(the integral is understood in the sense of the Cauchy principal value). The Plemelj–Privalov theorem (see [2]) asserts that $S \in B(H^{\mu}(L))$. Moreover it is an involution on $H^{\mu}(L)$ ([2]). Therefore from Proposition 1.1 and from Corollary 1.1 we conclude

PROPOSITION 4.1. *If* $A \in B(H^{\mu}(L))$, *then* $SA - AS = 0$ *if and only if* $A = SA_0 + A_0 S$, *where* $A_0 \in B(H(L))$.

PROPOSITION 4.2. *If* $A \in B(H^{\mu}(L))$ *then* $SA - AS$ *is compact if and only if* $A = SA_0 + A_0 S + T$, *where* $A_0 \in B(H^{\mu}(L))$, $T \in T(H^{\mu}(L))$.

The list of similar results can be easily prolongated.

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Hardy's inequality with weights

by

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Abstract. This paper is concerned with conditions on measures μ and ν that are both necessary and sufficient for the existence of a finite C such that

$$\Big[\int_0^\infty \Big|\int_0^x f(t)\,dt\Big|^p d\mu\Big]^{1/p} \leqslant C\Big[\int_0^\infty |f(x)|^p d\nu\Big]^{1/p},$$

where p is a fixed number satisfying $1 \leqslant p \leqslant \infty$. For absolutely continuous measures a new proof is given for a known condition, and a new condition is given that arises from an interpolation with change of measures. The case when $\int_0^x f(t)\,dt$ is replaced by $\int_x^\infty f(t)\,dt$ is sketched. For Borel measures a condition like the first one for absolutely continuous measures is proved. Estimates for C in terms of the constants of the conditions are also given.

1. Introduction. Hardy's inequality, [5], p. 20, states that if p and b satisfy $1 \leqslant p \leqslant \infty$ and $bp < -1$, then

$$\Big[\int_0^\infty \Big|x^b \int_0^x f(t)\,dt\Big|^p dx\Big]^{1/p} \leqslant \frac{-p}{bp+1}\Big[\int_0^\infty |x^{b+1} f(x)|^p\,dx\Big]^{1/p}, \tag{1.1}$$

and the indicated constant is the best possible. Several authors, Tomaselli [4], Talenti [3], and Artola [1], have recently investigated the problem of for what functions, $U(x)$ and $V(x)$, there is a finite constant, C, such that

$$\Big[\int_0^\infty \Big|U(x)\int_0^x f(t)\,dt\Big|^p dx\Big]^{1/p} \leqslant C\Big[\int_0^\infty |V(x)f(x)|^p\,dx\Big]^{1/p}; \tag{1.2}$$

this is, of course, just the inequality (1.1) with x^b and x^{b+1} replaced by the weight functions $U(x)$ and $V(x)$. Their principal result is the following theorem.

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THEOREM 1. *If $1 \leqslant p \leqslant \infty$, there is a finite C for which (1.2) is true if and only if*

$$B = \sup_{r>0}\Big[\int_r^\infty |U(x)|^p dx\Big]^{1/p}\Big[\int_0^r |V(x)|^{-p'} dx\Big]^{1/p'} < \infty, \tag{1.3}$$

where $1/p+1/p' = 1$. Furthermore, if C is the least constant for which (1.2) holds, then $B \leqslant C \leqslant p^{1/p}(p')^{1/p'}B$ for $1<p<\infty$ and $B = C$ if $p = 1$ or ∞.

In this theorem and throughout this paper $0\cdot\infty$ is to be taken as 0 and the usual convention is used for the integrals if p or p' is ∞.

The result about the constants is best possible since $C = p^{1/p}(p')^{1/p'}B$ in the case of the original Hardy inequality and $B = C = 1$ if U is taken to be characteristic function of $[1, 2]$ and V is taken to be 1 on $[0, 1]$ and ∞ elsewhere.

Although not explicitly stated by the cited authors, Theorem 1 also has the following dual.

THEOREM 2. *If $1 \leqslant p \leqslant \infty$, there is a finite C such that*

$$\Big[\int_0^\infty \Big|U(x)\int_x^\infty f(t)\,dt\Big|^p dx\Big]^{1/p} \leqslant C\Big[\int_0^\infty |V(x)f(x)|^p dx\Big]^{1/p} \tag{1.4}$$

if and only if

$$B = \sup_{r>0}\Big[\int_0^r |U(x)|^p dx\Big]^{1/p}\Big[\int_r^\infty |V(x)|^{-p'} dx\Big]^{1/p'} < \infty.$$

Furthermore, if C is the least constant for which (1.4) is true, then $B \leqslant C \leqslant p^{1/p}(p')^{1/p'}B$.

This paper consists of the proofs of four theorems about Hardy's inequality with weights. First in § 2 a new and simpler proof is given for Theorem 1 and the proof of Theorem 2 is sketched. In § 3 another necessary and sufficient condition is given for (1.2) to hold with a finite C. The result obtained is the following.

THEOREM 3. *If $1 < P < \infty$, there is a finite C for which (1.2) is true with p replaced by P if and only if there exist functions $U_0(x)$, $U_1(x)$, $V_0(x)$ and $V_1(x)$ such that $|U(x)| = |U_0(x)|^{1/P}|U_1(x)|^{1/P'}$, $|V(x)| = |V_0(x)|^{1/P}\times |V_1(x)|^{1/P'}$, $U_0(x)$ and $V_0(x)$ satisfy (1.2) with $p = 1$ and $U_1(x)$ and $V_1(x)$ satisfy (1.2) with $p = \infty$. Furthermore, if C_0, C_1 and C are the minimum constants in (1.2) for the pairs (U_0, V_0), (U_1, V_1) and (U, V) respectively with the appropriate values of p, then $C \leqslant C_0^{1/P}C_1^{1/P'}$ and there is a choice of the U_i's and V_i's such that $C_0^{1/P}C_1^{1/P'} \leqslant P^{1/P}(P')^{1/P'}C$.*

Theorem 3 is of interest since it shows that the strongest weighted form of Hardy's inequality can be obtained by using the interpolation with change of measures theorem proved in [2], p. 485. It also suggests an approach to the n dimensional problem where the result is much less clear.

Finally, in § 4 the question of general measures is considered. The result is the following.

THEOREM 4. *If μ and ν are Borel measures and $1 \leqslant p < \infty$, there is a finite C for which*

$$\Big[\int_0^\infty \Big|\int_0^x f(t)\,dt\Big|^p d\mu(x)\Big]^{1/p} \leqslant C\Big[\int_0^\infty |f(x)|^p d\nu(x)\Big]^{1/p} \tag{1.5}$$

if and only if

$$B = \sup_{r>0}\big[\mu([r,\infty))\big]^{1/p}\Big[\int_0^r \Big(\frac{d\nu^{||}}{dx}\Big)^{-p'/p} dx\Big]^{1/p'} < \infty$$

where $\nu^{||}$ denotes the absolutely continuous part of ν. Furthermore, if C is the least constant for which (1.5) is true, then $B \leqslant C \leqslant p^{1/p}(p')^{1/p'}B$ for $p>1$ and $B = C$ for $p = 1$.

The case when $p = \infty$ is not included in Theorem 4 since it is cumbersome to state and trivial to prove; the condition is that B, the least upper bound of all r such that $\mu([r,\infty)) > 0$, is finite and $\dfrac{d\nu^{||}}{dx} > 0$ for almost every x in $[0, B]$.

2. Proof of Theorems 1 and 2. For Theorem 1 it is sufficient to prove the asserted inequalities between B and C. The new proof is the proof that $C \leqslant Bp^{1/p}(p')^{1/p'}$. The proof given here that $B \leqslant C$ is standard; it is included for completeness and as a model for the proofs of the corresponding parts of Theorems 2 and 4.

To prove that $C \leqslant Bp^{1/p}(p')^{1/p'}$ for $1<p<\infty$, it will be shown that

$$\Big[\int_0^\infty \Big|U(x)\int_0^x f(t)\,dt\Big|^p dx\Big]^{1/p} \leqslant Bp^{1/p}(p')^{1/p'}\Big[\int_0^\infty |V(x)f(x)|^p dx\Big]^{1/p}. \tag{2.1}$$

To do this let $h(x) = \big[\int_0^x |V(t)|^{-p'}dt\big]^{1/pp'}$. By Hölder's inequality the pth power of the left side of (2.1) is bounded by

$$\int_0^\infty |U(x)|^p\Big[\int_0^x |f(t)V(t)h(t)|^p dt\Big]\Big[\int_0^x |V(u)h(u)|^{-p'}dt\Big]^{p/p'} dx;$$

simple special arguments justify this even if $V(t)h(t)$ is 0 or ∞ on a set of positive measure provided the right side of (2.1) is finite. Fubini's theorem shows that this equals

$$\int_0^\infty |f(t)V(t)h(t)|^p\Big(\int_t^\infty |U(x)|^p\Big[\int_0^x |V(u)h(u)|^{-p'}dt\Big]^{p-1}dx\Big)dt. \tag{2.2}$$

Now by performing the inner integration it is apparent that

$$\int_t^\infty |U(x)|^p\Big[\int_0^x |V(u)h(u)|^{-p'}\,du\Big]^{p-1}dx \tag{2.3}$$

equals

$$(p')^{p-1}\int_t^\infty |U(x)|^p\Big[\int_0^x |V(u)|^{-p'}\,du\Big]^{(p-1)/p'}dx.$$

By the definition of B this is bounded above by

$$(Bp')^{p-1}\int_t^\infty |U(x)|^p\Big[\int_x^\infty |U(u)|^p\,du\Big]^{-1/p'}dx. \tag{2.4}$$

Performing the outer integration shows that this equals

$$p(Bp')^{p-1}\Big[\int_t^\infty |U(x)|^p\,dx\Big]^{1/p}. \tag{2.5}$$

By the definition of B, this is bounded by

$$pB^p(p')^{p-1}|h(t)|^{-p}. \tag{2.6}$$

Now in (2.2) use the fact that (2.3) is bounded above by (2.6); this shows that (2.2) is bounded above by the pth power of the right side of (2.1) and completes the proof of (2.1) for $1 < p < \infty$.

For $p = 1$ and $p = \infty$, the fact that $C \leqslant B$ is proved by showing that

$$\Big[\int_0^\infty \Big|U(x)\int_0^x f(t)\,dt\Big|^p dx\Big]^{1/p} \leqslant B\Big[\int_0^\infty |V(x)f(x)|^p\,dx\Big]^{1/p}. \tag{2.7}$$

For $p = 1$ (2.7) follows just by interchanging the order of integration on the left side of the inequality. If $p = \infty$,

$$\Big|U(x)\int_0^x f(t)\,dt\Big| \leqslant [\operatorname*{ess\,sup}_{0\leqslant t\leqslant x}|f(t)V(t)|]\,|U(x)|\int_0^x |V(t)|^{-1}dt$$

and (2.7) follows immediately.

To prove that $B \leqslant C$, observe that for a non negative f, a reduction of the intervals of integration in (1.2) shows that for $r > 0$

$$\Big[\int_r^\infty |U(x)|^p\,dx\Big]^{1/p}\Big|\int_0^r f(t)\,dt\Big| \leqslant C\Big[\int_0^r |V(x)f(x)|^p\,dx\Big]^{1/p}. \tag{2.8}$$

It is sufficient to show that

$$\Big[\int_r^\infty |U(x)|^p\,dx\Big]^{1/p}\Big[\int_0^r |V(x)|^{-p'}dx\Big]^{1/p'} \leqslant C. \tag{2.9}$$

If $p \neq 1$ and $0 < \int_0^r |V(x)|^{-p'}dx < \infty$, (2.9) follows immediately from (2.8) by taking $f(x) = |V(x)|^{-p'}$. If $p = 1$ and $0 < \operatorname*{ess\,sup}_{0<x<r}\frac{1}{|V(x)|} < \infty$, (2.9) follows from (2.8) by letting f be the characteristic function of the set where $1/|V(x)| \geqslant -1/n + \operatorname*{ess\,sup}_{0<x<r} 1/|V(x)|$ and then letting $n \to \infty$. If $\Big[\int_0^r |V(x)|^{-p'}dx\Big]^{1/p'} = 0$, (2.9) is immediate. If $\Big[\int_0^r \Big|\frac{1}{V(x)}\Big|^{p'}dx\Big]^{1/p'} = \infty$, there exists an $f(x)$ such that $[\int_0^r |f(x)V(x)|^p dx]^{1/p} < \infty$ and $\int_0^r f(x)\,dx = \infty$. Then if $C < \infty$, (2.8) with this f shows that $[\int_r^\infty |U(x)|^p dx]^{1/p} = 0$ so (2.9) holds; if $C = \infty$, (2.9) is obviously true.

To prove Theorem 2, assume first that $0 < U(x) < \infty$ and $0 < V(x) < \infty$ almost everywhere on $[0, \infty)$. Let g be a function in $L^{p'}$. By Fubini's theorem

$$\int_0^\infty \Big[U(x)\int_x^\infty f(t)\,dt\Big]g(x)\,dx \tag{2.10}$$

equals

$$\int_0^\infty V(t)f(t)\Big[\frac{1}{V(t)}\int_0^t g(x)\,U(x)\,dx\Big]dt.$$

By Holder's inequality this is bounded above by

$$\Big[\int_0^\infty |f(t)\,V(t)|^p\,dt\Big]^{1/p}\Big[\int_0^\infty \Big|\frac{1}{V(t)}\int_0^t g(x)\,U(x)\,dx\Big|^{p'}dt\Big]^{1/p'}.$$

By Theorem 1 this is bounded by

$$p^{1/p}(p')^{1/p'}B\Big[\int_0^\infty |f(t)\,V(t)|^p\,dt\Big]^{1/p}\Big[\int_0^\infty |g(x)|^{p'}dx\Big]^{1/p'}.$$

and the converse of Hölder's inequality shows that $C \leqslant p^{1/p}(p')^{1/p'}B$. Simple limiting arguments take care of the cases where U and V are 0 or ∞ on a set of positive measure. The proof that $B \leqslant C$ is most easily done by imitating the corresponding proof in Theorem 1.

3. Proof of Theorem 3. If U and V can be written in the indicated form, then an interpolation with change of measure [2], p. 485 proves that (1.2) holds and $C \leqslant C_0^{1/P}C_1^{1/P'}$. To prove the rest of the theorem, assume that U and V satisfy (1.2) with p replaced by P and let $U_1(x) = [\int_x^\infty |U(t)|^P\,dt]^{1/P}$, $V_0(x) = [\int_0^x |V(t)|^{-P'}dt]^{-1/P'}$, $U_0(x) = |U(x)|^P|U_1(x)|^{1-P}$ and $V_1(x) = |V(x)|^{P'}|V_0(x)|^{1-P'}$. It is then sufficient by use of Theorem 1 to prove that U_0 and V_0 satisfy (1.3) with $p = 1$, to prove that U_1

and V_1 satisfy (1.3)with $p = \infty$, and to estimate the resulting B's in terms of C.

To prove that U_0 and V_0 satisfy (1.3), start with the fact that

$$\int_s^\infty U_0(x)\,dx = \int_s^\infty |U(x)|^P \Big[\int_x^\infty |U(t)|^P\,dt\Big]^{-1/P'} dx. \tag{3.1}$$

Performing the integration shows that the right side of (3.1) is bounded above by

$$P\Big[\int_s^\infty |U(t)|^P\,dt\Big]^{1/P}.$$

Theorem 1 then shows that this is bounded above by

$$BP\Big[\int_0^s |V(t)|^{-P'}\,dt\Big]^{1/P'} \leqslant CPV_0(s).$$

Therefore, $\int_s^\infty U_0(x)\,dx \leqslant CPV_0(s)$. Since $V_0(x)$ decreases as s increases, this proves that (1.3) is true for U_0 and V_0 for $p=1$ with constant CP.

A similar set of inequalities proves (1.3) is true for U_1 and V_1 for $p=\infty$ with constant CP'. Using Theorem 1 again, (1.2) is true for U_0 and V_0 for $p=1$ with minimum constant, $C_0 \leqslant CP$, and (1.2) is true for U_1 and V_1 for $p=\infty$ with minimum constant, $C_1 \leqslant CP'$. Combining these two inequalities proves that $C_0^{1/P} C_1^{1/P'} \leqslant CP^{1/P}(P')^{1/P'}$.

4. Proof of Theorem 4. To prove Theorem 4 observe that (1.5) is equivalent to the inequality

$$\Big[\int_0^\infty \Big|\int_0^x f(t)\,dt\Big|^p d\mu(x)\Big]^{1/p} \leqslant C\Big[\int_0^\infty |f(x)|^p \frac{d\nu^{||}}{dx}\,dx\Big]^{1/p} \tag{4.1}$$

since changing the values of f to 0 on the support of the singular part of ν does not affect the left side of (1.4) and the inequality must still hold for the modified function.

The proof that $C \leqslant Bp^{1/P}(p')^{1/p'}$ could be done in the same way as in Theorem 1, but one difficulty arises. The analogue of the passage from (2.4) to (2.5) requires the inequality

$$\int_{[t,\infty)} [\mu([x,\infty))]^{-1/p'}\,d\mu(x) \leqslant p\,[\mu([t,\infty))]^{1/p}.$$

This is true but the proof is not particularly simple.

Consequently, to prove that $C \leqslant Bp^{1/p}(p')^{1/p'}$ it is easier to proceed as follows. Let $g_n(x)$ be a sequence of monotone decreasing functions such that $0 \leqslant g_n(x) \leqslant \mu([x,\infty))$, $g_n(x)$ is absolutely continuous on $[0,\infty)$, $g_n(x) \leqslant g_{n+1}(x)$ and $\lim_{n\to\infty} g_n(x) = \mu([x,\infty))$ almost everywhere. Now

$$\int_0^\infty \Big|\int_0^x f(t)\,dt\Big|^p d\mu(x) = \int_0^\infty \mu([x,\infty))\,d\Big|\int_0^x f(t)\,dt\Big|^p; \tag{4.2}$$

if $\int_0^\infty |f(t)|\,dt < \infty$ and $\mu([0,\infty)) < \infty$, this is a well known fact about Riemann–Stieltjes integrals, and the other cases are easily obtained from this. By the monotone convergence theorem the right side of (4.2) equals

$$\sup_n \int_0^\infty g_n(x)\,d\Big|\int_0^x f(t)\,dt\Big|^p, \tag{4.3}$$

and by the reasoning used to obtain (4.2), (4.3) equals

$$\sup_n -\int_0^\infty \Big|\int_0^x f(t)\,dt\Big|^p g_n'(x)\,dx. \tag{4.4}$$

The conditions on the g_n's and the definition of B show that

$$\Big[\int_r^\infty -g_n'(x)\,dx\Big]^{1/p}\Big[\int_0^r \Big(\frac{d\nu^{||}}{dx}\Big)^{-p'/p} dx\Big]^{1/p'} \leqslant B.$$

Consequently, Theorem 1 shows that (4.4) is bounded above by

$$Bp^{1/p}(p')^{1/p'}\int_0^\infty |f(x)|^p \frac{d\nu^{||}}{dx}\,dx. \tag{4.5}$$

Therefore, (4.5) is an upper bound for the left side of (4.2); this completes the proof that $C \leqslant Bp^{1/p}(p')^{1/p'}$.

To show that $B \leqslant C$, a reduction of the intervals of integration in (4.1) shows that for $r > 0$

$$[\mu([r,\infty))]^{1/p}\Big|\int_0^r f(t)\,dt\Big| \leqslant C\Big[\int_0^r |f(x)|^p \frac{d\nu^{||}}{dx}\,dx\Big]^{1/p}. \tag{4.6}$$

The proof then is exactly the same as the corresponding part of the proof of Theorem 1 with $V(x)$ replaced by $\left(\frac{d\nu^{||}}{dx}\right)^{1/p}$ and $\int_r^\infty |U(x)|^p\,dx$ replaced by $\mu([r,\infty))$.

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Singular integrals and cardinal series

by

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Abstract. A cardinal series K is constructed with coefficients taken as the values of a singular integral kernel K_0 (of the Calderón-Zygmund type) at the non-zero lattice points of Euclidean space. It is shown that K is the kernel of an operator from L^p into L^p, and that when K is subjected to similarity transformations, the resulting operator K_t approaches K_0 in a weak sense. Special formulas are derived for the case when K_0 is the Weierstrass kernel, and from this pointwise convergence follows.

1. Introduction. In the approach of E. C. Titchmarsh [4] to the M. Riesz theory of the Hilbert transform, the theory is formulated first for discrete transforms and then extended by a limiting process to the Hilbert transform. Implicit in this work is the use of cardinal series.

In the present paper, we take a similar approach to the theory of singular integrals due to Calderón and Zygmund [1]. Our aim is more modest than that of [4] in that we shall accept their whole theory and not attempt to create an entirely new approach to singular integrals. In particular, we shall use their extension of the theory to discrete transforms (cf. [1]). From the discrete transform, a cardinal series is constructed as the kernel of a translation-invariant operator on $L^p(R_n)$ into itself. The operator is then subjected to similarity-transformations, which, in a weak limit sense, reproduces the original singular integral operator.

In the last section, the operator associated with the Weierstrass kernel is treated in some detail. In particular, a rather explicit formula for the associated cardinal series is obtained. From this, it is shown that pointwise convergence of the cardinal series to the original kernel follows.

2. Preliminaries. Let K_0 be a Calderón–Zygmund kernel on R_N (cf. [1]); i.e., $K_0(x) = \Omega(x')/|x|^N$ with x' the radial projection of x onto the unit sphere about the origin, where the integral of Ω over the unit sphere is 0, and Ω is continuous with modulus of continuity ω such that $\int_0^1 \frac{\omega(r)}{r}\,dr < \infty$. The singular convolution integral operator T_0 with

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