *F-norm* if it satisfies the following conditions

$$\begin{aligned} &\|x+y\| \leqslant \|x\| + \|y\| \quad \text{for all } x, y \in X, \\ &\|x\| = 0 \quad \text{if and only if } x = 0. \end{aligned} \tag{1.1.2}$$

An F-norm can be introduced on L^φ in such a way that convergence of a sequence, f_n, to 0 with respect to this norm implies $\int_\Omega \varphi(|f_n|)\,d\mu \to 0$, too. This norm (throughout the paper norm will actually mean F-norm) is defined by:

$$\|f\|_\varphi = \inf\left\{\varepsilon > 0\colon \int_\Omega \varphi\left(\frac{|f|}{\varepsilon}\right)d\mu \leqslant \varepsilon\right\} \tag{1.1.3}$$

and called the *norm generated by φ*. With this norm $L^\varphi(\Omega, \Sigma, \mu)$ becomes a Fréchet space, and we call $[L^\varphi(\Omega, \Sigma, \mu), \|\cdot\|_\varphi]$ a *generalized Orlicz space*.

PROPOSITION 1.1.2. *If $\varphi \in \Delta_2$, then simple functions on (Ω, Σ, μ), denoted by $\mathscr{L}(\Omega, \Sigma, \mu)$, are dense in $L^\varphi(\Omega, \Sigma, \mu)$.*

PROPOSITION 1.1.3. *If $\varphi \in \Delta_2$ and $f_n \in L_\varphi$ such that $\int_\Omega \varphi(|f_n|)\,d\mu \to 0$, then $\|f_n\|_\varphi \to 0$.*

If $\varphi(u) = \psi(u^r)$ where $0 < r \leqslant 1$, and ψ is a convex φ-function, define;

$$\|f\|_{r\varphi} = \inf\left\{\varepsilon > 0\colon \int_\Omega \varphi\left(\frac{|f|}{\varepsilon^{1/r}}\right)d\mu \leqslant 1\right\} \tag{1.1.4}$$

for $f \in L^\varphi(\Omega, \Sigma, \mu)$. Then $\|\cdot\|_{r\varphi}$ is an *r-homogeneous F-norm*, that is an F-norm with the additional property

$$\|af\|_{r\varphi} = |a|^r\|f\|_{r\varphi} \quad \text{for every scalar } a. \tag{1.1.5}$$

Remark. If $r = 1$, then $\|\cdot\|_{1\varphi}$ is the Minkowski norm on L^{φ}. ([31]).

PROPOSITION 1.1.4. *The F-norm $\|\cdot\|_{\varphi}$ is equivalent to $\|\cdot\|_{r\varphi}$.*

1.2. Hardy–Orlicz spaces. The following results from [16] are needed for later work.

DEFINITION 1.2.1. A φ-function φ which can be represented in the form $\varphi(u) = \Phi(\log u)$ for $u > 0$ where Φ is convex on the whole axis and which satisfies $\lim\limits_{u\to\infty} \dfrac{\Phi(u)}{u} = \infty$, will be called a *log-convex φ-function*. Every function φ such that $\varphi(u) = \psi(u^s)$, $0 < s \leqslant 1$ and ψ convex is a log-convex φ-function since $\varphi(u) = \Phi(\log u)$ where $\Phi(u) = e^{su}$. In the following, all φ-functions will be log-convex.

Let N denote the class of functions, F, analytic in the disk $\{z\colon |z| < 1\}$ such that

$$\sup_{0\leqslant r<1} \int_0^{2\pi} \log^+ |F(re^{i\theta})|\, d\theta < \infty. \tag{1.2.1}$$

Functions of this class have non-tangential limits at almost all points of $\{z\colon |z| = 1\}$ ([34]).

THEOREM 1.2.1. *The general function F of class N can be represented as:*

$$F(z) = B(z) \exp\left\{\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\lambda(t)\right\} \tag{1.2.2}$$

where $B(z)$ is a Blaschke product, and $\lambda(\cdot)$ is a real-valued function of bounded variation ([34]).

Let N^1 be the subclass of N made up of functions F, such that the function λ corresponding to F in (1.2.2) has its posititve variation absolutely continuous.

THEOREM 1.2.2. *A function $F \in N$, is in N^1 if and only if,*

$$\lim_{r\to 1} \int_0^{2\pi} \log^+ |F(re^{i\theta})|\, d\theta = \int_0^{2\pi} \log^+ |F(e^{i\theta})|\, d\theta. \tag{1.2.3}$$

THEOREM 1.2.3. *If $\varphi(u)$ is non-decreasing and convex in $(-\infty, \infty)$ and F is analytic in $\{z\colon |z| \leqslant 1\}$, then $\int_0^{2\pi} \varphi(\log |F(re^{i\theta})|)\, d\theta$ is a non-decreasing function of r, for $0 \leqslant r < 1$.*

THEOREM 1.2.4. *If φ is as in Theorem 1.2.3 and $F \in N^1$, then*

$$\int_0^{2\pi} \varphi(\log^+ |F(re^{i\theta})|)\, d\theta \leqslant \int_0^{2\pi} \varphi(\log^+ |F(e^{i\theta})|)\, d\theta.$$

The preceeding three theorems are from ([34]).

LEMMA 1.2.1. *Let f be continuous on $[0, 2\pi]$ and define;*

$$F(re^{i\theta}) = Pf(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t) f(t)\, dt \tag{1.2.4}$$

where $P_r(\theta - t)$ is the Poisson kernel. Then $F \in N^1$.

Proof. Recall that $P_r(\cdot)$ is defined as,

$$P_r(\theta) = \operatorname{Re}\left(\frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}\right) = \frac{1-r^2}{1-2r\cos(\theta)+r^2}.$$

Since f is continuous on $[0, 2\pi]$, it is bounded there. So let $|f(t)| \leqslant m$ for all $t \in [0, 2\pi]$. In addition $F(re^{i\theta}) \to f(\theta)$ uniformly in θ, F is analytic on $\{z\colon |z| < 1\}$, and F is continuous on $\{z\colon |z| \leqslant 1\}$. (See [13]). It follows from the Maximum Principle that $|F(re^{i\theta})| \leqslant m$ for all $0 \leqslant r < 1$ and all θ. Hence,

$$\log^+ |F(re^{i\theta})| \leqslant \log^+ m \quad \text{for all } r \in [0, 1) \text{ and all } \theta.$$

It follows that,

$$\int_0^{2\pi} \log^+ |F(re^{i\theta})|\, d\theta \leqslant \int_0^{2\pi} \log^+ m = 2\pi \log^+ m \quad \text{for all } r.$$

But then $F \in N$ by definition. By Theorem 1.2.2, F will be in N^1 if (1.2.3) holds. But by the Lebesgue Dominated Convergence Theorem and above,

$$\lim_{r\to 1} \int_0^{2\pi} \log^+ |F(re^{i\theta})|\, d\theta = \int_0^{2\pi} \lim_{r\to 1} \log^+ |F(re^{i\theta})|\, d\theta = \int_0^{2\pi} \log^+ |F(e^{i\theta})|\, d\theta$$

since $\log^+(\cdot)$ is continuous. Hence $F \in N^1$.

Let

$$\mu_\varphi(r; F) = \int_0^{2\pi} \varphi(|F(re^{i\theta})|)\, d\theta \quad \text{and} \quad \mu_\varphi(F) = \sup_{0\leqslant r<1} \mu_\varphi(r; F).$$

Note that since φ is log-convex, $\mu_\varphi(r; F)$ is non-decreasing in r by Theorem 1.2.3. Hence

$$\mu_\varphi(F) = \lim_{r\to 1} \mu_\varphi(r; F).$$

Now define;

$$H^{*\varphi} = \{F\colon F \text{ is analytic on } \{z\colon |z| < 1\} \text{ and } \mu_\varphi(F) < \infty\} \tag{1.2.5}$$

and;

$$H^{\varphi} = \{F\colon F \text{ is analytic on } \{z\colon |z| < 1\} \text{ and } \mu_\varphi(\lambda F) < \infty \text{ for some } \lambda > 0\}. \tag{1.2.6}$$

DEFINITION 1.2.2. $H^{*\varphi}$ and H^{φ} are called the *Hardy-Orlicz class* of functions.

PROPOSITION 1.2.1. *A necessary and sufficient condition for* $H^{*\varphi} = H^{\varphi}$ *is that* $\varphi \in \Delta_2$.

In the following $L^{\varphi} = L^{\varphi}(0, 2\pi)$.

THEOREM 1.2.5. *Let* $F \in N^1$ *and* $F(e^{i\cdot}) \in L^{*\varphi}$. *Then* $F \in H^{*\varphi}$.

COROLLARY 1.2.1. *Let* $\varphi \in \Delta_2$ *and* $F \in N^1$. *Then* $F \in H^{\varphi}$ *if* $F(e^{i\cdot}) \in L^{\varphi}$.

An F-norm can be defined on H^{φ} by

$$\|F\|_{H_\varphi} = \sup_{0 \leqslant r < 1} \|F(re^{i\cdot})\|_{\varphi} \tag{1.2.7}$$

where the norm on the right is the F-norm generated by φ on L^{φ}. The classes H^{φ} with the F-norm $\|\cdot\|_{H_\varphi}$ are Fréchet spaces, and are called the Hardy–Orlicz spaces. These spaces were defined for a convex φ by Weiss in [31].

LEMMA 1.2.2. *If* $F \in H_\varphi$ *then* $F(e^{i\cdot}) \in L_\varphi(0, 2\pi)$.

LEMMA 1.2.3. *If* $F \in H_\varphi$, *then* $\mu_\varphi(F) = \int_0^{2\pi} \varphi(|F(e^{i\theta})|)\,d\theta$.

THEOREM 1.2.6. *If* $F \in H_\varphi$, *then* $\|F\|_{H_\varphi} = \|F(e^{i\cdot})\|_\varphi$.

LEMMA 1.2.4. *If* $\varphi \in \Delta_2$ *and* $F \in H_\varphi$, *then there exists* $\{F_n\} \subset H_\varphi$ *such that* F_n *is continuous in* $\{z: |z| \leqslant 1\}$ *and* $\lim_{n\to\infty} \|F - F_n\|_{H_\varphi} = 0$.

LEMMA 1.2.5. *If* $\varphi \in \Delta_2$ *and* $F \in H_\varphi$, *then*

$$\lim_{R\to 1} \|F(\cdot) - F(R\cdot)\|_{H_\varphi} = 0. \tag{1.2.8}$$

THEOREM 1.2.7. *Polynomials are dense in* H_φ *if* $\varphi \in \Delta_2$.

In the case $\varphi(u) = \psi(u^s)$ where $0 < s \leqslant 1$ and ψ is a φ-function, an s-homogeneous norm can be defined in H_φ by means of the s-homogeneous norm in L^{φ} as;

$$\|F\|_{sH_\varphi} = \sup_{0 \leqslant r < 1} \|F(re^{i\cdot})\|_{s\varphi}.$$

The F-norm $\|\cdot\|_{sH_\varphi}$ is equivalent to the norm $\|\cdot\|_{H_\varphi}$ and Theorems 1.2.6 and 1.2.7 hold using $\|\cdot\|_{sH_\varphi}$ and $\|\cdot\|_{s_\varphi}$ instead of $\|\cdot\|_{H_\varphi}$ and $\|\cdot\|_\varphi$.

1.3. Results concerning subharmonic functions. In this section the three line theorem for subharmonic functions, and certain related results will be given.

THEOREM 1.3.1. (THE THREE LINE LEMMA FOR SUBHARMONIC FUNCTIONS). *Let* $f(z)$ *be non-negative, bounded and defined in* $S = \{z: 0 \leqslant \operatorname{Re} z \leqslant 1\}$ *such that* $\log f(z)$ *is subharmonic in* $\{z: 0 < \operatorname{Re} z < 1\}$ *and continuous in* S. *If* $f(0+iy) \leqslant M_1$, *and* $f(1+iy) \leqslant M_2$, *then* $f(t+iy) \leqslant M_1^{1-t} M_2^t$.

LEMMA 1.3.1. *If* $f(z)$ *is non-negative and* $\log f(z)$ *is subharmonic in a domain* D, *then* $(f(z))^a$ *is subharmonic in* D *for all* $a > 0$.

Proof. Let $z \in D$ and let $\{\xi: |z-\xi| \leqslant \varrho\} \subset D$. Then by definition:

$$\log f(z) \leqslant \frac{1}{2\pi} \int_0^{2\pi} \log f(z + \varrho e^{i\theta})\,d\theta.$$

Therefore;

$$\big(f(z)\big)^a = e^{a \log f(z)} \leqslant e^{a \frac{1}{2\pi} \int_0^{2\pi} \log f(z+\varrho e^{i\theta})\,d\theta}$$

$$= e^{\frac{1}{2\pi} \int_0^{2\pi} \log (f(z+\varrho e^{i\theta}))^a d\theta} \leqslant \frac{1}{2\pi} \int_0^{2\pi} \big(f(z + \varrho e^{i\theta})\big)^a d\theta,$$

by Jensen's inequality. Hence $\big(f(z)\big)^a$ is subharmonic.

DEFINITION 1.3.1. A function $I(\cdot)$ defined and continuous in the strip $0 \leqslant \operatorname{Re}(z) \leqslant 1$ will be called of *admissible growth* if

$$\sup_{|y| \leqslant r} \sup_{0 \leqslant x \leqslant 1} \log |I(x+iy)| \leqslant A e^{ar}, \quad a < \pi. \tag{1.3.1}$$

The following result is stated by Hirschman [11] for analytic functions, but an examination of the proof shows it proves actually the following result.

LEMMA 1.3.2. (Hirschman [11]): *Let* $I(z)$ *be non-negative and* $\log I(z)$ *be subharmonic and continuous in* $0 \leqslant \operatorname{Re} z \leqslant 1$. *If* $I(z)$ *is of admissible growth, and*

$$\log I(iy) \leqslant a_0(y) \quad \text{and} \quad \log I(1+iy) \leqslant a_1(y),$$

then for all $t \in [0, 1]$,

$$\log\big(I(t)\big) \leqslant \int_{-\infty}^{\infty} \omega(1-t, y)\, a^0(y)\,dy + \int_{-\infty}^{\infty} \omega(t, y)\, a_1(y)\,dy$$

where

$$\omega(t, y) = \frac{\frac{1}{2}\tan\left(\frac{\pi t}{2}\right)}{\left[\tan^2\left(\frac{\pi t}{2}\right) + \tanh^2\left(\frac{\pi y}{2}\right)\right] \cosh^2\left(\frac{\pi y}{2}\right)}.$$

2. INTERPOLATION OF SUBLINEAR OPERATORS

In this section, the M. Riesz convexity theorem [27] is generalized to sublinear operators on L^{φ} classes of Banach space valued functions and to H^{φ} classes of functions on the disc.

2.1. Interpolation in generalized Orlicz spaces. It is convenient to introduce the following:

DEFINITION 2.1.1. Let (Ω, Σ, μ) be a measure space, and X a Banach space. Let φ be a φ-function and

$$\mathscr{F} = \{f\colon f\colon \Omega \to X \text{ and } f \text{ is strongly measurable}\}.$$

Define $\|f\|_{\varphi(X)} = \big\|\|f\|_X\big\|_\varphi$ for $f \in \mathscr{F}$, and $L^{\varphi(X)} = \{f\colon f \in \mathscr{F} \text{ and } \|f\|_{\varphi(X)} < \infty\}$. Then $L^{\varphi(X)}$ is called the *Orlicz space of X-valued functions.*

All the properties of L^φ discussed in Section 1 hold for $L^{\varphi(X)}$.

DEFINITION 2.1.2. Let $(\Omega_1, \Sigma_1, \mu)$ and $(\Omega_2, \Sigma_2, \nu)$ be measure spaces and X, Y be Banach spaces. Let $\mathscr{F} = \{f\colon f\colon \Omega_1 \to Y$ and f is strongly measurable$\}$ and $\mathscr{G} = \{f\colon f\colon \Omega_2 \to X,\ f$ strongly measurable$\}$. Suppose T is a mapping of a subclass of $\mathscr{F}$ into $\mathscr{G}$. Then T is called a *sublinear operator* if it satisfies the following properties:

(i) If $f = f_1 + f_2$ and $Tf_i\,(i = 1, 2)$ are defined, then Tf is defined.

(ii) $\|T(f_1+f_2)\|_X \leqslant \|Tf_1\|_X + \|Tf_2\|_X$.

(iii) For any scalar a, $\|T(af)\|_X = |a|\,\|Tf\|_X$.

Remark. If $\varphi(u) = \psi(u^s)$ for $0 < s \leqslant 1$ and ψ a convex φ-function, then $\|f\|_{s\varphi} = \||f|^s\|_{1\psi}$, $f \in L^\varphi$. For:

$$\|f\|_{s\varphi} = \inf\left\{\varepsilon\colon \int_\Omega \varphi\left(\frac{|f|}{\varepsilon^{1/s}}\right) d\mu \leqslant 1\right\}$$

$$= \inf\left\{\varepsilon\colon \int_\Omega \psi\left(\frac{|f|^s}{\varepsilon}\right) d\mu \leqslant 1\right\} = \||f|^s\|_{1\psi}.$$

Similarly $\|f\|_{s\varphi(X)} = \big\|\|f\|_X^s\big\|_{1\psi}$.

The main result of this section is given in the following:

THEOREM 2.1.1. *Let $\varphi_i, Q_i\ (i = 1, 2)$ be φ-functions such that $\varphi_i(u) = \psi_i^{\sim}(u^{r_i})$; $Q_i(u) = S_i^{\sim}(u^{s_i})$ where $0 < r_i, s_i \leqslant 1$ and $\psi_i^{\sim}, S_i^{\sim}$ are convex. Let $\mathscr{F}$ and $\mathscr{G}$ be as in Definition 2.1.2, and let $L^{\varphi_i(Y)}$ be Orlicz spaces on $(\Omega_1, \Sigma_1, \mu)$ and $L^{Q_i(X)}$ be Orlicz spaces on $(\Omega_2, \Sigma_2, \nu)$. Suppose T is a sublinear mapping $L^{\varphi_i(Y)}$ into $L^{Q_i(X)}$ satisfying;*

$$\begin{aligned} &\|Tf\|_{rQ_1(X)} \leqslant M_1\|f\|_{r\varphi_1(Y)} \quad \textit{for all } f \in L^{\varphi_1(Y)}, \\ &\|Tf\|_{rQ_2(X)} \leqslant M_2\|f\|_{r\varphi_2(Y)} \quad \textit{for all } f \in L^{\varphi_2(Y)} \end{aligned} \tag{2.1.1}$$

for some $r \leqslant \min\{s_1, s_2, r_1, r_2\}$. If $\mathscr{L}(Y)$ is the class of simple functions on $\mathscr{F}$, then

$$\|Tf\|_{rQ_t(X)} \leqslant 4M_1^{1-t}M_2^t\|f\|_{r\varphi_t(Y)} \quad \textit{for all } f \in \mathscr{L}(Y) \tag{2.1.2}$$

where $Q_t^{-1} = (Q_1^{-1})^{1-t}(Q_2^{-1})^t$ and $\varphi_t^{-1} = (\varphi_1^{-1})^{1-t}(\varphi_2^{-1})^t$ with $0 \leqslant t \leqslant 1$.

Proof. Let $\psi_i(u) = \psi_i^{\sim}(u^{r_i/r})$ and $S_i(u) = S_i^{\sim}(u^{s_i/r})$. Since $r \leqslant r_i, s_i$; ψ_i and S_i are convex, and $\varphi_i(u) = \psi_i(u^r)$, $Q_i(u) = S_i(u^r)$. So the norms in (2.1.1) make sense. Let $\psi_t^{-1}(u) = (\psi_1^{-1}(u))^{1-t}(\psi_2^{-1}(u))^t$ and $S_t^{-1}(u) = (S_1^{-1}(u))^{1-t}(S_2^{-1}(u))^t$. Then if follows that $\varphi_t(u) = \psi_t(u^r)$, ψ_t convex and $Q_t(u) = S_t(u^r)$, S_t convex.

Let R_1, R_2 be the complementary functions to S_1, S_2 and $R_t(u) = (R_1^{-1}(u))^{1-t}(R_2^{-1}(u))^t$.

Let $a_i(u) = \varphi_i^{-1}(u)$, $a_t(u) = a_1^{1-t}(u)\,a_2^t(u)$, and define $a_z(u) = a_1^{1-z}(u) \times a_2^z(u)$, where $z = x + iy$ is a complex number. Then for each $u \neq 0$, since $a_i(\cdot)$ is positive, $a_z(u)$ is an analytic function of z in the strip $0 \leqslant x \leqslant 1$. Similarly let $\beta_i = R_i^{-1}(u)$ and then $\beta_z(u) = \beta_1^{1-z}(u)\beta_2^z(u)$ is also analytic in $0 \leqslant x \leqslant 1$. It follows that

$$|a_z(u)| = |a_1(u)|^{1-z}|a_2(u)|^z = a_1(u)^{1-x}a_2(u)^x$$

$$\leqslant \max_x\{a_1^{1-x}(u)\,a_2^x(u)\} \leqslant \max\{1, a_1(u)\}\cdot\max\{1, a_2(u)\}$$

i.e. $a_z(u)$ is bounded in $0 \leqslant x \leqslant 1$ for each u. Similarly $\beta_z(u)$ is bounded in $0 \leqslant x \leqslant 1$ for each u.

Let $f \in \mathscr{L}(Y) \subset L^{\varphi_i(Y)}$. Then Tf is well defined, so consider,

$$\big\|\|Tf\|_X^r\big\|_{S_t} = \sup\left\{\int_{\Omega_2} \|Tf\|_X^r |g|\,d\nu\colon \|g\|_{1R_t} \leqslant 1,\ g \in \mathscr{L}_\nu\right\} \tag{2.1.3}$$

where $\mathscr{L}_\nu$ is the set of simple functions on $(\Omega_2, \Sigma_2, \nu)$. The norm defined by (2.1.3) is equivalent to the Orlicz norm, $\|\cdot\|_{S_t}^{\circ} = \sup\{\int_\Omega fg\,d\mu\colon \|g\|_1 S_t^{\sim} \leqslant 1,\ g \in \mathscr{L}(\Omega, \Sigma, \mu)\}$ where $S_t^{\sim}$ is the complementary function to S_t (see [24]).

Suppose that $\|f\|_{r\varphi_t} = 1$, and fix $g \in \mathscr{L}_\nu$ such that $\|g\|_{1R_t} \leqslant 1$ and consider

$$I = \int_{\Omega_2} \|Tf\|_X^r |g|\,d\nu. \tag{2.1.4}$$

But $g \in \mathscr{L}_\nu$ implies

$$g = \sum_{l=1}^{m_2} b_l\chi_{G_l} = \sum_{l=1}^{m_2} |b_l|\,e^{i\theta_l}\chi_{G_l}.$$

Now define,

$$G_z = \beta_z(R_t|g|)\,e^{i\theta} = \sum_{l=1}^{m_2} \beta_z(R_t|b_l|)\,e^{i\theta_l}\chi_{G_l}. \tag{2.1.5}$$

Since $f \in \mathscr{L}(Y)$, $f = \sum_{j=1}^{m_1} a_j\chi_{F_j}$, $a_j \in Y$. Write $a_j = a_j u_j$ where $a_j = \|a_j\|_Y$ and $\|u_j\|_Y = 1$. Then $f = \sum_{j=1}^{m_1} a_j(u_j\chi_{F_j})$ and define

$$F_z = \sum_{j=1}^{m_1} a_z(\varphi_t(a_j))(u_j\chi_{F_j}). \tag{2.1.6}$$

Since the F_j's are disjoint,

$$\|f\|_Y = \sum_{j=1}^{m_1} a_j \|u_j\|_Y \chi_{F_j} = \sum_{j=1}^{m_1} a_j \chi_{F_j}$$

so that

$$\|F_z\|_Y = \sum_{j=1}^{m_1} \left| a_z(\varphi_t(a_j)) \right| \chi_{F_j} = \left| a_z(\varphi_t(\|f\|_Y)) \right|. \tag{2.1.7}$$

Since $F_z \in \mathscr{L}(Y)$ and $G_z \in \mathscr{L}_\nu$ the following extension of (2.1.4) can be defined,

$$I(z) = \int_{\Omega_2} \|T(F_z)\|_X^r |G_z| \, d\nu. \tag{2.1.8}$$

It is clear by construction that $I(t) = I$. The plan of the proof is to show that $I(z)$ satisfies the hypothesis of the three line theorem for subharmonic functions (Theorem 1.3.1) and obtain (2.1.2) as a consequence of that theorem. $I(z)$ can be simplified using (2.1.5) and (2.1.6) to yield;

$$I(z) = \sum_{l=1}^{m_2} \int_{G_l} \left\| T\left(\beta_z^{1/r}(R_t(|b_l|)) F_z\right) \right\|_X^r d\nu,$$

where the defining property (iii) for sublinear operators is used here. To simplify things, let

$$\left\{ \begin{aligned} \gamma_z^l &= \sum_{j=1}^{m_1} \beta_z^{1/r}(R_t(|b_l|)) \, a_z(\varphi_t(a_j)) \, u_j \chi_{F_j}, \\ \lambda_j^l(z) &= \sum_{j=1}^{m_1} \beta_z^{1/r}(R_t(|b_l|)) \, a_z(\varphi_t(a_j)), \\ \Gamma_l(z) &= \int_{G_l} \|T(\gamma_z^l)\|_X^r \, d\nu. \end{aligned} \right. \tag{2.1.9}$$

Then $\gamma_z^l = \sum_{j=1}^{m_1} \lambda_j^l(z) u_j \chi_{F_j}$ and $I(z) = \sum_{l=1}^{m_2} \Gamma_l(z)$. It is clear that $\lambda_j^l(z)$ is analytic, bounded, and continuous in $0 \leqslant x \leqslant 1$. From now on the proof proceeds in steps.

Step 1. $\Gamma_l(z)$ is continuous in $0 \leqslant x \leqslant 1$. For, consider

$$\begin{aligned} |\Gamma_l(z+\Delta z) - \Gamma_l(z)| &\leqslant \int_{G_l} \left| \|T(\gamma_{z+\Delta z}^l)\|_X^r - \|T(\gamma_z^l)\|_X^r \right| d\nu \\ &\leqslant \int_{G_l} \left| (\|T(\gamma_{z+\Delta z}^l)\|_X - \|T(\gamma_z^l)\|_X)^r \right| d\nu \\ &\qquad \text{(since } u^r,\ 0 < r \leqslant 1, \text{ is subadditive)}, \\ &\leqslant \int_{G_l} \|T(\gamma_{z+\Delta z}^l - \gamma_z^l)\|_X^r \, d\nu \\ &\qquad \text{(since } T \text{ is sublinear)}, \end{aligned} \tag{2.1.10}$$

$$\begin{aligned} &\leqslant 2 \left\| \|T(\gamma_{z+\Delta z}^l - \gamma_z^l)\|_X^r \right\|_{1S_1} \|\chi_{G_l}\|_{1R_1} \\ &\qquad \text{(by Hölder's inequality)}, \\ &\leqslant 2 \|T(\gamma_{z+\Delta z}^l - \gamma_z^l)\|_{rQ_1(X)} \|\chi_{G_l}\|_{1R_1} \\ &\qquad \text{(by the Remark above the statement of the Theorem)} \\ &\leqslant 2 M_1 \|\gamma_{z+\Delta z}^l - \gamma_z^l\|_{r\varphi_1(Y)} \|\chi_{G_l}\|_{1R_1}, \end{aligned}$$

by hypothesis, since $\gamma_z^l \in \mathscr{L}(Y) \subset L^{\varphi_1(Y)}$. But

$$\|\gamma_{z+\Delta z}^l - \gamma_z^l\|_{r\varphi_1(Y)} \leqslant \sum_{j=1}^{m_1} |\lambda_j^l(z+\Delta z) - \lambda_j^l(z)|^r \|u_j \chi_{F_j}\|_{r\varphi_1(Y)}.$$

Therefore

$$\lim_{\Delta z \to 0} \|\gamma_{z+\Delta z}^l - \gamma_z^l\|^{r\varphi_1(Y)} \leqslant \sum_{j=1}^{m_1} \lim_{\Delta z \to 0} |\lambda_j^l(z+\Delta z) - \lambda_j^l(z)|^r \|u_j \chi_{F_j}\|_{r\varphi_1(Y)} = 0$$

since $\lambda_j^l(z)$ is continuous and the sum is finite. It follows from (2.1.10) that

$$\lim_{\Delta z \to 0} |\Gamma_l(z+\Delta z) - \Gamma_l(z)| = 0$$

and therefore $\Gamma_l(z)$ is continuous in $0 \leqslant x \leqslant 1$.

Step 2. $\Gamma_l(z)$ is bounded in $0 \leqslant x \leqslant 1$. For, as in (2.1.10),

$$\begin{aligned} \Gamma_l(z) &= \int_{G_l} \|T\gamma_z^l\|_X^r d\nu \leqslant 2 \|T\gamma_z\|_{rQ_1(X)} \|\chi_{G_l}\|_{1R_1} \\ &\leqslant 2 M_1 \|\gamma_z\|_{r\varphi_1(Y)} \|\chi_{G_l}\|_{1R_1} \\ &\leqslant 2 M_1 \sum_{j=1}^{m_1} |\lambda_j^l(z)|^r \|u_j \chi_{F_j}\|_{r\varphi_1(Y)} \|\chi_{G_l}\|_{1R_1}. \end{aligned} \tag{2.1.11}$$

It follows that $\Gamma_l(z)$ is bounded in $0 \leqslant x \leqslant 1$ since $\lambda_j^l(z)$ is bounded in $0 \leqslant x \leqslant 1$ and the sum is finite.

Step 3. $\log \Gamma_l(z)$ is subharmonic in $0 < x < 1$. For, let $h(z)$ be any harmonic function in $0 < x < 1$, and let $H(z)$ be the analytic function whose real part is $h(z)$. If $e^{h(z)} \Gamma_l(z)$ is subharmonic for all such $h(z)$, then $\log \Gamma_l(z)$ will be subharmonic. Since a function is subharmonic in a region if it is subharmonic in a neighborhood of each point, fix $z \in \{0 < x < 1\}$, take $\varrho > 0$, and let $z_1, z_2, \ldots, z_p$ be a set of points equally spaced on the circle of radius ϱ about z. Then it is sufficient to show

$$e^{h(z)} \Gamma_l(z) \leqslant \frac{1}{2\pi} \int_0^{2\pi} e^{h(z+\varrho e^{i\theta})} \Gamma_l(z+\varrho e^{i\theta}) \, d\theta.$$

Let $\gamma_z^{*l} = e^{\frac{1}{r}H(z)} \gamma_z^l$ and $\lambda_j^{*l}(z) = e^{\frac{1}{r}H(z)} \lambda_j^l(z)$. Then $\gamma_z^{*l} = \sum_{j=1}^{m_1} \lambda_j^{*l}(z) (u_j \chi_{F_j})$, and

$$\Gamma_l^*(z) = e^{h(z)}\Gamma_l(z) = \int_{G_l} \|T\gamma_z^{*l}\|_X^r \, d\nu. \tag{2.1.12}$$

Now it will be sufficient to show that $\log\|T\gamma_z^{*l}\|_X$ is subharmonic for each ω, for if the latter holds, then by Lemma 1.3.1, $\|T(\gamma_z^{*l})\|_X^r$ would be subharmonic for each ω.

In this case, then

$$\|T(\gamma_z^{*l})\|_X^r \leqslant \frac{1}{2\pi}\int_0^{2\pi} \|T(\gamma_{z+\varrho e^{i\theta}}^{*l})\|_X^r \, d\theta, \tag{2.1.13}$$

for each ω, and then

$$\Gamma_l^*(z) = \int_{G_l} \|T(\gamma_z^{*l})\|_X^r \, d\nu \leqslant \int_{G_l}\left(\frac{1}{2\pi}\int_0^{2\pi} \|T(\gamma_{z+\varrho e^{i\theta}}^{*l})\|_X^r \, dl\right) d\nu$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_{G_l} \|T(\gamma_{z+\varrho e^{i\theta}}^{*l})\|_X^r \, d\nu \, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \Gamma_l^*(z+\varrho e^{i\theta})\, d\theta.$$

It then follows that $\Gamma_l^*(z)$ is subharmonic and so $\log\Gamma_l(z)$ would be subharmonic.

To show $\log\|T(\gamma_z^{*l})\|_X$ is subharmonic, it is sufficient to show $e^{k(z)}\|T\gamma_z^{*l}\|_X$ is subharmonic where $k(z)$ is any harmonic function. Let $K(z)$ be the analytic function whose real part is $k(z)$, $\gamma_z^{**l} = e^{K(z)}\gamma_z^{*l}$, and $\lambda_j^{**l}(z) = e^{K(z)}\lambda_j^{*l}(z)$. Then

$$\gamma_z^{**l} = \sum_{j=1}^{m_1} \lambda_j^{**l}(z)\, u_j \chi_{F_j}$$

and

$$e^{k(z)}\|T\gamma_z^{*l}\|_X = |e^{K(z)}|\,\|T\gamma_z^{*l}\|_X = \|T\gamma_z^{**l}\|_X. \tag{2.1.14}$$

Since $K(z)$, $H(z)$ and $\lambda_j^l(z)$ are analytic, it follows that $\lambda_j^{**l}(z)$ is analytic and therefore

$$\lambda_j^{**l}(z) = \frac{1}{2\pi}\int_0^{2\pi} \lambda_j^{**l}(z+\varrho e^{i\theta})\, d\theta = \lim_{p\to\infty}\frac{1}{p}\sum_{n=1}^{p}\lambda_j^{**l}(z_n) \tag{2.1.15}$$

where $z_n = \varrho e^{i\Delta\theta_n}$ and $\Delta\theta_n = \dfrac{2\pi}{p}$. Consider

$$\|T\gamma_z^{**l}\|_X \leqslant \left|\|T\gamma_z^{**l}\|_X - \left\|T\left(\frac{1}{p}\sum_{n=1}^{p}\gamma_{z_n}^{**l}\right)\right\|_X\right| + \left\|T\left(\frac{1}{p}\sum_{n=1}^{p}\gamma_{z_n}^{**l}\right)\right\|_X. \tag{2.1.16}$$

But

$$\left|\|T\gamma_z^{**l}\|_X - \left\|T\left(\frac{1}{p}\sum_{n=1}^{p}\gamma_{z_n}^{**l}\right)\right\|_X\right| \leqslant \left\|T\left(\gamma_z^{**l} - \frac{1}{p}\sum_{n=1}^{p}\gamma_{z_n}^{**l}\right)\right\|_X$$

$$\leqslant \sum_{j=1}^{m_1}\left|\lambda_j^{**l}(z) - \frac{1}{p}\sum_{n=1}^{p}\lambda_j^{**l}(z_n)\right| \|u_j\chi_{F_j}\|_X.$$

Therefore, for each ω

$$\lim_{p\to\infty}\left|\|T\gamma_z^{**l}\|_X - \left\|T\left(\frac{1}{p}\sum_{n=1}^{p}\gamma_{z_n}^{**l}\right)\right\|_X\right|$$

$$\leqslant \sum_{j=1}^{m_1}\lim_{p\to\infty}\left|\lambda_j^{**l}(z) - \frac{1}{p}\sum_{n=1}^{p}\lambda_j^{**l}(z_n)\right| \|u_j\chi_{F_j}\|_X = 0, \quad \text{by (2.1.15)}$$

It follows from (2.1.16) that

$$\|T\gamma_z^{**l}\|_X \leqslant \lim_{p\to\infty}\frac{1}{p}\sum_{n=1}^{p}\|T\gamma_{z_n}^{**l}\|_X = \frac{1}{2\pi}\int_0^{2\pi}\|T\gamma_{z+\varrho e^{i\theta}}^{**l}\|_X\, d\theta$$

for each ω. This is the same as

$$e^{k(z)}\|T\gamma_z^{*l}\|_X \leqslant \frac{1}{2\pi}\int_0^{2\pi} e^{k(z+\varrho e^{i\theta})}\|T\gamma_{z+\varrho e^{i\theta}}^{*l}\|_X\, d\theta$$

which implies $\log\|T\gamma_z^{*l}\|_X$ is subharmonic for each ω. Hence, by the previous argument $\log\Gamma_l(z)$ is subharmonic in $0 < x < 1$.

It thus follows that, since $I(z) = \sum_{l=1}^{m_2}\Gamma_l(z)$, $I(z)$ is bounded and continuous in $0 \leqslant x \leqslant 1$.

Step 4. $I(iy) \leqslant 2M_1$ and $I(1+iy) \leqslant 2M_2$. For, consider,

$$I(iy) = \int_\Omega \|TF_{iy}\|_X^r |G_{iy}|\, d\nu \leqslant 2\,\big\|\|TF_{iy}\|_X^r\big\|_{1S_t}\|G_{iy}\|_{1R_t} \tag{2.1.17}$$

$$= 2\,\|TF_{iy}\|_{rQ_t(X)}\|G_{iy}\|_{1R_t} \leqslant 2M_1\|F_{iy}\|_{r\varphi_t(Y)}\|G_{iy}\|_{1R_t}.$$

But $|G_{iy}| = \beta_1(R_t|g|)$ which implies

$$\int_{\Omega_2} R_1(|G_{iy}|)\, d\nu = \int_{\Omega_2} R_t(|g|)\, d\nu \leqslant 1$$

since $\|g\|_{1R_t} \leqslant 1$. Hence by definition,

$$\|G_{iy}\|_{1R_t} \leqslant 1. \tag{2.1.18}$$

Also

$$\|F_{iy}\|_y = |\alpha_{iy}\varphi_t(\|f\|_y)| = \alpha_1(\varphi_t(\|f\|_y))$$

by (2.1.7), and therefore

$$\int_{\Omega_1} \varphi_1(\|F_{iy}\|_y)\,d\mu = \int_{\Omega} \varphi_t(\|f\|_y)\,d\mu \leqslant 1$$

since $\|f\|_{r\varphi_t(Y)} = 1$. Hence

$$\|F_{iy}\|_{r\varphi_1(Y)} \leqslant 1. \tag{2.1.19}$$

But now (2.1.17), (2.1.18) and (2.1.19) give $I(iy) \leqslant 2M_1$. Similarly $I(1+iy) \leqslant 2M_2$.

It now follows from the three-line theorem,

$$I = I(t) \leqslant 2M_1^{1-t}M_2^t$$

and then by (2.1.3),

$$\big\|\|TF\|_X^r\big\|_{S_t} \leqslant 2M_1^{1-t}M_2^t. \tag{2.1.20}$$

But

$$\big\|\,\|Tf\|_X^r\big\|_{1S_t} \leqslant \big\|\,\|Tf\|_X^r\big\|_{S_t}^{\circ} \leqslant 2\,\big\|\|Tf\|_X^r\big\|_{1S_t}$$

(see [31]) and (2.1.20) becomes,

$$\|Tf\|_{rQ_t(X)} = \big\|\,\|Tf\|_X^r\big\|_{1S_t} \leqslant 4M_1^{1-t}M_2^t \tag{2.1.21}$$

for all $f \in \mathscr{L}(Y)$ such that $\|f\|_{r\varphi_t(Y)} = 1$. Now let $f \in \mathscr{L}(Y)$ be arbitrary and let

$$f' = \frac{1}{\|f\|_{r\varphi_t(Y)}^{1/r}} f.$$

Then $\|f'\|_{r\varphi_t(Y)} = 1$ and $\|Tf'\|_{rQ_t(X)} \leqslant 4M_1^{1-t}M_2^t$. But then

$$\|Tf\|_{rQ_t(X)} \leqslant 4M_1^{1-t}M_2^t\|f\|_{r\varphi_t(Y)}.$$

Thus the theorem is completely proved. In the following the operator T can be extended to the whole space under certain conditions.

COROLLARY 2.1.1. *Let the hypothesis of Theorem 2.1.1 hold. If, in addition, T is linear and $\varphi_i \in \Delta_2$, $i = 1, 2$ then T can be extended to all of $L^{\varphi_t(X)}$ with the same bound as in the above theorem.*

COROLLARY 2.1.2. *Let the hypothesis of Theorem 2.1.1 hold. If, in addition, $\varphi_1 < \varphi_2$ (i.e. there exists constants c_1, c_2 and u_0 such that $\varphi_1(c_1u) \leqslant c_2\varphi_2(u)$ for $u \geqslant u_0 \geqslant 0$) $\varphi_1, \varphi_2, Q_1, Q_2 \in \Delta_2$, and $\mu(\Omega_1) < \infty$, then T can be extended to all of $L^{\varphi_t(Y)}$ with the some bound as in the theorem.*

Proof. Since $\varphi_1 < \varphi_2$, it follows that $\varphi_1 < \varphi_t < \varphi_2$ (see [24]). But then there exists constants c_1, c_2 such that

$$\varphi_1(c_1u) \leqslant c_2\varphi_t(u) \quad \text{for } u \geqslant u_0$$

which implies

$$\psi_1(c_1^r u^r) \leqslant c_2\psi_t(u^r) \quad \text{for } u \geqslant u_0$$

or, equivalently

$$\psi_1(c_1' u) \leqslant c_2\psi_t(u) \quad \text{for } u \geqslant u_0^{1/r}$$

and so $\psi_1 < \psi_t$. But since $\mu(\Omega_1) < \infty$ there exists a constant q (depending on $\mu(\Omega_1)$, u_0 and the φ's and Q's) such that

$$\|f\|_{1\psi_1} \leqslant q\|f\|_{1\psi_t} \quad \text{for all } f \in L^{\psi_t} \text{ (see [14]);} \tag{2.1.22}$$

or equivalently

$$\|f\|_{r\varphi_1(Y)} \leqslant q\|f\|_{r\varphi_t(Y)} \quad \text{for all } f \in L^{\varphi_t(Y)}. \tag{2.1.23}$$

Let $f \in L^{\varphi_t(Y)}$. Since $\varphi_1 \in \Delta_2$, there exists $f_n \in \mathscr{L}(Y)$ such that

$$\|f - f_n\|_{r\varphi_t(Y)} \to 0.$$

Since $\varphi_1 < \varphi_t$, and $\mu(\Omega) < \infty$, $L^{\varphi_t(Y)} \subset L^{\varphi_1(Y)}$ and so $f, f_n \in L^{\varphi_1(Y)}$. Consider

$$\begin{aligned} \big\|\|Tf\|_X - \|Tf_n\|_X\big\|_{rQ_1} &\leqslant \big\|\|T(f-f_n)\|_X\big\|_{rQ_1} = \|T(f-f_n)\|_{rQ_1(X)} \\ &\leqslant M_1\|f-f_n\|_{r\varphi_1(Y)} \leqslant M_1q\|f-f_n\|_{r\varphi_t(Y)} \end{aligned} \tag{2.1.24}$$

by (2.1.23). Therefore

$$\lim_{n\to\infty}\big\|\,\|Tf\|_X - \|Tf_n\|_X\big\|_{rQ_1} \leqslant M_1q\lim_{n\to\infty}\|f-f_n\|_{r\varphi_t(Y)} = 0.$$

Since $Q_1 \in \Delta_2$, it follows that

$$\lim_{n\to\infty}\int_{\Omega_2}(\|Tf\|_X - \|Tf_n\|_X)\,d\nu = 0.$$

Consequently, there exists a subsequence $\{f_{nk}\}$ such that $Q_1(\|Tf\|_X - \|Tf_{nk}\|_X) \to 0$, a.e., and this implies $\|Tf_{nk}\|_X \to \|Tf\|_X$ a.e. since Q_1 is continuous. Since $\|f-f_n\|_{r\varphi_t(Y)} \to 0$ it follows that $\|f_{nk}\|_{r\varphi_t(Y)} \to \|f\|_{r\varphi_t(Y)}$. So consider

$$\begin{aligned} \int_{\Omega_1} Q_t\left(\frac{\|Tf\|_X}{(4M_1^{1-t}M_2^t\|f\|_{r\varphi_t(Y)})^{1/r}}\right)d\nu &= \int_{\Omega_1}\lim_{n_k\to\infty} Q_t\left(\frac{\|Tf_{nk}\|_X}{(4M_1^{1-t}M_2^t\|f_{nk}\|_{r\varphi_t(Y)})^{1/r}}\right)d\nu \\ &\leqslant \varliminf_{n_k\to\infty}\int_{\Omega_1} Q_t\left(\frac{\|Tf_{nk}\|_X}{(4M_1^{1-t}M_2^t\|f_{nk}\|_{r\varphi_t(Y)})^{1/r}}\right)d\nu \\ &\leqslant 1 \end{aligned}$$

by Fatou's lemma, the fact that $Q_t \in \Delta_2$, and Theorem 2.1.1. So by definition

$$\|Tf\|_{rQ_t(X)} \leqslant 4M_1^{1-t}M_2^t\|f\|_{r\varphi_t(Y)}.$$

An extension of the theorem in an infinite measure space can be given if the hypothesis is strengthened. This will be presented in the following proposition.

Proposition 2.1.1. *Let the hypothesis of Theorem 2.1.1 hold, with $\varphi_1 < \varphi_2$ and $\varphi_1, \varphi_2, Q_1, Q_2 \in \Delta_2$. If, in addition, there exists constants K_1, K_2 and u_1 such that*

$$\varphi_2(K_1 u) \leqslant K_2 \varphi_t(u) \quad \text{for } u \leqslant u_1, \tag{2.1.25}$$

then T can be extended to all of $L^{\varphi_t(Y)}$ with the same bound.

Proof. It follows from $\varphi_1 < \varphi_2$ that $\varphi_1 < \varphi_t < \varphi_2$, so that, there exist constants K_3, K_4, u_0 such that

$$\varphi_1(K_3 u) \leqslant K_4 \varphi_t(u) \quad \text{for } u \geqslant u_0. \tag{2.1.26}$$

Let $f \in L^{\varphi_t(Y)}$ and write $f = f'_m + f''_m$, where

$$f'_m(\omega) = f(\omega) \quad \text{if } \|f(\omega)\|_Y \leqslant m u_0 \tag{2.1.27}$$

and 0 otherwise. Hence

$$\|f''_m\|_Y > m u_0 \quad \text{or } = 0, \tag{2.1.28}$$

and

$$\|f''_m\|_{r\varphi_t(Y)} \to 0 \quad \text{as } m \to \infty, \text{ since } \varphi_t \in \Delta_2.$$

$$E_m = \left\{\omega\colon \|f'_m(\omega)\|_Y \geqslant \frac{u_1}{m}\right\}. \tag{2.1.29}$$

Then $\mu(E_m) < \infty$ and if $g_m = f'_m \chi_{E_m}$, g_m is bounded and has finite support. Therefore there exists $f_m \in \mathscr{L}(Y)$ such that

$$\|f_m - g_m\|_Y < \frac{u_1}{m} \quad \text{for all } \omega$$

and $f_m = 0$ where $g_m = 0$. But then

$$\|f_m - f'_m\|_Y < \frac{u_1}{m} \quad \text{for all } \omega \tag{2.1.30}$$

since $\|f'_m\|_Y < \dfrac{u_1}{m}$ outside E_m. Now consider

$$\big|\,\|Tf\|_X - \|Tf_m\|_X\big| \leqslant \|T(f - f_m)\|_X \leqslant \|T(f'_m - f_m)\|_X + \|Tf''_m\|_X. \tag{2.1.31}$$

The idea is to show that there exists a subsequence such that $\big|\,\|Tf\|_X - \|Tf_{m_j}\|_X\big| \to 0$. Then as in Corollary 2.1.2, T can be extended to all of $L^{\varphi_t(Y)}$ with the same bound. If follows from (2.1.25) and (2.1.30) that $f_m - f'_m \in L^{\varphi_2(Y)}$. So by hypothesis,

$$\|T(f_m - f'_m)\|_{rQ_2(X)} \leqslant M_1 \|f_m - f'_m\|_{r\varphi_2(Y)}. \tag{2.1.32}$$

Let $A_k = \{\omega\colon \|f(\omega)\|_Y \geqslant 1/k\}$ and pick k such that for $\varepsilon > 0$ arbitrary,

$$\int_{A_k^c} \varphi_t(\|f\|_Y)\, d\mu < \varepsilon$$

(this can be done since $\int_{\Omega_1} \varphi_t(\|f\|_Y)\, d\mu < \infty$).

Then

$$\int_{\Omega_1} \varphi_t(\|f_m - f'_m\|_Y)\, d\mu = \int_{A_k} \varphi_t(\|f_m - f'_m\|_Y)\, d\mu + \int_{A_k^c} \varphi_t(\|f_m - f'_m\|_Y)\, d\mu.$$

But for m large enough, $f_m = 0$ and $f'_m = f$ on A_k^c. So for large m,

$$\int_{\Omega_1} \varphi_t(\|f_m - f'_m\|_Y)\, d\mu = \int_{A_k} \varphi_t(\|f_m - f'_m\|_Y)\, d\mu + \int_{A_k^c} \varphi_t(\|f\|_Y)\, d\mu \leqslant \varphi_t\left(\frac{u_1}{m}\right)\mu(A_k) + \varepsilon.$$

But $\mu(A_k) < \infty$ and so

$$\lim_{m\to\infty} \int_{\Omega_1} \varphi_t(\|f_m - f'_m\|_Y)\, d\mu \leqslant \lim_{m\to\infty} \varphi_t\left(\frac{u_1}{m}\right)\mu(A_k) + \varepsilon = \varepsilon$$

because φ_t is continuous and $\varphi_t(0) = 0$. Since $\varepsilon > 0$ was arbitrary,

$$\int_{\Omega_1} \varphi_t(\|f_m - f'_m\|_Y)\, d\mu \to 0.$$

But by (2.1.25) and (2.1.30)

$$\lim_{m\to\infty} \int_{\Omega_1} \varphi_2(K_1 \|f_m - f'_m\|_Y)\, d\mu \leqslant K_2 \lim_{m\to\infty} \int_{\Omega_1} \varphi_t(\|f_m - f'_m\|_Y)\, d\mu = 0. \tag{2.1.33}$$

It follows, since $\varphi_2 \in \Delta_2$, that $\|f_m - f'_m\|_{r\varphi_2(Y)} \to 0$, and so by (2.1.32) $\lim\limits_{m\to\infty} \|T(f_m - f'_m)\|_{rQ_2(X)} = 0$. But then there exists a subsequence such that

$$\lim_{m_k\to\infty} \|T(f_{m_K} - f'_{m_K})\|_X = 0 \quad \text{a.e.} \tag{2.1.34}$$

By (2.1.26) and (2.1.28)

$$\lim_{m\to\infty} \int_{\Omega_1} \varphi_1(K_3 \|f''_m\|_Y)\, d\mu \leqslant K_4 \lim_{m\to\infty} \int_{\Omega_1} \varphi_t(\|f''_m\|_Y)\, d\mu = 0$$

since $\|f''_m\|_{r\varphi_t(Y)} \to 0$ and $\varphi_t \in \Delta_2$. Hence $\lim\limits_{m\to\infty} \|f''_m\|_{r\varphi_1(Y)} = 0$ and therefore,

$$\lim_{m\to\infty} \|Tf''_m\|_{rQ_1(X)} \leqslant M_1 \lim_{m\to\infty} \|f''_m\|_{r\varphi_1(Y)} = 0.$$

But then there exists another subsequence such that

$$\lim_{m_l\to 0} \|Tf''_{m_l}\|_X = 0 \quad \text{a.e.} \tag{2.1.35}$$

since $Q_1 \in \Delta_2$. But the equations (2.1.34), (2.1.35) and (2.1.31) imply there exists a subsequence such that

$$\lim_{m_j\to\infty} |\,\|Tf\|_X - \|Tf_{m_j}\|_X| = 0 \quad \text{a.e.}$$

This completes the proof of the proposition.

2.2. Interpolation in Hardy–Orlicz spaces. In this section, the convexity theorem for linear and sublinear operators on Hardy–Orlicz spaces will be given.

PROPOSITION 2.2.1. *Let $\varphi_i, Q_i, \varphi_t, Q_t$ and r be as in Theorem* 2.1.1. *In addition let $Q_i, \varphi_i \in \Delta_2$. If T is a linear operator such that $T\colon H_{\varphi_i} \to H_{Q_i}$ $(i = 1, 2)$ and*

$$\|TF\|_{rH_{Q_i}} \leqslant M_i \|F\|_{rH_{\varphi_i}} \quad \textit{for all } F \in H_{\varphi_i} \tag{2.2.1}$$

then $T\colon H_{Q_t} \to H_{\varphi_t}$ with

$$\|TF\|_{rH_{Q_t}} \leqslant 4M_1^{1-t}M_2^t \|F\|_{rH_{\varphi_t}} \quad \textit{for all } F \in H_{\varphi_t}. \tag{2.2.2}$$

Proof. Let $Pr(\theta - t)$ be the Poisson kernel and define for f, continuous on $[0, 2\pi]$,

$$Pf(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} Pr(\theta - t) f(t)\,dt, \quad 0 \leqslant r < 1. \tag{2.2.3}$$

Then the operator P is linear and by Lemma 1.2.1, $Pf \in N^1$. Since f is continuous, it follows from Theorem 1.2.3 that $Pf \in H_\varphi$, for any log-convex φ-function φ, and by Theorem 1.2.4,

$$\|Pf\|_{H\varphi} = \|f\|_\varphi.$$

If $\varphi(u) = \psi(u^r)$, $0 < r \leqslant 1$, ψ convex, then

$$\|Pf\|_{rH\varphi} = \|f\|_{r\varphi}. \tag{2.2.4}$$

For $F \in H_\varphi$, let $f(\cdot) = F(e^{i\cdot})$. Then by Theorem 1.2.4, $f \in L^\varphi(0, 2\pi)$ and $\|F\|_{H_\varphi} = \|f\|_\varphi$. Define

$$RF(\cdot) = F(e^{i\cdot}). \tag{2.2.5}$$

Then $R\colon H_\varphi \to L_\varphi$, R is linear, and

$$\|RF\|_\varphi = \|F\|_{H_\varphi}. \tag{2.2.6}$$

Let f be continuous and define T^* by

$$T^*(f) = RTP(f) \tag{2.2.7}$$

then T^* is well defined, linear, and $T^*\colon C(0, 2\pi) \to L^{Q_i}(0, 2\pi)$ $(i = 1, 2)$ with

$$\begin{aligned} \|T^*f\|_{rQ_i} &= \|RTPf\|_{rQ_i} = \|TP(f)\|_{rH_{Q_i}} \\ &\leqslant M_i \|Pf\|_{rH_{\varphi_i}} = M_i \|f\|_{r\varphi_i} \quad (i = 1, 2), \end{aligned} \tag{2.2.8}$$

by (2.2.4) (2.2.5) and the hypothesis. Since continuous functions are dense in $L^{\varphi_i} T^*$ can be extended to all of $L^{\varphi_i}(0, 2\pi)$, preserving the bounds (2.2.8). But now the hypotheses of Theorem 2.1.1 are satisfied, and so $T^*\colon L^{\varphi_t} \to L^{Q_t}$ with

$$\|Tf\|_{rQ_t} \leqslant 4M_1^{1-t}M_2^t \|f\|_{r\varphi_t}. \tag{2.2.9}$$

Let $F(z)$ be a polynomial. Then by [13], p. 33 there exists a continuous function f such that

$$F(re^{i\theta}) = Pf = \frac{1}{2\pi}\int_0^{2\pi} Pr(\theta - t) f(t)\,dt.$$

But then,

$$\begin{aligned} \|TF\|_{rH_{Q_t}} &= \|TPf\|_{rH_{Q_t}} = \|RTPf\|_{rQ_t} = \|T^*f\|_{rQ_t} \leqslant 4M_1^{1-t}M_2^t \|f\|_{r\varphi_t} \\ &= 4M_1^{1-t}M_2^t \|Pf\|_{rH_{\varphi_t}} = 4M_1^{1-t}M_2^t \|F\|_{rH_{\varphi_t}}. \end{aligned} \tag{2.2.10}$$

Since polynomials are dense in $H_{\varphi_t}(\varphi_t \in \Delta_2)$ and T is linear T can be extended to all of H_{φ_t} with the same bound.

PROPOSITION 2.2.2. *Let the hypothesis of Proposition* 2.2.1 *hold. If, instead, T is sublinear and $\varphi_1 < \varphi_2$, then the conclusion of Proposition* 2.2.1 *holds.*

Proof. In Theorem 2.1.1, it is only necessary for T to be defined on the continuous functions for the conclusion to hold. In this case, the proof of Proposition 2.2.1 is valid up to the point where T is defined on all polynomials. But since $\varphi_1 < \varphi_2$ and the measure is finite, T can be extended to all of H_{φ_t} with the same bound by a method similar to the proof of Corollary 2.1.2.

3. INTERPOLATION WITH FACTORS

In this section the interpolation theorem with factors is proven for sublinear operators on generalized Orlicz spaces. The theorem is similar to Theorem 2.1.1, but with the hypothesis that φ_i, Q_i's are Δ_2 functions and the functions are not B-space valued (because of the factors). With these conditions the previous theorem is a special case of this theorem but the proofs are interesting enough for both to be included. All the known theorems on interpolation with change of measures can be shown to be special cases of these theorems. A deduction of this will be given later.

3.1. Interpolation with factors in generalized Orlicz spaces. The following theorem is proved in [15].

THEOREM 3.1.1. *Let φ_1, φ_2 be convex φ-functions and*

$$\varphi_t^{-1}(\omega) = (\varphi_1^{-1}(\omega))^{1-t}(\varphi_2^{-1}(\omega))^t.$$

Let (Ω, Σ, μ) be a measure space and u, v be measurable functions. Let $a = u^{1-t}v^t$. Also let $\varphi_i = \psi_i(u^r)$, ψ_i convex, $0 \leqslant r < 1$. Then for each f such that $uf \in L^{\varphi_1}$ and $vf \in L^{\varphi_2}$, one has $af \in L^{\varphi_t}$. In fact

$$\|af\|_{r\varphi_t} \leqslant 4\|uf\|_{r\varphi_1}^{1-t}\|vf\|_{r\varphi_2}^t.$$

This leads to the main result of this section:

THEOREM 3.1.2. *Let $\varphi_i, Q_i (1 = 1, 2)$ be Δ_2 φ-functions such that $\varphi_i(u) = \psi^{\sim}(u^{r_i})$ and $Q_i(u) = S_i^{\sim}(u^{s_i})$ with $0 < r_i, s_i, \leqslant 1$ and $\psi_i^{\sim}$, $S_i^{\sim}$ convex. Let u_1, u_2 be non-negative measurable functions on a measure space (Ω, Σ, μ) such that $u_i\chi_E \in L^{\varphi_i}(\Omega_1, \Sigma_1, \mu)$ for $E \in \Sigma$, with $\mu(E) < \infty$. Also let k_1, k_2 be non-negative measurable functions on a measure space $(\Omega_2, \Sigma_2, \nu)$. Let T be a sublinear mapping of $L^{\varphi_i}(\Omega_1, \Sigma_2, \mu)$ into $L^{Q_i}(\Omega_2, \Sigma_2, \nu)$ such that for some $0 < r \leqslant \min\{r_1, r_2, s_1, s_2\}$*

$$\|k_1 T(f)\|_{rQ_1} \leqslant M_1\|fu_1\|_{r\varphi_1}$$

for all f such that $u_1 f \in L_{\varphi_1}$, and

$$\|k_2 T(f)\|_{rQ_2} \leqslant M_2\|fu_2\|_{r\varphi_2} \tag{3.1.1}$$

for all f such that $u_2 f \in L_{\varphi_2}$. If $(\varphi_t^{-1}) = (\varphi_1^{-1})^{1-t}(\varphi_2^{-1})^t$, $Q_t^{-1} = (Q_1^{-1})^{1-t}(Q_2^{-1})^t$, $k_t = k_1^{1-t}k_2^t$, and $u_t = u_1^{1-t}u_2^t$, then

$$\|k_t T(f)\|_{rQ_t} \leqslant 4M_1^{1-t}M_2^t\|fu_t\|_{r\varphi_t}$$

for all $f \in \mathscr{L}_1$, the class of simple functions on $(\Omega_1, \Sigma_1 1\mu)$.

Proof. Let ψ_i, S_i, ψ_t and S_t be defined as in Theorem 2.1.1. Assumptions will be made on u_1 and u_2 that will be removed later. So assume $u_1, u_2 \geqslant \varepsilon_1 > 0$. Let $f \in \mathscr{L}_1$ be such that $\|f\|_{r\varphi_t} = 1$ and $g \in \mathscr{L}_2$, (the simple functions on $(\Omega_2, \Sigma_2, \nu)$) be such that $\|g\|_{1R_t} \leqslant 1$, where R_t is as in Theorem 2.1.1. Consider

$$I = \int_{\Omega_2} |k_t|^r |T(u_t^{-1}f)|^r |g|\,d\nu. \tag{3.1.2}$$

Let

$$f = \sum_{j=1}^{m_1} |a_j| e^{i\theta_j}\chi_{F_j}, \qquad g = \sum_{l=1}^{m_2} |b_l| e^{i\theta'_l}\chi_{G_l}$$

and define

$$F_z = \alpha_z(\varphi_t|f|)e^{i\theta} = \sum_{j=1}^{m_1} \alpha_z(\varphi_t|a_j|)e^{i\theta_j}\chi_{F_j}; \tag{3.1.3}$$

and

$$G_z = \beta_z(R_t(|g|))e^{i\theta'} = \sum_{l=1}^{m_2} \beta_z(R_t|b_l|)e^{i\theta_l}\chi_{G_l} \tag{3.1.4}$$

where α_z and β_z are defined in Theorem 2.1.1. Let $k_z = k_1^{1-z}k_2^z$ and $v_z = (u_1^{-1})^{1-z}(u_2^{-1})^z$. Note that

$$|v_z| = |u_1^{-1}|^{1-x}|u_2^{-1}|^x \leqslant \left(\frac{1}{\varepsilon_1}\right)^{1-x}\left(\frac{1}{\varepsilon_1}\right)^x = \frac{1}{\varepsilon_1}, \quad (\chi = \operatorname{Re} z). \tag{3.1.5}$$

Now define

$$I(z) = \int_{\Omega_2} |k_z|^r |T(v_z F_z)|^r |G_z|\,d\nu. \tag{3.1.6}$$

The plan, as in Theorem 2.1.1, is to show that $I(z)$ satisfies the hypothesis of the three-line theorem for subharmonic functions, since $I(t) = I$. The desired result will follow easily from this theorem. $I(z)$ can be simplified. Let

$$\lambda_z^l = \sum_{j=1}^{m_2} \beta_z^{1/r}(R_t(|b_l|))\,\alpha_z(\varphi_t(|a_j|))e^{i(\theta_j+\theta'_l)}\chi_{F_j} \tag{3.1.7}$$

then $\lambda_z \in \mathscr{L}_1$ for each z. Let $\gamma_z^l = v_z\lambda_z^l$. Then

$$u_1\gamma_z^l \in L^{\varphi_1} \quad \text{and} \quad u_2\gamma_z^l \in L^{\varphi_2}. \tag{3.1.8}$$

This is true since

$$u_1\gamma_z^l \leqslant u_1|\gamma_z| = u_1|v_z||\gamma_z| \leqslant \frac{u_1}{\varepsilon_1}|\gamma_z|.$$

But $\dfrac{1}{\varepsilon_1}|\gamma_z| \in \mathscr{L}_1$ and so by hypothesis $\dfrac{u_1}{\varepsilon_1}|\gamma_z| \in L^{\varphi_1}(u_1\chi_E \in L^{\varphi_1}$ for all $E \in \Sigma$ such that $\mu(E) < \infty$ clearly implies $u_1 f \in L^{\varphi_1}$ for all simple functions $f)$. Similarly $u_2\gamma_z \in L^{\varphi_2}$. $I(z)$ can now be written,

$$I(z) = \sum_{l=1}^{m_2} \int_{G_l} |k_z|^r |T(\gamma_z)|^r\,d\nu = \sum_{l=1}^{m_2} \Gamma_l(z). \tag{3.1.9}$$

From now on, the proof proceeds in steps.

Step 1. $\Gamma_l(z)$ is subharmonic in $0 < x < 1$. For this it is necessary to show $\log|T(\gamma_z^l)|$ is subharmonic. For this it is sufficient to show $e^{h(z)}|T(\gamma_z)|$ is subharmonic for any harmonic h in $0 < x < 1$. The proof is similar to that of Theorem 2.1.1, to which it reduces if $u_1 = 1 = v_1$. There are, however, some complications since these functions are not constants. Since a function is subharmonic in a region if it is subharmonic in a neighborhood of each point, fix z, take $\varrho > 0$, and let $z_1, \ldots, z_p$ be a set of points equally spaces on the circle of radius ϱ about z. Then it is sufficient to show

$$(3.1.10) \qquad e^{h(z)}\Gamma_l(z) \leqslant \frac{1}{2\pi}\int_0^{2\pi} e^{h(z+\varrho e^{i\theta})}\Gamma_l(z+\varrho e^{i\theta})\,d\theta .$$

Let $H(z)$ be the analytic function whose real part is $h(z)$, and

$$(3.1.11) \qquad \gamma_z^{*l} = e^{H(z)}\gamma_z^l \text{ and } \lambda_z^{*l} = e^{H(z)}\lambda_z^l$$

then $\gamma_z^{*l} = v_z\lambda_z^{*l}$ and

$$(3.1.12) \qquad u_1\gamma_z^{*l}\,\epsilon\, L^{\varphi_1}, \qquad u_2\gamma_z^{*l}\,\epsilon\, L^{\varphi_2}$$

since $h(z)$ is bounded in $\{\xi\colon |z-\xi|\leqslant\varrho\}$. (The proof is the same as for (3.1.8). It is clear that γ_z^{*l} is analytic for each $\omega\,\epsilon\,\Omega_1$ and therefore

$$(3.1.13) \qquad \gamma_z^{*l} = \frac{1}{2\pi}\int_0^{2\pi}\gamma_{z+\varrho e^{i\theta}}^{*l}\,d\theta = \lim_{p\to\infty}\sum_{n=1}^{p}\gamma_{z_n}^{*l},$$

where $z_n = \varrho e^{i\Delta\theta_n}$ and $\Delta\theta_n = \dfrac{1\pi}{p}$. By the sublinearity of T,

$$(3.1.14) \qquad |T(\gamma_z^{*l})| = |T(\gamma_z^{*l})| - \Big|T\Big(1/p\sum_{n=1}^{p}\gamma_{z_n}^{*l}\Big)\Big| + \Big|T\Big(1/p\sum_{n=1}^{p}\gamma_{z_n}^{*l}\Big)\Big|$$
$$\leqslant \Big|T\Big(\gamma_z^{*l}-1/p\sum_{n=1}^{p}\gamma_{z_n}^{*l}\Big)\Big| + \Big|T\Big(1/p\sum_{n=1}^{p}\gamma_{z_n}^{*l}\Big)\Big|$$
$$\leqslant \Big|T\Big(\gamma_z^{*l}-1/p\sum_{n=1}^{p}\gamma_{z_n}^{*l}\Big)\Big| + 1/p\sum_{n=1}^{p}|T\gamma_{z_n}^{*l}|.$$

The plan is to show $|T(\gamma_z^{*l})|$ is subharmonic for each ω. For this it is sufficient to show that there exists a subsequence such that

$$(3.1.15) \qquad \lim_{p_k\to\infty}\Big|T\Big(\gamma_z^{*l}-1/p_k\sum_{n=1}^{p_k}\gamma_z^{*l}\Big)\Big| = 0 \qquad \text{a.e.}$$

for, (3.1.14) becomes

$$(3.1.16) \qquad |T(\gamma_z^{*l})| \leqslant \lim_{p_k\to\infty}\frac{1}{p_k}\sum_{n=1}^{p_k}|T(\gamma_{z_n}^{*l})| = \frac{1}{2\pi}\int_0^{2\pi}|T(\gamma_{z+\varrho e^{i\theta}}^{*l})|\,d\theta .$$

The first step is to show

$$(3.1.17) \qquad \lim_{p\to\infty}\Big\|u_1\Big(\gamma_z^{*l}-1/p\sum_{n=1}^{p}\gamma_{z_n}^{*l}\Big)\Big\|_{r\varphi_1} = 0$$

since

$$(3.1.18) \qquad \Big\|k_1T\Big(\gamma_z^{*l}-1/p\sum_{n=1}^{p}\gamma_{z_n}^{*l}\Big)\Big\|_{rQ_1} \leqslant M_1\Big\|u_1\Big(\gamma_z^{*l}-1/p\sum_{n=1}^{p}\gamma_{z_n}^{*l}\Big)\Big\|_{r\varphi_1}$$

by hypothesis and (3.1.12). But since $\varphi_1\,\epsilon\,\Delta_2$, (3.1.17) will follow from

$$(3.1.19) \qquad \lim_{p\to\infty}\int_{\Omega_1}\varphi_1\Big(u_1\Big(\gamma_z^{*l}-1/p\sum_{n=1}^{p}\gamma_{z_n}^{*l}\Big)\Big)d\mu = 0 .$$

Consider,

$$(3.1.20) \qquad |\gamma_z^{*l}| \leqslant \sum_{j=1}^{m_1}|\beta_z(R_l(b_l)|^{1/r}|\alpha_z(\varphi_l(|a_j|))|\,|v_z|\,e^{h(z)}\chi_{F_j} \leqslant M'\sum_{j=1}^{m_1}\chi_{F_j}$$

since $|\beta_z|$, $|\alpha_z|$, $(v_z|$ and $e^{h(z)}$ are bounded for all $z\,\epsilon\,\{\xi\colon |\xi-z|\leqslant\varrho\}$ and all $\omega\,\epsilon\,\Omega_1$. Therefore

$$(3.1.21) \qquad \Big|\gamma_z^{*l}-1/p\sum_{n=1}^{p}\gamma_{z_n}^{*l}\Big| \leqslant |\gamma_z^{*l}| + 1/p\sum_{n=1}^{p}|\gamma_{z_n}^{*l}|$$
$$\leqslant M'\sum_{j=1}^{m_1}\chi_{F_j} + 1/p\sum_{n=1}^{p}M'\sum_{j=1}^{m_1}\chi_{F_j}$$
$$= 2M'\sum_{j=1}^{m_1}\chi_{F_j}.$$

But $2M'\sum_{j=1}^{m_1}\chi_{F_j}\,\epsilon\,\mathscr{L}_1$ and so by hypothesis $u_1\big(2M'\sum_{j=1}^{m_1}\chi_{F_j}\big)\,\epsilon\, L^{\varphi_1}$ and therefore

$$(3.1.22) \qquad \varphi_1\Big(u_1\Big|\gamma_z^{*l}-1/p\sum_{n=1}^{p}\gamma_{z_n}^{*l}\Big|\Big) \leqslant \varphi_1\Big(2M\sum_{j=1}^{p}\chi_{F_j}\Big)\,\epsilon\, L^1,$$

since $\varphi_1\,\epsilon\,\Delta_2$. So by the Lebesque Dominated Convergence Theorem,

$$(3.1.23) \qquad \lim_{p\to\infty}\int_{\Omega_1}\varphi_1\Big(u_1\Big|\gamma_z^{*l}-1/p\sum_{n=1}^{p}\gamma_{z_n}^{*l}\Big|\Big)d\mu$$
$$= \int_{\Omega_1}\lim_{p\to\infty}\varphi_1\Big(u_1\Big|\gamma_z^{*l}-1/p\sum_{n=1}^{p}\gamma_{z_n}^{*l}\Big|\Big)d\mu = 0$$

(3.1.13) since φ_1 is continuous. Hence (3.1.17) holds, and therefore since $Q_1\,\epsilon\,\Delta_2$

$$(3.1.24) \qquad \lim_{p\to\infty}\int_{\Omega_2}Q_1\Big(k_1T\Big(\gamma_z^{*l}-1/p\sum_{n=1}^{p}\gamma_{z_n}^{*l}\Big)\Big)d\nu = 0$$

which implies the existence of a subsequence such that

$$\Big|T\Big(\gamma_z^{*l}-1/p_k\sum_{n=1}^{p_k}\gamma_{z_n}^{*l}\Big)\Big| \to 0 \qquad \text{a.e.}$$

So (3.1.15) holds and therefore $|T(\gamma_z^{*l})|$ is subharmonic for each ω. Bu

$$(3.1.25) \qquad |T(\gamma_z^{*l})| = |T(e^{H(z)}\gamma_z^l)| = |e^{H(z)}|\,|T(\gamma_z^l)| = e^{h(z)}|T(\gamma_z^l)|$$

and so $\log|T(\gamma_z^l)|$ is subharmonic. Since k_z is analytic, $\log|k_z|$ is subharmonic also. So if $\Lambda(z) = |k_z|^r|T(\gamma_z^l)|^r$,

$$\log\Lambda(z) = r\log|k_z| + r\log|T(\gamma_z^l)| \tag{3.1.26}$$

and therefore $\log\Lambda(z)$ is subharmonic. But then if $h(z)$ is any harmonic function, $e^{h(z)}\Lambda(z)$ is subharmonic, and so

$$e^{h(z)}\Lambda(z) \leqslant \frac{1}{2\pi}\int_0^{2\pi} e^{h(z+\varrho e^{i\theta})}\Lambda(z+\varrho e^{i\theta})\,d\theta \tag{3.1.27}$$

for each ω. Hence,

$$\begin{aligned} e^{h(z)}\Gamma_l(z) &= \int_{G_l} e^{h(z)}\Lambda(z)\,dv \leqslant \int_{G_l}\left(\frac{1}{2\pi}\int_0^{2\pi} e^{h(z+\varrho e^{i\theta})}\Lambda(z+\varrho e^{i\theta})\,d\theta\right)dv \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(\int_{G_l} e^{h(z+\varrho e^{i\theta})}\Lambda(z+\varrho e^{i\theta})\,dv\right)d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{h(z+\varrho e^{i\theta})}\Gamma_l(z+\varrho e^{i\theta})\,d\theta. \end{aligned} \tag{3.1.28}$$

It follows that $\log\Gamma_l(z)$ is subharmonic.

Step 2. $\Gamma_l(z)$ is continuous. Consider

$$\begin{aligned} &|\Gamma_l(z+\Delta z) - \Gamma_l(z)| \\ &\leqslant \int_{G_l}\big|\,|k_{z+\Delta z}|^r|T(\gamma_{z+\Delta z}^l)|^r - |k_z|^r|T(\gamma_z^l)|^r\big|\,dv \\ &\leqslant \int_{G_l}\big|\,|k_{z+\Delta z}|^r|T(\gamma_{z+\Delta z}^l)|^r - |k_{z+\Delta z}|^r|T(\gamma_z^l)|^r + |k_{z+\Delta z}|^r|T(\gamma_z^l)|^r - |k_z|^r|T(\gamma_z^l)|^r\big|\,dv \\ &\leqslant \int_{G_l}\big|\,|k_{z+\Delta z}|^r\big(|T(\gamma_{z+\Delta z}^l)|^r - |T(\gamma_z^l)|^r\big)\,dv + \int_{G_l}\big|\big(|k_{z+\Delta z}|^r - |k_z|^r\big)\big|\,|T(\gamma_z^l)|^r\,dv \\ &\leqslant \int_{G_l}|k_{z+\Delta z}|^r|T(\gamma_{z+\Delta z}^l - \gamma_z^l)|^r\,dv + \int_{G_l}\big(|k_{z+\Delta z}| - |k_z|\big)^r|T(\gamma_z^l)|^r\,dv \end{aligned} \tag{3.1.29}$$

by the sublinearity of T and subadditivity of w^r. Note that for all $z = x+iy$ such that $0 \leqslant x \leqslant 1$,

$$|k_z| = |k_1^{1-z}k_2^z| = k_1^{1-x}k_2^x \leqslant \max_x\left\{k_1\left(\frac{k_2}{k_1}\right)^x\right\} = \max\{k_1, k_2\} \leqslant k_1 + k_2. \tag{3.1.30}$$

Therefore, since $0 < r \leqslant 1$ and $0 \leqslant \operatorname{Re}(z+\Delta z) \leqslant 1$,

$$\begin{aligned} &\int_{G_l}|k_{z+\Delta z}|^r|T(\gamma_{z+\Delta z}^l - \gamma_z^l)|^r\,dv \\ &\quad\leqslant \int_{G_l}k_1^r|T(\gamma_{z+\Delta z}^l - \gamma_z^l)|^r\,dv + \int_{G_l}k_2^r|T(\gamma_{z+\Delta z}^l - \gamma_z^l)|^r\,dv \\ &\quad\leqslant \|k_1T(\gamma_{z+\Delta z}^l - \gamma_z^l)\|_{rQ_1}\|\chi_{G_l}\|_{R_1}^{\circ} + \|k_2T(\gamma_{z+\Delta z}^l - \gamma_z^l)\|_{rQ_2}\|\chi_{G_l}\|_{R_2}^{\circ} \\ &\quad\leqslant M_1\|u_1(\gamma_{z+\Delta z}^l - \gamma_z^l)\|_{r\varphi_1}\|\chi_{G_l}\|_{R_1}^{\circ} + M_2\|u_2(\gamma_{z+\Delta z}^l - \gamma_z^l)\|_{r\varphi_2}\|\chi_{G_l}\|_{R_2}^{\circ} \end{aligned} \tag{3.1.31}$$

by hypothesis. It can be shown that $\|u_i(\gamma_{z+\Delta z}^l - \gamma_z^l)\|_{r\varphi_i} \to 0$ as $\Delta z \to 0$ by an argument similar to that used in Step 1 to show

$$\Big\|u_1\Big(\gamma_z^l - 1/p\sum_{n=1}^{p}\gamma_{z_n}^l\Big)\Big\|_{r\varphi_1} \to 0$$

using the Dominated Convergence Theorem. Hence, the first integral on the right of (3.1.29) $\to 0$ as $\Delta z \to 0$.

As for the second term of (3.1.29), consider

$$\begin{aligned} \int_{G_l}\big(|k_{z+\Delta z}| - |k_z|\big)^r|T(\gamma_z^l)|\,dv &\leqslant \|(|k_{z+\Delta z}| - |k_z|)^r|T(\gamma_z^l)|^r\|_{1S_1}\|\chi_{G_l}\|_{R_1}^{\circ} \\ &= \|(|k_{z+\Delta z}| - |k_z|)|T(\gamma_z^l)|\,\|_{rQ_1}\|\chi_{G_l}\|_{R_1}^{\circ}. \end{aligned} \tag{3.1.32}$$

The first term on the right can be shown to converge to zero as $\Delta z \to 0$, again using the Dominated Convergence Theorem. Hence the second integral on the right of (3.1.29) $\to 0$ as $\Delta z \to 0$, i.e.

$$\lim_{\Delta z\to 0}|\Gamma_l(z+\Delta z) - \Gamma_l(z)| = 0,$$

and so $\Gamma_l(z)$ is continuous in $0 \leqslant x \leqslant 1$.

Step 3. $\Gamma_l(z)$ is bounded. Consider

$$\begin{aligned} \int_{G_l}|k_r|^r|T(\gamma_z^l)|^r\,dv &\leqslant \int_{G_l}(k_1^r + k_2^r)|TT(\gamma_z^l)|^r\,dv \\ &= \int_{G_l}k_1^r|T(\gamma_z^l)^r\,dv + \int_{G_l}k_2^r|T(\gamma_z^l)|^r\,dv \\ &\leqslant \|k_1T(\gamma_z^l)\|_{rQ_1}\|\chi_{G_l}\|_{R_1}^{\circ} + \|k_2T(\gamma_z^l)\|_{rQ_2}\|\chi_{G_l}\|_{R_2}^{\circ} \\ &\leqslant M_1\|u_1\gamma_z^l\|_{r\varphi_1}\|\chi_{G_l}\|_{R_1}^{\circ} + M_2\|u_2\gamma_z^l\|_{r\varphi_2}\|\chi_{G_l}\|_{R_2}^{\circ} \end{aligned} \tag{3.1.33}$$

by Holder's inequality and (3.1.30). But by (3.1.21) this becomes,

$$\begin{aligned} &\int_{G_l}|k_z|^r|T(\gamma_z^l)|^r\,d\mu \\ &\quad\leqslant M_1M'\sum_{j=1}^{m_1}\|\chi_{F_j}\|_{r\varphi_1}\|\chi_{G_l}\|_{R_1}^{\circ} + M_1M'\sum_{j=1}^{m_1}\|\chi_{F_j}\|_{r\varphi_2}\|\chi_{G_l}\|_{R_2}^{\circ} < \infty. \end{aligned}$$

Hence $\Gamma_l(z)$ is bounded. Now, since $I(z) = \sum_{l=1}^{m_2}\Gamma_l(z)$. It follows that $I(z)$ is continuous and bounded in $0 \leqslant x \leqslant 1$, and $\log I(z)$ is subharmonic in $0 < x < 1$.

Step 4. $I(iy) \leqslant 2M_1$ and $I(1+iy) \leqslant 2M_2$. Consider

$$\begin{aligned} I(iy) &= \int_{\Omega_2}|k_{iy}|^r|T(v_{iy}F_{iy})|^r|G_{iy}|\,dv \\ &\leqslant 2\,\|k_1^r|k_1^{-iy}k_2^{iy}|^r|T(v_{iy}F_{iy})|^r\|_{1S_1}\|G_{iy}\|_{1R_1} \\ &= 2\,\big\|k_1T\big(u_1^{-1}(u_1^{iy}u_2^{-iy}F_{iy})\big)\big\|_{rQ_1}\|G_{iy}\|_{1R_1} \\ &\leqslant 2M_1\|F_{iy}\|_{r\varphi_1}\|G_{iy}\|_{1R_1}. \end{aligned} \tag{3.1.34}$$

But $\|F_{iy}\|_{r\varphi_1} \leqslant 1$ and $\|G_{iy}\|_{1R_1} \leqslant 1$, and so $I(iy) \leqslant 2M_1$. Similarly $I(1+iy) \leqslant 2M_2$. Now in view of the three-line theorem,

(3.1.35) $$I = I(t) \leqslant 2M_1^{1-t}M_2^t.$$

So as in Theorem 2.1.1,

(3.1.36) $$\|k_tT(u_t^{-1}f)\|_{rQ_t} \leqslant 4M_1^{1-t}M_2^t\|f\|_{r\varphi_t}$$

for all $f \in \mathscr{L}_1$, and therefore

(3.1.37) $$\|k_tT(f)\|_{rQ_t} \leqslant 4M_1^{1-t}M_2^t\|u_tf\|_{r\varphi_t}$$

for all $f \in \mathscr{L}$.

Now the assumptions on u_1, u_2 will be removed. Let

(3.1.38) $$u_i^n = u_i \text{ if } u_i \geqslant 1/n \quad\text{and}\quad u_i^n = 1/n \quad\text{otherwise } (i=1,2).$$

Then $u_1^n, u_2^n \geqslant 1/n$ and $u_i \leqslant u_i^n$. But then $u_if \leqslant u_i^nf$, and so by hypothesis,

(3.1.39) $$\|k_i(Tf)\|_{rQ_i} \leqslant M_i\|u_if\|_{r\varphi_i} \leqslant M_i\|u_i^nf\|_{r\varphi_i}.$$

But since $u_1, u_2 \geqslant 1/n$, the above proof applies, and therefore for all n,

(3.1.40) $$\|k_tT(f)\|_{rQ_t} \leqslant \|u_t^nf\|_{r\varphi_t}$$

for each $f \in \mathscr{L}_1$ where $u_t^n = (u_1^n)^{1-t}(u_2^n)^t$. It is now necessary to show

(3.1.41) $$\lim_{n\to\infty}\|u_t^nf\|_{r\varphi_t} = \|u_tf\|_{r\varphi_t}.$$

Consider

(3.1.42) $$|u_t^nf-u_tf| = (u_t^n-u_t)|f| \leqslant (u_t^1-u_t)|f| \leqslant (u_t^1+u_t)|f| \leqslant 2u_t^1|f|$$

since u_t^n decreases with n. But $u_1^1f = u_1\chi_Af+\chi_{A^c}f \in L^{\varphi_1}$ where $A = \{\omega\colon u_1(\omega) \geqslant 1\}$, and $u_2^1f = u_2\chi_Bf+\chi_{B^c}f \in L^{\varphi_2}(B = \{\omega\colon u_2(\omega) \geqslant 1\})$, by hypothesis since $f \in \mathscr{L}_1$. So by Theorem 3.1.1, $u_t^1f \in L^{\varphi_t}$, and therefore

(3.1.43) $$\varphi_t(|u_t^nf-u_tf|) \leqslant \varphi_t(2u_t^1|f|) \in L^1$$

and so by the Dominated Convergence Theorem,

(3.1.44) $$\lim_{n\to\infty}\int_{\Omega_1}\varphi_t(|u_t^nf-u_tf|)\,d\mu = \int_{\Omega_1}\lim_{n\to\infty}\varphi_t(|u_t^nf-u_tf|)\,d\mu = 0$$

and so be the Δ_2 condition,

$$\|u_t^nf-u_tf\|_{r\varphi_t} \to 0.$$

Hence (3.1.40) holds and so for all $f \in \mathscr{L}_1$,

(3.1.45) $$\|k_tT(f)\|_{rQ_t} \leqslant \lim_{n\to\infty}\|u_t^nf\|_{r\varphi_t} = \|u_tf\|_{r\varphi_t}.$$

This completes the proof.

Remark. Corollaries similar to those after Theorem 2.1.1 also hold here.

4. FURTHER RESULTS

A theorem on the interpolation of a smooth family of operators which generalizes Stein's result on an analytic family [29] is given in subsection 4.1. Subsection 4.2 deals with change of measures.

4.1. Smooth families of operators. Let $L(\Omega_1, \Sigma_1, \mu)$ denote the class of simple functions on a measure space $(\Omega_1, \Sigma_1, \mu)$ and $\mathscr{M}(\Omega_2, \Sigma_2, \nu)$ the class of measurable functions on a measure space $(\Omega_2, \Sigma_2, \nu)$.

Definition 4.1.1. Suppose $\gamma_z(\omega)$ is a function of z and ω such that $\gamma_z(\cdot) \in L(\Omega_1, \Sigma_1, \mu)$ for each z and $\gamma_\cdot(\omega)$ is analytic for each ω. Then $\gamma_z(\omega)$ is called an *analytic simple function* on $(\Omega_1, \Sigma_1, \mu)$.

Definition 4.1.2. A family of operators T_z (depending on the complex parameter z), is called a *smooth family* if the following conditions hold:

(i) $T_z\colon L(\Omega_1, \Sigma_1, \mu) \to \mathscr{M}(\Omega_2, \Sigma_2, \nu)$ for each z.

(ii) If $\gamma_z(\omega)$ is an analytic simple function on $(\Omega_1, \Sigma_1, \mu)$, then $|T_z(\gamma_z)|$ is subharmonic for each ω.

A smooth family, T_z, is of *admissible growth* if for all $0 < r \leqslant 1$,

$$I(z) = \int_{\Omega_2}|T_z(\gamma_z)|^r|\lambda_z|\,d\nu$$

is of admissible growth (see Definition 1.3.1, for each analytic simple function λ_z on $(\Omega_2, \Sigma_2, \nu)$ and each analytic simple function γ_z on $(\Omega_1, \Sigma_1, \mu)$.

Theorem 4.1.1. *Let T_z be a smooth family of sublinear operators of admissible growth, defined in $0 \leqslant \operatorname{Re} z \leqslant 1$. Suppose that φ_i, Q_i, $(i = 1, 2)$, are φ-functions such that $\varphi_i(u) = \varphi_i^\sim(u^{r_i})$ and $Q_i(u) = S_i^\sim(u^{s_i})$ where $S_i^\sim$ and $\varphi_i^\sim$ are convex and $0 < r_i, s_i \leqslant 1$, and let $\varphi_t^{-1} = (\varphi_1^{-1})^{1-t}(\varphi_2^{-1})^t$ and $Q_t^{-1} = (Q_1^{-1})^{1-t}(Q_2^{-1})^t$. Finally suppose*

$$\|T_{iy}(f)\|_{rQ_1} \leqslant A_1(y)\|f\|_{r\varphi_1},$$
$$\|T_{(1+iy)}(f)\|_{rQ_2} \leqslant A_2(y)\|r\|_{r\varphi_2}$$

for each $f \in L(\Omega_1, \Sigma_1, \mu)$, where $0 < r \leqslant \min(r_1, r_2, s_1, s_2)$, and $\log|A_i(y)| \leqslant Ae^a|r|$, $a < \pi$, $i = 1, 2$. Then,

$$\|T_t(f)\|_{rQ_t} \leqslant 2A_t\|f\|_{r\varphi_t}$$

for all $f \in L(\Omega_1, \Sigma_1, \mu)$ and where

$$\log A_t = \int_{-\infty}^{\infty}\omega(1-t, y)\log\big(2A_1(y)\big)\,dy + \int_{-\infty}^{\infty}\omega(t, y)\log\big((2A_2(y)\big)\,dy$$

and $\omega(t, y)$ is defined in Section 1.

Proof. Let $\psi_i(u) = \varphi_i^\sim(u^{r_i/r})$, $S_i(u) = S_i(u^{s_i/r})$, $\psi_t^{-1} = (\psi_1^{-1})^{1-t}(\psi_2^{-1})^t$ and $S_t^{-1} = (S_1^{-1})^{1-t}(S_2^{-1})^t$. Then $\varphi_i(u) = \psi_i(u^r)$, $Q_i(u) = S_i(u^r)$, $\varphi_t(u)$

$=\psi_t(u^r)$ and $Q_t(u)=\psi_t(u^r)$ with ψ_i, ψ_t, Q_i and Q_t all convex. Let R_1 and R_2 be the complementary functions to S_1 and S_2 and $R_t^{-1} = (R_1^{-1})^{1-t}(R_2^{-1})^t$. Let α_z and β_z be as in Theorem 2.1.1. Let $f \in L(\Omega_1, \Sigma_1, \mu)$ with $\|f\|_{r\varphi_t}=1$ and $g \in L(\Omega_2, \Sigma_2, \mu)$ with $\|g\|_{1R_t} \leqslant 1$, and consider

(4.1.1) $$I = \int_{\Omega} |T_t(f)|^r |g|\, d\nu .$$

Suppose $f = \sum_{j=1}^{m_1} a_j \chi_{F_j}$ and $g = \sum_{k=1}^{m_2} b_k \chi_{G_k}$ and define

(4.1.2) $$F_z = \alpha_z\big(\varphi_t(|b|)\big)e^{i\theta} = \sum_{j=1}^{m_1} \alpha_z\big(\varphi_t(|a_j|)\big)e^{i\theta_j}\chi_{F_j}$$

and

(4.1.3) $$G_z = \beta_z\big(R_t(|g|)\big)e^{i\theta'} = \sum_{k=1}^{m_2} \beta_z\big(R_t(|b_k|)\big)e^{i\theta_k}\chi_{G_k}$$

and finally,

(4.1.4) $$\gamma_z^k = \sum_{j=1}^{m_1} \beta_z^{1/r}\big(R_t(|b_k|)\big)\alpha_z\big(\varphi_t(|a_j|)\big)e^{i(\theta_j+\theta_k')}\chi_{F_j}.$$

Note that γ_z^k is an analytic simple function for each $k=1,2,\ldots,m_2$.

Consider the following extension of I,

(4.1.5) $$I(z) = \int_{\Omega_2} |T_z(F_z)|^r |G_z|\, d\nu = \sum_{k=1}^{m_2} \int_{G_k} |T_z(\gamma_z^k)|^r d\nu .$$

$I(z)$ has the following properties:

(i) $I(z) \geqslant 0$, $\log I(z)$ is subharmonic in $0<\operatorname{Re} z<1$, and $I(z)$ is continuous on $0 \leqslant \operatorname{Re} z \leqslant 1$.

(ii) $I(z)$ is of admissible growth in $0 \leqslant \operatorname{Re} z \leqslant 1$.

These properties shown using the methods in the proof of Theorem 2.1.2 since T_z is a smooth family.

(iii) $I(iy) \leqslant 2A_1(y)$ and $I(1+iy) \leqslant 2A_2(y)$. For consider

$$I(iy) = \int_{\Omega_2} |T_{iy}(F_{iy})|^r |G_{iy}|\, d\nu \leqslant 2\|T_{iy}(F_{iy})\|_{rQ_1}\|G_{iy}\|_{1R_1}$$
$$\leqslant 2A_1(y)\|F_{iy}\|_{r\varphi_1}\|G_{iy}\|_{1R_1} \leqslant 2A_1(y)$$

by construction of F_{iy} and G_{iy}. Similarly $I(1+iy) \leqslant 2A_2(y)$.

(iv) $I = I(t)$.

So by the lemma of Hirchman (Lemma 1.3.3) $I(t) \leqslant A_t$ where A_t is defined in the statement of the theorem. Therefore, as before,

(4.1.6) $$\|T_t f\|_{rQ_t} \leqslant 2A_t\|f\|_{r\varphi_t}$$

for each $f \in L(\Omega_1, \Sigma_1, \mu)$. This proves the theorem.

Remark. Defining $T_z(f) = k^z T(u^{-z}f)$, T_z can be shown to be a smooth family of operators using the methods of the proof of Theorem 3.1.2. In this case Theorem 3.1.2 is a corollary of Theorem 4.1.1.

4.2. Change of measures. Let the φ-function of Theorem 3.1.2 be $\varphi_i(u) = |u|^{p_i}$ and $Q_i(u) = |u|^{q_i}$ where $1 < p_i, q_i, < \infty$. Let $\frac{1}{p_t} = \frac{1-t}{p_1} + \frac{t}{p_2}$ and $\frac{1}{q_t} = \frac{1-t}{q_1} + \frac{t}{q_2}$. Then the theorem takes the following form.

If

(4.2.1) $$\|k_i(Tf)\|_{q_i,\nu} \leqslant M_i\|u_i f\|_{p_i,\mu}, \qquad f \in L^{p_i}, \qquad i=1,2$$

then

(4.2.2) $$\|k_t(Tf)\|_{q_t,\nu} \leqslant M_1^{1-t}M_2^t\|u_t f\|_{p_t,\mu}, \qquad f \in L^{p_t}$$

where $\|f\|_{p,\mu} = \big(\int_{\Omega} |f|^p d\mu\big)^{1/p}$. (The constant 4 can be removed in the L^p case.)

Suppose μ_1, μ_2, ν_1 and ν_2 are measures with the following property. There exist positive measurable functions $\alpha_1, \alpha_1, \beta_1$ and β_2 such that

(4.2.3) $$u_i(A) = \int_A \alpha_i\, d\mu, \qquad \text{all } A \in \Sigma_1,$$

and

(4.2.4) $$\nu_i(A) = \int_A \beta_i\, d\nu, \qquad \text{all } A \in \Sigma_2.$$

Define

(4.2.5) $$\mu_s(A) = \int_A \alpha_1^{1-s}\alpha_2^s\, d\mu, \qquad \text{all } A \in \Sigma_1$$

and

(4.2.6) $$\nu_s(A) = \int_A \beta_1^{1-s}\beta_2^s\, d\nu, \qquad \text{all } A \in \Sigma_2.$$

If $u_i = \alpha_1^{1/p_i}$, $k_i = \beta_i^{1/q_i}$, $s(t) = \frac{tp_t}{p_1}$ and $r(t) = \frac{tq_t}{q_1}$, then (see [30] for details) equations (4.2.1) and (4.2.2) take the following form,

(4.2.7) $$\|Tf\|_{q_i,\nu_i} \leqslant M_i\|f\|_{p_i,\mu_i}, \qquad f \in L^{p_i,\mu_i}$$

and

(4.2.8) $$\|Tf\|_{q_t,\nu_{r(t)}} \leqslant M_1^{1-t}M_2^t\|f\|_{p_t,\mu_{s(t)}}$$

which is the result proved in [30].

Now suppose the φ-functions in Theorem 2.1.2 are all convex and all satisfy the following condition

(4.2.9) $$\varphi(uv) \leqslant \varphi(u)\varphi(v) \qquad \text{for } u, v \geqslant 0.$$

Also let μ and ν satisfy (4.2.3) and (4.2.4). Then Theorem 3.1.2 can be shown to include the result of Rao [24].

At this time, it is not clear that Theorem 3.1.2 can be used to extend the change of measures theorem further (i.e. to more general φ-functions). The problem seems to depend on some multiplicative property of the φ-functions, as in (4.2.9).

Using Theorem 2.1.1 it is possible to prove a result on positive operators similar to ([1], Theorem 1), and then it seems an extension of the Dunford–Schwartz–Hopf ergodic theorem can be obtained.

Acknowledgment. This paper constitutes part of the authors doctoral dissertation written at Carnegie–Mellon University under the guidance of Professor M. M. Rao. The author gratefully acknowledges Dr. Rao's interest and many helpful suggestions.

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Received May 16, 1971 (345)