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Singular integrals and cardinal series

by

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Abstract. A cardinal series K is constructed with coefficients taken as the values of a singular integral kernel K_0 (of the Calderón-Zygmund type) at the non-zero lattice points of Euclidean space. It is shown that K is the kernel of an operator from L^p into L^p, and that when K is subjected to similarity transformations, the resulting operator K_t approaches K_0 in a weak sense. Special formulas are derived for the case when K_0 is the Weierstrass kernel, and from this pointwise convergence follows.

1. Introduction. In the approach of E. C. Titchmarsh [4] to the M. Riesz theory of the Hilbert transform, the theory is formulated first for discrete transforms and then extended by a limiting process to the Hilbert transform. Implicit in this work is the use of cardinal series.

In the present paper, we take a similar approach to the theory of singular integrals due to Calderón and Zygmund [1]. Our aim is more modest than that of [4] in that we shall accept their whole theory and not attempt to create an entirely new approach to singular integrals. In particular, we shall use their extension of the theory to discrete transforms (cf. [1]). From the discrete transform, a cardinal series is constructed as the kernel of a translation-invariant operator on $L^p(R_n)$ into itself. The operator is then subjected to similarity-transformations, which, in a weak limit sense, reproduces the original singular integral operator.

In the last section, the operator associated with the Weierstrass kernel is treated in some detail. In particular, a rather explicit formula for the associated cardinal series is obtained. From this, it is shown that pointwise convergence of the cardinal series to the original kernel follows.

2. Preliminaries. Let K_0 be a Calderón–Zygmund kernel on R_N (cf. [1]); i.e., $K_0(x) = \Omega(x')/|x|^N$ with x' the radial projection of x onto the unit sphere about the origin, where the integral of Ω over the unit sphere is 0, and Ω is continuous with modulus of continuity ω such that $\int_0^1 \frac{\omega(r)}{r}\,dr < \infty$. The singular convolution integral operator T_0 with

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kernel K_0 then maps L^p into L^p, $1 < p < \infty$. Furthermore, as shown in [1], T_0 maps l^p into l^p in the following sense. Let $\{x_n\}$ be a multisequence in l^p with index n ranging over the lattice points of R_N. Let

$$y_n = \sum_{m \neq n} K_0(m-n) x_m.$$

Then $\{y_n\}$ also belongs to l^p with norm not exceeding a constant multiple of that of $\{x_n\}$. From the values of K_0 at the non-zero lattice points, we form a cardinal series; i.e., let

$$K_W(x) = K_W(\xi_1, \xi_2, \ldots, \xi_N) = \frac{\sin \pi \xi_1}{\pi \xi_1} \frac{\sin \pi \xi_2}{\pi \xi_2} \cdots \frac{\sin \pi \xi_N}{\pi \xi_N}.$$

The Fourier transform of K_W is the characteristic function of S, the hyper-rectangle of side 2π symmetric about the origin and with sides parallel to the coordinate axes. Let

$$K(x) = \sum{}' K_0(m) K_W(x+m)$$

where the prime indicates the term corresponding to the zero lattice point is omitted. The series is sometimes known as a Whittaker cardinal series (cf. [2] for the general theory). The function K is entire of exponential type, and the series interpolates K at the non-zero lattice points; i.e., $K(m) = K_0(m)$, $m \neq 0$. Now we form the convolution operator T with kernel K. Thus

$$(T\varphi)(x) = \int K(y-x)\varphi(y)\,dy.$$

T also maps L^p into itself as can be seen from the following argument. First, $T\varphi = T(K_W * \varphi)$ since both $\hat{K}$ and $\hat{K}_W$ have support in S, where K_W has the value one. Since the Fourier transform of $K_W * \varphi$ also has support in S, $\|K_W * \varphi\|_p \leqq C\|\varphi\|_p$. Thus, it may be assumed that $K_W * \varphi = \varphi$. In this case, the L^p norm of φ is equivalent to the l^p norm of $\{\varphi(n)\}$. The same is true of $K * \varphi$, and for smooth φ

$$(K * \varphi)(n) = \sum K(m-n)\varphi(m) = \sum{}' K_0(m-n)\varphi(m).$$

For the first equality, we are using the fact that the function

$$b_n(x) = \sum K(m-n+x)\varphi(m+x)$$

is periodic of period one in each variable, and that the series converges uniformly to b_n. The Fourier coefficients can be computed directly to show that b_n is the constant function $(K * \varphi)(n)$ (cf. [2, p. 576] for a similar argument). Hence by the result of [1] cited above,

$$\sum |(K * \varphi)(n)|^p \leqq C \sum |\varphi(n)|^p.$$

From the equivalence of the L^p and l^p norms of the functions and sequences involved, it follows, as stated, that

$$\|K * \varphi\|_p \leqq C\|\varphi\|_p.$$

Now we propose to subject the translation-invariant operator T to similarity transformations in the following way. For $\lambda > 0$ and for φ in the Schwartz space $\mathscr{S}$, let $\varphi_\lambda(x) = \lambda^N \varphi(\lambda x)$. Let K_λ be defined similarly. As a tempered distribution, K_λ may be defined by the relation $K_\lambda(\varphi) = \lambda^N K(\varphi_{1/\lambda})$. K_λ is the kernel of an operator T_λ which maps L^p into itself and with the same operator norm as that of T. Explicitly

$$(1) \qquad (T_\lambda \varphi)(x) = \int K_\lambda(y-x)\varphi(y)\,dy.$$

Translation-invariant operators mapping L^p into itself form a Banach space L_p^p (under the operator norm) which has a weakly closed unit ball (cf. [3]). For us, this means that, if for each φ of $\mathscr{S}$, $\lim_{\lambda\to\infty} K_\lambda(\varphi)$ exists, then there is a distribution J such that

$$\lim_{\lambda\to\infty} K_\lambda(\varphi) = J(\varphi)$$

and such that J is the kernel of a translation-invariant operator T_J in L_p^p with operator norm not exceeding that of T. It will be shown that J exists, and in fact that T_J is, apart from a constant multiple of the identity, the original operator T_0. This exception is explained in the next section.

It is possible that one can show, independently of the theory of Calderón and Zygmund, that the kernel K_0 evaluated at the non-zero lattice points defines a discrete operator bounded from l^p into itself. Hence, by the procedure outlined above, one would obtain an independent approach to singular integrals. However, this does not now seem like a reasonable way to treat singular integrals.

3. The principal theorem. It is known [1] that, if K_0 is a Calderón-Zygmund kernel, the spherical patrial sums

$$(2) \qquad \sum_{0<|m|<r} K_0(m)$$

will converge as r goes to ∞. Very often the limit is zero, as is clearly the case if K_0 is an odd function. However this is not always so, and we denote the limiting value of (2) by $\Gamma(K_0)$. With $\Gamma(K_0)$ thus defined, we are prepared to state our principal theorem.

THEOREM 1. *Let T_0 be a Calderón-Zygmund singular integral operator with kernel K_0. Then T_λ, defined by* (1), *converges weakly, as λ goes to ∞, to the operator*

$$\Gamma(K_0)I + T_0.$$

As explained above, it is enough to show that, for each φ of $\mathscr{S}$,

$$\lim_{\lambda\to\infty} K_\lambda(\varphi) = \Gamma(K_0)\varphi(0)+K_0(\varphi) \tag{3}$$

where both K_λ and K_0 are considered as distributions. First, let φ be of exponential type; i.e., let supp $\hat{\varphi}$ be compact. Then

$$K_\lambda(\varphi) = \lambda^N \int K(\lambda x)\varphi(x)\,dx = \sum{}' K_0(m)\Big\{\lambda^N\int K_W(\lambda x - m)\varphi(x)\,dx\Big\}.$$

Since φ is of exponential type, the term in brackets on the right is $\varphi(m/\lambda)$ for λ sufficiently large, and

$$K_\lambda(\varphi) = \sum{}' K_0(m)\varphi(m/\lambda) = \sum{}' K_0(m/\lambda)\varphi(m/\lambda)\lambda^{-N}.$$

The second equality follows from the homogeneity of K_0. If $\varphi(0) = 0$, the sum on the right is a Riemann sum for the absolutely convergent integral $\int K_0(x)\varphi(x)\,dx$. If $\varphi(0) \neq 0$, choose δ small but temporarily fixed. Then

$$K_\lambda(\varphi) = \Big\{\sum_{0<|m|<\lambda\delta} + \sum_{\lambda\delta\leqslant|m|}\Big\} K_0(m/\lambda)\varphi(m/\lambda)\lambda^{-N}. \tag{4}$$

The second term on the right is a Riemann sum for the absolutely convergent integral $\int\limits_{\delta\leqslant|x|} K_0(x)\varphi(x)\,dx$, and for small δ, this is close to $K_0(\varphi)$. For the first term, write $\varphi(m/\lambda) = \varphi(0)+\psi(m/\lambda)$ where $\psi(m/\lambda) = O(|m|/\lambda) = O(\delta)$. Thus the first sum in (4) is

$$\varphi(0)\sum_{0<|m|<\lambda\delta} K_0(m) + \sum_{0<|m|<\lambda\delta} K_0(m)\psi(m/\lambda).$$

The limit of the first term above is $\Gamma(K_0)\varphi(0)$. Since $K_0(m) = O(|m|^{-N})$ and $\psi(m/\lambda) = O(|m|/\lambda)$, the second term above is $O(\delta)$. This verifies (3) for φ of exponential type.

For general φ, write $\varphi = \hat{\psi}_1+\hat{\psi}_2$ where ψ_1 has compact support and ψ_2 has small L^1 norm. This can be accomplished by multiplying $\hat{\varphi}$ by a smooth localizing function. Then

$$K_\lambda(\varphi) = K_\lambda(\hat{\psi}_1)+\hat{K}_\lambda(\psi_2).$$

Since $\hat{K}_\lambda$ is a function bounded uniformly in λ, the second term is small for all λ. Since $\hat{\psi}_1$ is of exponential type, the limit of the first term is $\Gamma(K_0)\hat{\psi}_1(0)+K_0(\hat{\psi}_1)$ which equals

$$\Gamma(K_0)\varphi(0)+K_0(\varphi)-\{\Gamma(K_0)\hat{\psi}_2(0)+\hat{K}_0(\psi_2)\}.$$

Since ψ_2 has small L^1 norm and $\hat{K}_0$ is bounded, the bracketed term is small, and the theorem follows.

4. The Weierstrass kernel. If, for $N = 2$, $K_0(x, y)$ is taken to be $(x+iy)^{-2}$, the Weierstrass kernel, an especially important singular integral is obtained (cf. [1, p. 269]). In this case, an explicit expression for K can be obtained, from which the pointwise convergence of K_λ to K_0 will follow. Because of the evenness of K_0,

$$\hat{K}(x, y) = \mathscr{X}_S(x, y)\sum{}'(m+in)^{-2}e^{-i(mx+ny)}.$$

Let D be the differential operator $-\partial^2/\partial x^2-2i\,\partial^2/\partial x\partial y+\partial^2/\partial y^2$. For φ in $\mathscr{S}$

$$(D\hat{K})(\varphi) = \hat{K}(D\varphi) = (2\pi)^{+2}\sum{}'(m+in)^{-2}c_{m,n}(D\varphi)$$

where $c_{m,n}(D\varphi)$ denotes the Fourier coefficient of index (m, n) of the function $D\varphi$. The restriction of $D\varphi$ to S does not lead to a smooth function on the torus, but it is, at least, in L^2, and the above series converges.

Since we shall separate terms later, we introduce a summability method (double Cesaro sums) so that

$$(2\pi)^{-2}(D\hat{K})(\varphi) = \lim_{R\to\infty}\sum{}'\sigma(m, n; R)(m+in)^{-2}c_{m,n}(D\varphi)$$

where $\sigma(m, n; R) = (1-|m|/R)(1-|n|/R)$ for $0 \leqq |m|, |n| < R$. Integration by parts leads to the formula

$$c_{m,n}(D\varphi) = (m+in)^2[c_{m,n}(\varphi)+\hat{L}_{m,n}(\varphi)]$$

where $\hat{L}_{m,n}$ is a distribution with support on the boundary of S. Thus

$$(2\pi)^{-2}(D\hat{K})(\varphi) = \lim_{R\to\infty}\sum{}'\sigma(m, n; R)c_{m,n}(\varphi)+\lim_{R\to\infty}\sum{}'\sigma(m, n; R)\hat{L}_{m,n}(\varphi).$$

Since φ is smooth near the origin, its Fourier series is thus summable to $\varphi(0, 0)$, and

$$(2\pi)^{-2}(D\hat{K})(\varphi) = \varphi(0, 0)-c_{0,0}(\varphi)+(2\pi)^{-2}\hat{L}(\varphi)$$

where $\hat{L}(\varphi)$ indicates the second limit above so that $\hat{L}$ is a distribution with support on the boundary of S. Hence

$$D\hat{K} = (2\pi)^2\delta-\mathscr{X}_S+\hat{L}$$

where $\mathscr{X}_S$ is the characteristic function of S. Taking inverse Fourier transforms gives

$$K(x, y) = K_0(x, y)[1-K_W(x, y)+L(x, y)]. \tag{5}$$

By examining in detail the structure of L, we may prove our final theorem.

THEOREM 2. *Let K be defined by* (5) *with $K_0(x, y) = (x+iy)^{-2}$. For $x \neq 0$, and $y \neq 0$, $K_\lambda(x, y)$ converges to $K_0(x, y)$.*

Since $(K_0)_\lambda = K_0$, it is enough to show that $[K_0(K_W - L)]_\lambda$ converges to 0. Now

$$(K_0 K_W)_\lambda(x, y) = K_0(x, y) K_W(\lambda x, \lambda y)$$

which clearly converges to 0 as λ goes to ∞.

For similar reasons, it suffices to show that $L(\lambda x, \lambda y)$ converges to zero. By direct computation we may show that $\hat{L}$ is composed of several terms of which there are three typical types. The first is

$$\{\varphi(\pi, \pi) - \varphi(-\pi, \pi) - \varphi(\pi, -\pi) + \varphi(-\pi, -\pi)\} \times \\ \times \lim_{R\to\infty} \sum{}' \sigma(m, n; R)(-1)^{m+n} K_0(m, n).$$

Write

$$K_0(m, n) = \frac{m^2 - n^2}{|m+in|^4} - 2i\frac{mn}{|m+in|^4} = a_{m,n} + b_{m,n}.$$

Since $a_{m,n} = -a_{n,m}$, and since $b_{m,n} = -b_{m,-n}$, we have, by the symmetry of $\sigma(m, n; R)$, that this term is zero.

To estimate other terms in $\hat{L}$, we introduce the following:

$$\bar{a}_n(R) = \sum_m \sigma(m, n; R)(-1)^m (m + 2in) K_0(m, n);$$
$$\bar{\beta}_n(R) = \sum_m \sigma(m, n; R)(-1)^m K_0(m, n).$$

By elementary means, it is possible to show that $\bar{a}_n(R) = O(1/|n|)$ and $\bar{\beta}_n(R) = O(1/n^2)$ uniformly in R. These sequences converge, as R goes to ∞, in the l^2 sense to sequences, $\bar{a}_n$ and $\bar{\beta}_n$, respectively, satisfying the same order condition. Let a_n and β_n be the sequences of Fourier coefficients of the functions $\hat{g}$ and $\hat{h}$, respectively.

A term of the second type in $\hat{L}$ arises as

$$\lim_{R\to\infty} \sum \bar{a}_n(R) \int_{-\pi}^{\pi} \varphi(\pi, y) e^{-iny} dy = 2\pi \sum \bar{a}_n \bar{d}_n = \int_{-\pi}^{\pi} \varphi(\pi, y) \bar{\hat{g}}(y) dy.$$

The corresponding function in L is thus $e^{i\pi x} g(y)$. Since g is in L^2, and $\hat{g}$ has compact support, then $g(y)$ goes to 0 at ∞ as desired.

A term of the third type in $\hat{L}$ is

$$\lim_{R\to\infty} \sum \bar{\beta}_n(R) \int_{-\pi}^{\pi} \frac{\partial\varphi}{\partial x}(\pi, y) e^{-iny} dy.$$

By an argument similar to the preceding but involving the order condition on $\beta_n(R)$, it may be shown that the corresponding term of L also goes to 0 at ∞.

The hypothesis that both $x \neq 0$ and $y \neq 0$ is essential. For example, if $K_\lambda(x, 0) = \lambda^2 K(\lambda x, 0)$ were bounded for any $x \neq 0$, then as a function of one variable, $K(x, 0)$ would be in L^1. But it is easily verified that its Fourier transform is not continuous.

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