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Investigations in the powersum theory II

by

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To the memory of L. J. Mordell

1. The second named author, partly in collaboration with S. Knapowski based a number of applications on the following theorem. Let be $b_1, \ldots, b_n$ complex numbers, m nonnegative integer, further

$$1 = |z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_n| > 0 \tag{1.1}$$

and

$$g(\nu) = \sum_{j=1}^{n} b_j z_j^{\nu}. \tag{1.2}$$

Then the theorem in question asserts the existence of an integer ν_0 satisfying

$$m+1 \leqslant \nu_0 \leqslant m+n \tag{1.3}$$

for which the inequality

$$|g(\nu_0)| \geqslant \left(\frac{n}{8e(m+n)}\right)^n \min_{l=1,\ldots,n} |b_1 + \ldots + b_l| \tag{1.4}$$

holds.

In (1.1) the fact that $|z_1| = 1$ is of course only a normalization. It means "essentially" that $|g(\nu)|$ is estimated for a proper choice of ν in a "narrow" interval from below by its maximal term (which explains its applicability).

In the case when the b_j numbers are in a half-plane then the application of the above theorem goes smoothly; this holds of course in the important case when all b_j's are 1. In the first paper of this series([1]) (quoted as I in the sequel) we have seen how one can reduce very considerably the range of l in (1.4) which helps a lot in the applications since it permits to replace the last inconvenient factor essentially by $\left|\sum_{j=1}^{n} b_j\right|$. In many

([1]) Ann. Univ. Sci. Budapest. Eötvös Sect. Math., to appear.

cases however sums of the form

$$\Big|\sum_{j=l_1}^{l_2} b_j\Big|$$

or even the single b_j's (e.g. when the b_j's are values of characters belonging to the multiplicative group of reduced residue classes $\bmod k$) are better tractable; to get such a theorem one has to take the normalization $|z_k| = 1$ with a $k \neq 1$, i.e.

$$|z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_k| = 1 \geqslant |z_{k+1}| \geqslant \ldots \geqslant |z_n|.$$

Then we assert the

THEOREM. *Let m nonnegative integer and the indices k_1 and k_2 are defined by*

(1.5) $$|z_1| \geqslant \ldots \geqslant |z_{k_1-1}| > \frac{m+2n}{m+n} \geqslant |z_{k_1}| \geqslant \ldots \geqslant |z_k| = 1 \geqslant \ldots$$
$$\geqslant |z_{k_2}| \geqslant \frac{m}{m+n} > |z_{k_2+1}| \geqslant \ldots \geqslant |z_n|$$

(if no such k_1 exists let $k_1 = 1$ and if no such k_2 exists let $k = n$). Then there is an integer ν_1 with

$$m+1 \leqslant \nu_1 \leqslant m+n$$

such that the inequality

(1.6) $$|g(\nu_1)| \geqslant \frac{1}{n}\left(\frac{n}{24e(m+n)}\right)^n \min_{k_1+1 \leqslant h_1 \leqslant h_2 \leqslant k_2} \Big|\sum_{\mu=h_1}^{h_2} b_\mu\Big|$$

holds.

It is natural to ask whether or not the dependence of the right side of (1.6) can be replaced by $|b_1 + \ldots + b_n|$ or by $\min_j |b_j|$? In the case $k = n$ the second of us showed indeed the inequality

$$\max_{m+1 \leqslant \nu \leqslant m+n} |g(\nu)| \geqslant \left(\frac{n}{2e(m+n)}\right)^n |b_1 + \ldots + b_n|,$$

which had already several applications.

For $1 \leqslant k \leqslant n-1$ the trivial example

$$z_1 = \ldots = z_k = 1, \quad z_{k+1} = \ldots = z_n = 0,$$
$$b_1 = \ldots = b_k = 0, \quad b_{k+1} = \ldots = b_n = 1$$

shows that the last factor in (1.6) cannot be replaced by $|b_1 + \ldots + b_n|$; the example

$$z_1 = \ldots = z_n = 1,$$
$$b_1 = 1, \quad b_2 = \ldots = b_n = -\frac{1}{n-1}$$

shows that even for $1 \leqslant k \leqslant n$ it cannot be replaced by $\min_j |b_j|$ either.

As to the proof it is algebraic in the sense of I, i.e., avoiding complex integration; using it one could make the proof somewhat shorter, and instead of (1.6) could get the stronger inequality

(1.7) $$|g(\nu_1)| \geqslant \frac{1}{n}\left(\frac{n}{8e(m+2n)}\right)^n \min_{k_1+1 \leqslant h_1 \leqslant h_2 \leqslant k_2} \Big|\sum_{\mu=h_1}^{h_2} b_\mu\Big|.$$

2. For the proof we shall need a lemma.

Let $\pi_n(z) = a_0 z^n + \ldots + a_n$ be an arbitrary polynomial and

(2.1) $$A = \prod_j |\xi_j|$$

where ξ_j's stand for the zeros of $\pi_n(z)$ in $|z| > 1$ empty product meaning 1 (throughout the whole paper!).

Then we have(2) the

LEMMA. *We have the inequality*

$$\max_{-1 \leqslant x \leqslant 1} |\pi_n(x)| \geqslant \frac{A}{2^{n-1}} |a_0|.$$

Actually we shall need the following easy

COROLLARY. *If $a < b$ then the inequality*

$$\max_{a \leqslant x \leqslant b} |\pi_n(x)| \geqslant 2|a_0| \left(\frac{b-a}{4}\right)^n {\prod_j}' \left|\frac{2\eta_j - (a+b)}{b-a}\right|$$

holds where the η_j's stand for all zeros of $\pi_n(z)$ outside the disc

$$\left|z - \frac{a+b}{2}\right| < \frac{b-a}{2}.$$

3. We may suppose

$$z_\mu \neq z_\nu \quad \text{if} \quad \mu \neq \nu;$$

let m and $N > m+1$ be natural numbers and for abbreviation

(3.1) $$\frac{n}{m+n} = \delta.$$

The index $q \leqslant n-1$ be defined (if it exists) by

(3.2) $$|z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_q| > 1 + \delta \geqslant |z_{q+1}| \geqslant \ldots \geqslant |z_k| = 1 \geqslant \ldots \geqslant |z_n|;$$

(2) P. Turán, *On an inequality of Chebyshev*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 11 (1968), pp. 15–16.

if not let $q_0 = 0$. Let further be

$$f(z) = \prod_{j=1}^{q}\left(\frac{z}{z_j}-1\right)\cdot\prod_{j=q+1}^{n}(z-z_j), \tag{3.3}$$

and

$$g_n(x) = \prod_{j=1}^{q}\left(\frac{x}{|z_j|}-1\right)\cdot\prod_{j=q+1}^{n}(x-|z_j|). \tag{3.4}$$

We apply the corollary to $g_n(x)$ in $[1-\delta, 1]$. This gives an R_1 with

$$1-\delta \leqslant R_1 < 1 \tag{3.5}$$

such that

$$|g_n(R_1)| \geqslant 2\left(\frac{\delta}{4}\right)^n \prod_{j=1}^{q}\frac{1}{|z_j|}\cdot\prod{}'\frac{2}{\delta}\left||z_j|-\left(1-\frac{\delta}{2}\right)\right|$$

where the last product is extended to all z_j's with

$$\left||z_j|-\left(1-\frac{\delta}{2}\right)\right| \geqslant \frac{\delta}{2}.$$

All factors in the last product are $\geqslant 1$; we keep only those with $|z_j| \geqslant 1+\delta$, i.e.,

$$|g_n(R_1)| \geqslant 2\left(\frac{\delta}{4}\right)^n \prod_{j=1}^{q}\frac{2}{\delta}\left|1-\frac{1-\delta/2}{|z_j|}\right| \geqslant 2\left(\frac{\delta}{4}\right)^n\left(\frac{3}{1+\delta}\right)^q.$$

But this means evidently that on the circle $|z| = R_1$ the inequality

$$|f(z)| \geqslant 2\left(\frac{\delta}{4}\right)^n\left(\frac{3}{1+\delta}\right)^q \tag{3.6}$$

holds *a fortiori*.

Next we apply the corollary to $g_n(x)$ in $[1, 1+\delta]$. This gives an R_2 with

$$1 < R_2 \leqslant 1+\delta \tag{3.7}$$

such that

$$|g_n(R_2)| \geqslant 2\left(\frac{\delta}{4}\right)^n \prod_{j=1}^{q}\frac{1}{|z_j|}\prod{}'\frac{2}{\delta}\left||z_j|-\left(1+\frac{\delta}{2}\right)\right|$$

where the last product is extended to all z_j's with

$$\left||z_j|-\left(1+\frac{\delta}{2}\right)\right| \geqslant \frac{\delta}{2}.$$

Again as before

$$|g_n(R_2)| \geqslant 2\left(\frac{\delta}{4}\right)^n \prod_{j=1}^{q}\frac{2}{\delta}\left|1-\frac{1+\delta/2}{|z_j|}\right| \geqslant 2\left(\frac{\delta}{4}\right)^n\frac{1}{(1+\delta)^q}$$

and *a fortiori* the inequality

$$|f(z)| \geqslant 2\left(\frac{\delta}{4}\right)^n\frac{1}{(1+\delta)^q} \tag{3.8}$$

holds on the circle $|z| = R_2$.

4. With these R_1 and R_2 let

$$F(z) = \sum_{j=1}^{n}\frac{f(z)}{f'(z_j)(z-z_j)}\left(\frac{z}{z_j}\right)^{m+1}\frac{R_2^N}{R_2^N-z_j^N}\cdot\frac{z_j^N}{z_j^N-R_1^N}. \tag{4.1}$$

Obviously

$$F(z) = \sum_{\nu=m+1}^{m+n} d_\nu z^\nu \tag{4.2}$$

and for $\mu = 1, 2, \ldots, n$

$$F(z_\mu) = \frac{R_2^N}{R_2^N-z_\mu^N}\cdot\frac{z_\mu^N}{z_\mu^N-R_1^N}. \tag{4.3}$$

Hence for $\mu = 1, 2, \ldots, n$

$$\sum_{\nu=m+1}^{m+n} d_\nu z_\mu^\nu = \frac{R_2^N}{R_2^N-z_\mu^N}\cdot\frac{z_\mu^N}{z_\mu^N-R_1^N};$$

multiplying by b_μ and summation with respect to μ we get the important identity

$$\sum_{\nu=m+1}^{m+n} d_\nu g(\nu) = \sum_{\mu=1}^{n} b_\mu\frac{R_2^N}{R_2^N-z_\mu^N}\cdot\frac{z_\mu^N}{z_\mu^N-R_1^N}. \tag{4.4}$$

Thus

$$\max_{\nu=m+1,\ldots,m+n}|g(\nu)| \geqslant \frac{\left|\sum_{\mu=1}^{n} b_\mu\frac{R_2^N}{R_2^N-z_\mu^N}\cdot\frac{z_\mu^N}{z_\mu^N-R_1^N}\right|}{\sum_{\nu=m+1}^{m+n}|d_\nu|}. \tag{4.5}$$

5. Finally we need the upper bound for

$$\sum_{\nu=m+1}^{m+n}|d_\nu| = \|F(z)\|. \tag{5.1}$$

Let

$$f_0(z) \equiv 1, \tag{5.2}$$

for $\nu \leqslant q$

$$f_\nu(z) = \prod_{j=1}^{\nu} \left(\frac{z}{z_j} - 1\right) \tag{5.3}$$

and for $q < \nu \leqslant n$

$$f_\nu(z) = \prod_{j=1}^{q} \left(\frac{z}{z_j} - 1\right) \prod_{j=q+1}^{\nu} (z - z_j). \tag{5.4}$$

We need the identity

$$\frac{1}{z-z_j} = \frac{1}{z-z_1} + \frac{z_j - z_1}{(z-z_1)(z-z_2)} + \frac{(z_j-z_1)(z_j-z_2)}{(z-z_1)(z-z_2)(z-z_3)} + \\ + \ldots + \frac{(z_j-z_1)(z_j-z_2)\ldots(z_j-z_{j-1})}{(z-z_1)(z-z_2)\ldots(z-z_j)}. \tag{5.5}$$

We introduce the $f_\nu(z)$'s. Let first $j \leqslant q$. Then

$$\frac{1}{z-z_j} = \frac{f_0(z_j)}{f_1(z)} \frac{1}{z_1} + \frac{f_1(z_j)}{f_2(z)} \frac{1}{z_2} + \ldots + \frac{f_{j-1}(z_j)}{f_j(z)} \cdot \frac{1}{z_j} \tag{5.6}$$

and for $q < j \leqslant n$

$$\frac{1}{z-z_j} = \sum_{i=1}^{q} \frac{f_{i-1}(z_j)}{f_i(z)} \frac{1}{z_i} + \sum_{i=q+1}^{j} \frac{f_{i-1}(z_j)}{f_i(z)}. \tag{5.7}$$

Since $f_n(z) = f(z)$ we get from (4.1)

$$F(z) = \sum_{j=1}^{q} \frac{f_n(z)}{f_n'(z_j)} \left(\frac{z}{z_j}\right)^{m+1} \frac{R_2^N}{R_2^N - z_j^N} \cdot \frac{z_j^N}{z_j^N - R_1^N} \sum_{l=1}^{q} \frac{f_{l-1}(z_j)}{f_l(z)} \cdot \frac{1}{z_l} + \\ + \sum_{j=q+1}^{n} \frac{f_n(z)}{f_n'(z_j)} \left(\frac{z}{z_j}\right)^{m+1} \frac{R_2^N}{R_2^N - z_j^N} \frac{z_j^N}{z_j^N - R_1^N} \sum_{l=q+1}^{j} \frac{f_{l-1}(z_j)}{f_l(z)}.$$

Changing the order of summations we get

$$F(z) = \sum_{l=1}^{q} \frac{f_n(z)}{f_l(z)} \frac{z^{m+1}}{z_l} \sum_{j=l}^{n} \frac{f_{l-1}(z_j)}{f_n'(z_j)} \cdot \frac{R_2^N}{R_2^N - z_j^N} \cdot \frac{z_j^{N-m-1}}{z_j^N - R_1^N} + \\ + \sum_{l=q+1}^{n} \frac{f_n(z)}{f_l(z)} z^{m+1} \sum_{j=l}^{n} \frac{f_{l-1}(z_j)}{f_n'(z_j)} \cdot \frac{R_2^N}{R_2^N - z_j^N} \cdot \frac{z_j^{N-m-1}}{z_j^N - R_1^N}. \tag{5.8}$$

Since — with the notation (5.1) — for $l \leqslant q$ we have

$$\left\| \frac{f_n(z)}{f_l(z)} \cdot \frac{z^{m+1}}{z_l} \right\| \leqslant \prod_{j=l+1}^{q} \left(1 + \frac{1}{|z_j|}\right) \cdot \prod_{j=q+1}^{n} (1 + |z_j|) \leqslant (2+\delta)^{n-l} \frac{1}{(1+\delta)^{q-l}}$$

and for $l > q$

$$\left\| \frac{f_n(z)}{f_l(z)} z^{m+1} \right\| \leqslant \prod_{j=l+1}^{n} (1 + |z_j|) \leqslant (2+\delta)^{n-l}$$

we get from (5.8)

$$\|F(z)\| \leqslant \sum_{l=1}^{n} (2+\delta)^{n-l} \left| \sum_{j=l}^{n} \frac{f_{l-1}(z_j)}{f_n'(z_j)} \cdot \frac{R_2^N}{R_2^N - z_j^N} \cdot \frac{z_j^{N-m-1}}{z_j^N - R_1^N} \right|. \tag{5.9}$$

6. We have to estimate the absolute value of

$$U_l = \sum_{j=l}^{n} \frac{f_{l-1}(z_j)}{f_n'(z_j)} \cdot \frac{R_2^N}{R_2^N - z_j^N} \cdot \frac{z_j^{N-m-1}}{z_j^N - R_1^N}. \tag{6.1}$$

Using the representation $\left(\varepsilon = \exp\left(\frac{2\pi i}{N}\right)\right)$

$$\frac{z^{N-m-1}}{(R_2^N - z^N)(z^N - R_1^N)} = \frac{1}{N(R_2^N - R_1^N)} \sum_{p=1}^{n} \left(R_1^{-m} \frac{\varepsilon^{-pm}}{z - R_1 \varepsilon^p} - R_2^{-m} \frac{\varepsilon^{-pm}}{z - R_2 \varepsilon^p}\right)$$

and reversing the order of summation we get

$$U_l = \frac{R_2^N}{R_2^N - R_1^N} \{R_1^{-m} U_l' - R_2^{-m} U_l''\} \tag{6.2}$$

where

$$U_l' = -\frac{1}{N} \sum_{p=1}^{N} \varepsilon^{-pm} \sum_{j=l}^{n} \frac{f_{l-1}(z_j)}{f_n'(z_j)} \cdot \frac{1}{R_1 \varepsilon^p - z_j} \tag{6.3}$$

and

$$U'' = -\frac{1}{N} \sum_{p=1}^{n} \varepsilon^{-pm} \sum_{j=l}^{N} \frac{f_{l-1}(z_j)}{f_n'(z_j)} \cdot \frac{1}{R_2 \varepsilon^p - z_j}. \tag{6.4}$$

Now let us observe that the inner sums in (6.3) resp. (6.4) are

$$\frac{f_{l-1}(R_1 \varepsilon^p)}{f_n(R_1 \varepsilon^p)} \quad \text{resp.} \quad \frac{f_{l-1}(R_2 \varepsilon^p)}{f_n(R_2 \varepsilon^p)}$$

and hence

$$|U_l'| \leqslant \max_{|z|=R_1} \left| \frac{f_{l-1}(z)}{f_n(z)} \right| \tag{6.5}$$

resp.

$$|U_l''| \leqslant \max_{|z|=R_2} \left| \frac{f_{l-1}(z)}{f_n(z)} \right|. \tag{6.6}$$

7. We shall estimate $|U_l'|$ resp. $|U_l''|$ on using (3.6) resp. (3.8). Evidently for $l \leqslant q+1$

$$\max_{|z|=R_1} |f_{l-1}(z)| \leqslant \prod_{j=1}^{l-1}\left(1+\frac{1}{|z_j|}\right) < (2+\delta)^{l-1}$$

resp.

$$\max_{|z|=R_2} |f_{l-1}(z)| \leqslant \prod_{j=1}^{l-1}\left(1+\frac{1+\delta}{|z_j|}\right) < (2+\delta)^{l-1},$$

further for $q+1 < l \leqslant n$

$$\max_{|z|=R_1} |f_{l-1}(z)| \leqslant \prod_{j=1}^{q}\left(1+\frac{1}{|z_j|}\right)\prod_{j=q+1}^{l-1}(1+|z_j|) \leqslant (2+\delta)^{l-1}$$

resp.

$$\max_{|z|=R_2} |f_{l-1}(z)| \leqslant \prod_{j=1}^{q}\left(1+\frac{1+\delta}{|z_j|}\right)\prod_{j=q+1}^{l-1}(1+\delta+|z_j|) \leqslant (2+\delta)^{l-1}.$$

Hence from (3.6) we have for $l = 1, \ldots, n$

$$(7.1)\qquad |U_l'| \leqslant \max_{|z|=R_1}\left|\frac{f_{l-1}(z)}{f_n(z)}\right| \leqslant \frac{1}{2}\left(\frac{4}{\delta}\right)^n\left(\frac{1+\delta}{3}\right)^q(2+\delta)^{l-1},$$

resp. from (3.8)

$$(7.2)\qquad |U_l''| \leqslant \max_{|z|=R_2}\left|\frac{f_{l-1}(z)}{f_n(z)}\right| \leqslant \frac{1}{2}\left(\frac{4}{\delta}\right)^n(1+\delta)^q(2+\delta)^{l-1}.$$

Thus from (6.2), (3.5) and (3.7) we have for $l = 1, 2, \ldots, n$

$$(7.3)\qquad |U_l| \leqslant \frac{1}{2}\left(\frac{4}{\delta}\right)^n(2+\delta)^{l-1}\frac{R_2^N}{R_2^N-R_1^N}(1+\delta)^q\left\{\frac{1}{(1-\delta)^m}\cdot\frac{1}{3^q}+1\right\}.$$

(5.9) gives then

$$\|F(z)\| \leqslant \frac{n}{2}\left(\frac{4}{\delta}\right)^n(2+\delta)^{n-1}(1+\delta)^q\frac{R_2^N}{R_2^N-R_1^N}\left\{\frac{3^{-q}}{(1-\delta)^m}+1\right\}$$
$$\leqslant n\left(\frac{24}{\delta}\right)^n(1-\delta)^{-m}\frac{R_2^N}{R_2^N-R_1^N}.$$

Hence (4.5) and (3.1) give, using also $(1-\delta)^m \geqslant e^{-n}$ that

$$(7.4)\qquad \max_{m+1\leqslant\nu\leqslant m+n}|g(\nu)|$$
$$\geqslant \frac{1}{n}\left(\frac{n}{24e(m+n)}\right)^n\frac{R_2^N-R_1^N}{R_2^N}\left|\sum_{\mu=1}^{n} b_\mu\frac{R_2^N}{R_2^N-z_\mu^N}\cdot\frac{z_\mu^N}{z_\mu^N-R_1^N}\right|.$$

8. Let us remark that R_1 and R_2 are independent of N. Let N go to $+\infty$. If the indices l_1 and l_2 are defined by

$$(8.1)\qquad |z_1| \geqslant \ldots \geqslant |z_{l_1}| > R_2 > |z_{l_1+1}| \geqslant \ldots \geqslant |z_{l_2}| > R_1 > |z_{l_2+1}| \geqslant \ldots \geqslant |z_n|$$

then (7.4) gives

$$(8.2)\qquad \max_{m+1\leqslant\nu\leqslant m+n}|g(\nu)| \geqslant \frac{1}{n}\left(\frac{n}{24e(m+n)}\right)^n\left|\sum_{\mu=l_1+1}^{l_2} b_\mu\right|.$$

Since

$$R_2 \leqslant 1+\delta = \frac{m+2n}{m+n},\qquad R_1 \geqslant 1-\delta = \frac{m}{m+n}$$

(1.5) gives

$$k_1 \leqslant l_1,\qquad k_2 \geqslant l_2,$$

i.e.,

$$\left|\sum_{\mu=l_1+1}^{l_2} b_\mu\right| \geqslant \min_{k_1+1\leqslant h_1\leqslant h_2\leqslant k_2}\left|\sum_{\mu=h_1}^{h_2} b_\mu\right|$$

which proves Theorem.

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