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ON MONOTONE MAPPINGS OF CONTINUA

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A *continuum* is a compact connected space. Let X be a Hausdorff continuum. We write

$$X\setminus z = U|V$$

if U and V are disjoint non-empty open sets whose union is $X\setminus z$. In any such separation, the set $A = \overline{U} = U\cup z$ is a *nodal* set. A nodal set meets any subcontinuum of X in a continuum. A mapping $f: X\to X$ is *totally monotone* if $f^{-1}C$ is a continuum for each subcontinuum C of X.

Ward [3] has shown that if $f: X\to X$ is a continuous monotone surjection, then X contains a subcontinuum which is invariant under f and contains no cutpoints. Ward pointed out that this theorem fails if f is not surjective. Nevertheless, it appears that in most applications involving non-surjective monotone f, it is enough to know that there is a subcontinuum $K\subset X$ which satisfies $fK\subset K$ and contains no cutpoints. Thus our purpose here is to prove the following

THEOREM. *Let Y be a Hausdorff continuum and $f: Y\to Y$ be totally monotone. Then there is a subcontinuum $X\subset Y$ which is minimal with respect to being a non-empty subcontinuum of Y for which*

(a) $fX\subset X$,

(b) $f{\downharpoonleft}X$ *is totally monotone.*

Furthermore, X contains no cutpoints of itself.

LEMMA 1. *Let X be a Hausdorff continuum and $f: X\to X$ be a totally monotone function. Let A be a nodal set of X and*

$$L = \bigcap\{f^{-k}A:\ k = 0, 1, 2, \ldots\}.$$

Then

(a) *L is a continuum,*

(b) *L is the largest subset of A which is mapped into itself by f.*

Proof. (a) is proved by induction using the fact that A is a nodal set and f is totally monotone. (b) follows from the fact that $fL\subset L$ and every subset of A which is mapped into itself by f is contained in L.

LEMMA 2. *Let X be a Hausdorff continuum and $f: X \to X$ be a totally monotone function and suppose that*

$$X \setminus z = U \mid V.$$

Let $A = \bar{U}$, and $B = \bar{V}$. *If* $f^{-1}B \cap B = \emptyset$, *then*

$$L = \bigcap \{f^{-k}A: k = 0, 1, 2, \ldots\}$$

is a non-empty continuum.

Proof. By Lemma 1, L is a continuum. We assume that $L = \emptyset$ so that no non-empty subset of A is mapped into itself by f.

Suppose that there exists a least integer $k \geqslant 1$ for which $f^{-k}B = \emptyset$. Then for $j \geqslant k$ we get $f^{-j}B = \emptyset$ so that $f^jA \subset A, j \geqslant k$. Then $\bigcup \{f^jA: j \geqslant k\}$ is a non-empty subset of A which is mapped into itself by f and so $L \neq \emptyset$. This contradiction shows that $f^{-k}B \neq \emptyset$, $k \geqslant 1$.

Now $f^{-1}B \cap B = \emptyset$ implies that $f^{-1}B \subset A$ and so we have $f^{-k}A \neq \emptyset$ for every $k \geqslant 1$. Since X is connected, it follows that $f^{-k}z \neq \emptyset$, $k \geqslant 1$. We further observe that $f^{-(k+1)}z \subset f^{-(k+1)}B \subset f^{-k}A$ so that

$$f^{-k}A \cap f^{-(k+1)}A \neq \emptyset, \quad k \geqslant 0. \tag{1}$$

If $f^{-k}B \cap B = \emptyset$ for every $k \geqslant 1$, then

$$L' = \bigcup \{f^{-k}B: k \geqslant 1\} \subset A$$

satisfies $f^{-1}L' \subset L'$ and so L' contains a non-empty subset which is mapped into itself by f. Thus, since $L = \emptyset$, there exists a least integer k for which $f^{-k}B \cap B \neq \emptyset$. Then $k > 1$ and

$$f^{-1}B \cup \ldots \cup f^{-(k-1)}B \subset A, \qquad B \cup f^{-1}B \cup \ldots \cup f^{-k}B \subset f^{-1}A,$$
$$f^{-(j-1)}B \cup f^{-(j+1)}B \cup \ldots \cup f^{-(k+j-1)}B \subset f^{-j}A,$$

the last inclusion holding for $j \geqslant 2$.

These inclusions imply

$$f^{-k}B \subset f^{-1}A \cap \ldots \cap f^{-(k-1)}A \tag{2}$$

and

$$f^{-(k-2)}B \subset A \cap f^{-2}A \cap f^{-4}A \cap \ldots \cap f^{-(k-1)}A, \tag{3_0}$$

$$f^{-(k-1)}B \subset A \cap f^{-1}A \cap f^{-3}A \cap \ldots \cap f^{-(k-2)}A \tag{4_0}$$

if k is odd and

$$f^{-(k-1)}B \subset A \cap f^{-2}A \cap f^{-4}A \cap \ldots \cap f^{-(k-2)}A, \tag{3_e}$$

$$f^{-(k-2)}B \subset A \cap f^{-1}A \cap f^{-3}A \cap \ldots \cap f^{-(k-1)}A \tag{4_e}$$

if k is even. If k is odd, from (2)

$$C = (f^{-2}A \cup \ldots \cup f^{-(k-1)}A) \cup (f^{-1}A \cap \ldots \cap f^{-(k-2)}A)$$

is a continuum. Since A is a nodal set, $A \cap C$ is also a continuum. Then (3_0) and (4_0) imply that

$$M = A \cap f^{-1}A \cap \ldots \cap f^{-(k-1)}A \neq \emptyset. \tag{5}$$

Likewise, if k is even, (5) holds. From the above inclusions we also get

$$\emptyset \neq f^{-(k-1)}B \subset f^{-1}A \cap f^{-k}A \tag{6}$$

if $k > 2$.

Suppose $z \in f^{-k}z$. Then of course $f^k z = z$, and if $j = 1, \ldots, k-1$, we deduce from $B \cap f^j B = \emptyset$ that $f^j z \in A$. Hence $L' = \{z, f' z, \ldots, f^{k-1}z\} \subset A$ and $fL' = L'$ and this contradicts $L = \emptyset$. Therefore $z \notin f^{-k}z$.

Suppose $z \in f^{-k}V$. Then $f^{-k}A$ is a continuum which does not contain z, and so $f^{-k}A \subset U$ or $f^{-k}A \subset V$. We deal first with the case $f^{-k}A \subset V$. Assume there is a least integer $m > k$ for which $A \cap f^{-m}A \neq \emptyset$. Then, by (6) and (1), we find that

$$D = f^{-1}A \cup f^{-k}A \cup f^{-(k+1)}A \cup \ldots \cup f^{-m}A$$

is a continuum. Furthermore

$$A \cap D = f^{-1}A \cap A \bigcup f^{-m}A \cap A$$

is a continuum (since A is a nodal set) so that $f^{-1}A \cap f^{-m}A = \emptyset$. But then $A \cap f^{-(m-1)}A = \emptyset$, and this contradicts the choice of m. Hence if $m \geqslant k, f^{-m}A \cap A = \emptyset$. But then

$$L' = \bigcup \{f^{-m}A \colon m \geqslant k\} \subset B$$

satisfies $f^{-1}L' \subset L'$ and this contradiets $B \cap f^{-1}B = \emptyset$. Therefore, we must conclude that $f^{-k}A \subset U \subset A$. By applying f^{-k} to both sides of inclusions (2), (3_0), (4_0), (3_e) and (4_e) and using the fact that A is a nodal set and f^k is monotone, we find that $\emptyset \neq f^{-k}M \subset M$. By induction we conclude that $f^{-nk}M \neq \emptyset$, $n = 0, 1, 2, \ldots$ Since

$$L = \bigcap \{f^{-nk}M \colon n = 0, 1, 2, \ldots\}, \tag{6}$$

we once again find $L \neq \emptyset$ and deduce that $z \notin f^{-k}V$.

Our final alternative is $z \in f^{-k}U$. Then $f^{-k}B$ is a continuum not containing z so that by choice of k, $f^{-k}B \subset V$. Then A does not meet $f^{-k}z$ and since $z \in A \cap f^{-k}U$, we have $A \subset f^{-k}A$. Then M, as defined by (5), satisfies $M \subset f^{-k}M$. By (6), $L \neq \emptyset$, and this contradiction completes the proof.

Proof of the theorem. The existence of a subcontinuum X which is minimal with respect to (a) and (b) follows from Zorn's lemma. We prove that X has no cutpoints.

Assume z is a cutpoint of X and define U, V, A, B and L as in Lemma 2. Also let

$$M = \bigcap \{f^{-k}B \colon k = 0, 1, 2, \ldots\}.$$

Then $fL \subset L, fM \subset M$ and $f|L, f|M$ are totally monotone: for example, if C is a subcontinuum of L, then $L \cap f^{-1}C = A \cap f^{-1}C$.

We may assume that $z \neq fz$ by minimality of X; we may suppose that $z \epsilon f^{-1}U$. We consider the possible cases.

Case I. $f^{-k}B = \emptyset$ for some $k \geqslant 1$.

In this case, we may proceed as in the proof of Lemma 2 to show that $L \neq \emptyset$.

Case II. $f^{-k}B \neq \emptyset$ for every $k \geqslant 1$.

Then $f^{-1}B$ is a non-empty continuum not containing z so that $f^{-1}B \subset U$ or $f^{-1}B \subset V$. If $f^{-1}B \subset U$, Lemma 2 tells us that $L \neq \emptyset$. If $f^{-1}B \subset B$, then $M \neq \emptyset$.

In any case, we have contradicted the minimality assumptions on X, and so X has no cutpoints. The proof is complete.

The theorem of this paper is a result of preliminary investigations of the following open

Problem. Let X be a Hausdorff continuum and S be an abelian semigroup of continuous monotone surjections of X onto X. Does there exist a subcontinuum K of X which satisfies $SK = K$ and does not contain a cutpoint of itself? (**P 770**)

An affirmative answer to the above question would generalize a theorem of Wallace [2]; most likely it would also lead to the solution of Problem 2 of [1], p. 286.

REFERENCES

[1] William J. Gray, *Fixed points in spaces with cutpoints*, Archiv der Mathematik 20 (1969), p. 283-286.

[2] A. D. Wallace, *Group invariant continua*, Fundamenta Mathematicae 36 (1949), p. 119-124.

[3] L. E. Ward, Jr., *Continua invariant under monotone transformations*, The Journal of the London Mathematical Society 31 (1956), p. 114-119.

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