Natural sums of ordinals

by

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1. Introduction. We characterize the Hessenberg natural sum, [3], as well as a more general notion of natural sum, given below. Our results are formalizible within Gödel-Bernays set theory — for example, within the axiom system consisting of groups A, B, C, and D of [2].

Thus an *ordinal number* can be considered as a set that is transitive and ϵ-connected. "On" will denote the class of all ordinal numbers. We will refer to ordinal numbers simply as "ordinals". (In this respect we differ from the terminology of [2]; there On is included among the ordinals.) We assume familiarity with the basic properties of ordinals and of their arithmetic [2, Chapter III] and [4, Chapter XIV].

Greek letters, sometimes with subscripts, will denote ordinals; we also use Roman letters these ordinals are natural numbers. "$\alpha < \beta$" means $\alpha \epsilon \beta$; "$\alpha \leqslant \beta$", "$\alpha > \beta$", and "$\alpha \geqslant \beta$" are defined in terms of "$\alpha < \beta$" in the usual way.

Braces will be used to designate proper classes as well as sets.

For any ordinal numbers α and β, the *Hessenberg sum* of α and β, $\alpha(+)\beta$, is defined as follows: α and β admit representations of the form

$$\alpha = \omega^{\alpha_1}\cdot m_1 + \omega^{\alpha_2}\cdot m_2 + \ldots + \omega^{\alpha_r}\cdot m_r\,,$$

$$\beta = \omega^{\alpha_1}\cdot n_1 + \omega^{\alpha_2}\cdot n_2 + \ldots + \omega^{\alpha_r}\cdot n_r\,,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_r$ are decreasing ordinals and $m_1, m_2, \ldots, m_r, n_1, n_2, \ldots, n_r$ are natural numbers. Let

$$\alpha(+)\beta = \omega^{\alpha_1}\cdot(m_1+n_1) + \omega^{\alpha_2}\cdot(m_2+n_2) + \ldots + \omega^{\alpha_r}\cdot(m_r+n_r)\,.$$

(This definition is independent of the above representations of α and β, since different representations differ only in "zero terms".)

A binary operation on On, $\oplus$, is called a *natural sum* if for all ordinals α, β, and γ,

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1) $\alpha \oplus \beta = \beta \oplus \alpha$;

2) $(\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma)$;

3) $\alpha \oplus 0 = \alpha$;

4) $\beta < \gamma$ if and only if $\alpha \oplus \beta < \alpha \oplus \gamma$.

This definition is (essentially) due to Carruth, [1].

The Hessenberg sum is clearly a natural sum. Moreover, Carruth proved that it is the "smallest" natural sum; in fact, if $\oplus$ is an arbitrary natural sum, then for all ordinals α and β,

$$\alpha + \beta \leqslant \alpha (+) \beta \leqslant \alpha \oplus \beta .$$

Let λ be a nonzero ordinal. A binary operation, $\oplus_\lambda$, on λ is said to be a *λ-natural sum* if for all $\alpha, \beta, \gamma < \lambda$,

0) $\alpha \oplus_\lambda \beta < \lambda$;

1)-4) of the definition of "natural sum" hold for $\oplus_\lambda$. λ-natural sums are discussed in [5].

A nonzero ordinal γ is said to be a *prime component* if there exists no decomposition, $\gamma = \alpha + \beta$, where $\alpha < \gamma$ and $\beta < \gamma$. The prime components are just the ordinals of the form ω^α for some ordinal α [4, pp. 319–320].

A necessary and sufficient condition for a λ-natural sum to exist is for λ to be a prime component [5, Theorem 3, p. 50].

Let λ be a prime component and let $\oplus$ $(\oplus_\lambda)$ be a natural sum (λ-natural sum). A nonzero ordinal γ $(\gamma < \lambda)$ will be called a *$\oplus$-prime* ($\oplus_\lambda$-prime) if there exists no decomposition, $\gamma = \alpha \oplus \beta$ $(\gamma = \alpha \oplus_\lambda \beta)$, where $\alpha < \gamma$ and $\beta < \gamma$. It follows from the definition of "natural sum" ("λ-natural sum") that a necessary and sufficient condition for λ to be a $\oplus$-prime ($\oplus_\lambda$-prime) is that $\gamma = \alpha \oplus \beta$ $(\gamma = \alpha \oplus_\lambda \beta)$ iff $\{\alpha, \beta\} = \{0, \gamma\}$. In particular, it is easily established that

(1) the $(+)$-primes are precisely the prime components.

A nonzero ordinal γ $(\gamma < \lambda)$ is said to be *$\oplus$-irreducible* ($\oplus_\lambda$-irreducible) if $\alpha < \gamma$ and $\beta < \gamma$ together imply $\alpha \oplus \beta < \gamma$ $(\alpha \oplus_\lambda \beta < \gamma)$. γ $(\gamma < \lambda)$ is said to be *$\oplus$-reducible* ($\oplus_\lambda$-reducible) if it is not $\oplus$-irreducible ($\oplus_\lambda$-irreducible). Every $\oplus$-irreducible ($\oplus_\lambda$-irreducible) element is a $\oplus$-prime ($\oplus_\lambda$-prime); the converse is false, in general. (See [5, p. 52].) However, $(+)$-primes are irreducible. Moreover, a necessary and sufficient condition for a nonzero ordinal γ to be a prime component is for $\alpha < \gamma$ and $\beta < \gamma$ together to imply $\alpha + \beta < \gamma$.

If $\diamond$ is a binary operation on On and $\diamond_\lambda$ is a binary operation on λ, then $\diamond_\lambda$ is said to be the *restriction of* $\diamond$ *to* λ and $\diamond$ is said to be an *extension of* $\diamond_\lambda$ if $\alpha \diamond \beta = \alpha \diamond_\lambda \beta$ whenever $\alpha < \lambda$ and $\beta < \lambda$. If λ is a prime component, if $\oplus_\lambda$ is a λ-natural sum and if $\oplus$ is a natural sum that is an extension of $\oplus_\lambda$, we say that $\oplus$ is a *natural extension of* $\oplus_\lambda$. We denote the restriction of $(+)$ to λ by "$(+)_\lambda$".

2. The main results.

THEOREM 1. a) *Let $\oplus$ be a natural sum. For each nonzero ordinal λ, the restriction of $\oplus$ to λ is a λ-natural sum if and only if λ is a $\oplus$-irreducible element.*

b) *Let λ be a prime component and let $\oplus_\lambda$ be a λ-natural sum. For arbitrary ordinals α and β, let*

$$\alpha = \lambda \cdot \alpha_1 + \alpha_2 , \quad \alpha_2 < \lambda$$

and

$$\beta = \lambda \cdot \beta_1 + \beta_2 , \quad \beta_2 < \lambda$$

be the unique representations of this form. Then for every $\mu \geqslant \lambda$ and for every natural sum $\oplus'$, the binary operation, $\oplus$, on On, *defined by*

$$\alpha \oplus \beta = \mu \cdot (\alpha_1 \oplus' \beta_1) + (\alpha_2 \oplus_\lambda \beta_2) ,$$

is a natural extension of $\oplus_\lambda$.

c) *Let λ be a prime component, let $\oplus_\lambda$ be a λ-natural sum, and let $\oplus$ be a natural extension of $\oplus_\lambda$. Then for every $\gamma < \lambda$, γ is a $\oplus$-prime if and only if γ is a $\oplus_\lambda$-prime, and γ is $\oplus$-irreducible if and only if γ is $\oplus_\lambda$-irreducible.*

THEOREM 2. a) *Let $\oplus$ be a natural sum. A necessary and sufficient condition that $\oplus = (+)$ is that*

(2) *for every ordinal γ, γ is a $\oplus$-prime if and only if γ is a $(+)$-prime.*

b) *Let λ be a prime component and let $\oplus_\lambda$ be a λ-natural sum. A nececessary and sufficient condition that $\oplus_\lambda = (+)_\lambda$ is that*

$(2)_\lambda$ *for every ordinal γ, γ is a $\oplus_\lambda$-prime if and only if γ is a $(+)_\lambda$-prime.*

Proof. a) Clearly $(+)$ satisfies (2).

Let $\oplus$ be an arbitrary natural sum that satisfies (2). Let

$$A = \{\delta \in \mathrm{On}: \text{for all } \alpha, \beta \in \mathrm{On},\ \alpha (+) \beta = \delta \text{ iff } \alpha \oplus \beta = \delta\} .$$

By (1) and clause 3) of the definition of "natural sum", every ordinal of the form ω^α is in A.

Also, $0 \in A$ because $\alpha (+) \beta = 0$ iff $\alpha = \beta = 0$ iff $\alpha \oplus \beta = 0$.

Suppose $\gamma \subseteq A$ but $\gamma \notin A$. Then there are ordinals α and β such that $\alpha (+) \beta = \gamma < \alpha \oplus \beta$. Since γ is not a $\oplus$-prime, there exist $\mu \notin \{0, \gamma\}$ such that for some ν, $\mu \oplus \nu = \gamma$. It easily follows from 1), 3), and 4) of the definition of "natural sum" that $\nu \notin \{0, \gamma\}$.

$\mu(+)\nu \not< \gamma$, by our inductive hypothesis. Since $\mu(+)\nu \leqslant \mu\oplus\nu = \gamma$, we must have $\mu(+)\nu = \gamma = \alpha(+)\beta$. Let the Cantor normal form of γ be

$$\gamma = \omega^{\varrho_1}\cdot J_1 + \omega^{\varrho_2}\cdot J_2 + \ldots + \omega^{\varrho_r}\cdot J_r\,,$$

where $\varrho_1 > \varrho_2 > \ldots > \varrho_r \geqslant 0$, and where $J_1, J_2, \ldots, J_r$ are nonzero natura numbers. Then

$$\alpha = \omega^{\varrho_1}\cdot J_1' + \omega^{\varrho_2}\cdot J_2' + \ldots + \omega^{\varrho_r}\cdot J_r'$$

and

$$\beta = \omega^{\varrho_1}\cdot J_1'' + \omega^{\varrho_2}\cdot J_2'' + \ldots + \omega^{\varrho_r}\cdot J_r''\,,$$

where for each $i = 1, 2, \ldots, r$, $J_i' + J_i'' = J_i$. Similarly,

$$\mu = \omega^{\varrho_1}\cdot J_1^* + \omega^{\varrho_2}\cdot J_2^* + \ldots + \omega^{\varrho_r}\cdot J_r^*$$

and

$$\nu = \omega^{\varrho_1}\cdot J_1^{**} + \omega^{\varrho_2}\cdot J_2^{**} + \ldots + \omega^{\varrho_r}\cdot J_r^{**}\,,$$

where for each $i = 1, 2, \ldots, r$, $J_i^* + J_i^{**} = J_i$. By means of the inductive hypothesis together with the associativity and commutativity of natural sums, it is easily established that

$$\alpha\oplus\beta = \mu\oplus\nu = \gamma\,.$$

This contradiction establishes that $\gamma \subseteq A$ implies $\gamma \in A$, and hence that $A = \mathrm{On}$.

b) $(+)_\lambda$ satisfies $(3)_\lambda$. For any λ-natural sum, $\oplus_\lambda$, let $A_\lambda = \{\delta < \lambda$: for all $\alpha, \beta \in \mathrm{On}$, $\alpha(+)_\lambda\beta = \delta$ iff $\alpha\oplus_\lambda\beta = \delta\}$.

The argument of part a), with minor modifications, shows that $A_\lambda = \lambda$.

A natural sum, $\oplus$, is said to be *continuous* provided that for all ordinals α, ζ_0, and β_ζ, where $\zeta < \zeta_0$,

$$\alpha\oplus\bigcup_{\zeta<\zeta_0}\beta_\zeta = \bigcup_{\zeta<\zeta_0}(\alpha\oplus\beta_\zeta)\,.$$

Let λ be a prime component. A λ-natural sum $\oplus_\lambda$ is said to be *continuous* provided that for all ordinals α, ζ_0, and β_ζ, where $\zeta < \zeta_0$, whenever $\alpha < \lambda$, $\beta_\zeta < \lambda$ for all $\zeta < \zeta_0$, and $\bigcup_{\zeta<\zeta_0}\beta_\zeta < \lambda$, then

$$\alpha\oplus_\lambda\bigcup_{\zeta<\zeta_0}\beta_\zeta = \bigcup_{\zeta<\zeta_0}(\alpha\oplus_\lambda\beta_\zeta)\,.$$

$(+)$ is continuous and for every prime component λ, $(+)_\lambda$ is continuous; thus (2) $\big((2)_\lambda\big)$ is a sufficient condition for a natural sum (λ-natural sum) to be continuous. The example of [5, pp. 58–59] shows that this condition is not necessary. The main result of [5] is that a continuous λ-natural sum, $\oplus_\lambda$, coincides with $(+)_\lambda$ if and only if the $\oplus_\lambda$-primes are precisely the $\oplus_\lambda$-irreducible elements. We now show that the assumption of continuity can be removed.

THEOREM 3. a) *Let $\oplus$ be a natural sum. A necessary and sufficient, condition that $\oplus = (+)$ is that every $\oplus$-prime be $\oplus$-irreducible.*

b) *Let λ be a prime component and let $\oplus_\lambda$ be a λ-natural sum. A necessary and sufficient condition that $\oplus_\lambda = (+)_\lambda$ is that every $\oplus_\lambda$-prime be $\oplus_\lambda$-irreducible.*

Proof. The verifications of a) and b) are similar; we prove only part a).

a) It suffices to show that if every $\oplus$-prime is $\oplus$-irreducible, then $\oplus = (+)$. According to Theorem 2, we need only show that if every $\oplus$-prime is $\oplus$-irreducible, then for every β, β is a $\oplus$-prime iff β is a $(+)$-prime.

Assume that every $\oplus$-prime is $\oplus$-irreducible.

Suppose β is not a $(+)$-prime. Then

$$\beta = \omega^\alpha + \omega^{\beta_1}\cdot N_1 + \omega^{\beta_2}\cdot N_2 + \ldots + \omega^{\beta_r}\cdot N_r\,,$$

where $\alpha \geqslant \beta_1 > \beta_2 > \ldots > \beta_r$, $0 < N_i < \omega$ for $1 \leqslant i \leqslant r$, and $r \geqslant 1$. Also, $\omega^\alpha < \beta$ and $\omega^{\beta_1}\cdot N_1 + \omega^{\beta_2}\cdot N_2 + \ldots + \omega^{\beta_r}\cdot N_r < \beta$; yet

$$\beta = \omega^\alpha + (\omega^{\beta_1}\cdot N_1 + \omega^{\beta_2}\cdot N_2 + \ldots + \omega^{\beta_r}\cdot N_r)$$
$$\leqslant \omega^\alpha\oplus(\omega^{\beta_1}\cdot N_1 + \omega^{\beta_2}\cdot N_2 + \ldots + \omega^{\beta_r}\cdot N_r)\,.$$

Thus, if β is not a $(+)$-prime, it is $\oplus$-reducible; hence it is not a $\oplus$-prime.

We now proceed to show that for every ordinal ϱ, ω^ϱ is a $\oplus$-prime.

Clearly, ω^0 is a $\oplus$-prime.

Suppose that for all $\gamma \leqslant \delta$, ω^γ is a $\oplus$-prime. Then the restriction of $\oplus$ to ω^δ, which we denote by "$\oplus_{\omega^\delta}$", is an ω^δ-natural sum for which $(2)_{\omega^\delta}$ is true. By Theorem 3b), $\oplus_{\omega^\delta} = (+)_{\omega^\delta}$. Furthermore, we claim that

(3) for all $\alpha \leqslant \omega^\delta$, $\omega^\delta\oplus\alpha = \omega^\delta(+)\alpha$.

Surely, $\omega^\delta\oplus 0 = \omega^\delta = \omega^\delta(+)0$.

Let $\alpha > 0$ and suppose $\omega^\delta\oplus\gamma = \omega^\delta(+)\gamma$ for all $\gamma < \alpha \leqslant \omega^\delta$. Then $\omega^\delta(+)\alpha$ cannot be a $\oplus$-prime; thus we have $\alpha_1\oplus\alpha_2 = \omega^\delta(+)\alpha$, where $\{\alpha_1, \alpha_2\} \neq \{0, \omega^\delta(+)\alpha\}$. We can assume that $\omega^\delta \leqslant \alpha_1 < \omega^\delta\oplus\alpha$ and that $0 \leqslant \alpha_2 \leqslant \alpha$. Then $\alpha_1 = \omega^\delta + \alpha_3 = \omega^\delta\oplus\alpha_3$, where $\alpha_3 < \alpha$. It follows that

$$\omega^\delta(+)\alpha = \alpha_1\oplus\alpha_2 = (\omega^\delta\oplus\alpha_3)\oplus\alpha_2 = \omega^\delta\oplus(\alpha_3\oplus\alpha_2)\,.$$

Clearly, $\alpha_3\oplus\alpha_2 = \alpha$; therefore $\omega^\delta\oplus\alpha = \omega^\delta(+)\alpha$. This establishes (3).

In particular, we have $\omega^\delta\oplus\omega^\delta = \omega^\delta\cdot 2$; by induction it can be established that

$$\underbrace{\omega^\delta\oplus\omega^\delta\oplus\ldots\oplus\omega^\delta}_{n\ \text{times}} = \omega^\delta\cdot N\,,\qquad \text{for all natural numbers } N \geqslant 1\,.$$

Now let ϱ be such that for all $\gamma < \varrho$, ω^γ is a $\oplus$-prime. If $\mu < \omega^\varrho$ and $\nu < \omega^\varrho$, there are $\delta < \varrho$ and $M < \omega$ for which $\mu < \omega^\delta \cdot M$ and $\nu < \omega^\delta \cdot M$. Thus

$$\alpha \oplus \beta < \omega^\delta \cdot M \oplus \omega^\delta \cdot M < \omega^\varrho.$$

Consequently, ω^ϱ is a $\oplus$-prime.

COROLLARY. a) *A sufficient condition for a natural sum $\oplus$ to be continuous is for every $\oplus$-prime to be $\oplus$-irreducible.*

b) *Let λ be a prime component. A sufficient condition for a λ-natural sum $\oplus_\lambda$ to be continuous is for every $\oplus_\lambda$-prime to be $\oplus_\lambda$-irreducible.*

References

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Some remarks on selectors (I)

by

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The Axiom of Choice states that every family of pairwise disjoint non-void sets $\mathfrak{X} = \langle X_i \rangle_{i \in I}$ has a selector, i.e. there is a set S such that $|S \cap X_i| = 1$ for every $i \in I$. The situation is quite different when we consider non-disjoint families. In the present paper we will study the problem of the existence of selectors of the families which have large subfamilies with selectors. (Of course the Axiom of Choice is assumed throughout.) So our problem has rather a "compactness" character.

We say that a family $\mathfrak{X} = \langle X_\alpha \rangle_{\alpha < \varkappa}$ has *partial selectors* if for every $\beta < \varkappa$ the family $\mathfrak{X} \upharpoonright \beta = \langle X_\alpha \rangle_{\alpha < \beta}$ has a selector. $\mathbf{E}(\varkappa, \lambda)$ (or respectively $\mathbf{F}(\varkappa, \lambda)$ (1)) will denote the following statement: *For every family $\mathfrak{X} = \langle X_\alpha \rangle_{\alpha < \varkappa}$ of sets of powers $< \lambda$ (or $= \lambda$ respectively) if $\mathfrak{X}$ has partial selectors then $\mathfrak{X}$ has a selector.*

It is easy to see that for each infinite cardinal $\varkappa$, the statement $\mathbf{E}(\varkappa, \omega_0)$ is provable in **ZFC**. In [1], P. Erdös and A. Hajnal ask: *Does* $\mathbf{E}(\omega_2, \omega_1)$ *hold?* We give a partial answer (Corollary 4.6) to the question. The main result is contained in § 4 (Theorem 4.4). It states that under the assumption of **GCH**, the property $\mathbf{E}(\varkappa, \varkappa)$ is equivalent to the weak compactness of $\varkappa$.

The paper is arranged as follows: in § 0 we give some neccessary definitions, and in § 1 we prove some simplest properties of the statements **E** and **F**. In particular, from 1.1.5 it follows that the investigations of the statement **F** can be reduced to **E** with respective parameters. In § 2, we give the proof of a part of 4.3, namely that the weak compactness of $\varkappa$ implies $\mathbf{E}(\varkappa, \varkappa)$. In § 3, we study connections between $\mathbf{E}(\varkappa, \varkappa)$ and the *tree property* of $\varkappa$. From the results of §§ 3 and 4 it follows that if we do not assume the strong inaccessibility of $\varkappa$, then the property $\mathbf{E}(\varkappa, \varkappa)$ is a better approximation of the weak compactness than the tree property of $\varkappa$. Finally, in § 4, we prove two theorems which have rather a combinatorial character.

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(1) $\mathbf{F}(\varkappa, \lambda)$ denotes $S(\varkappa, \lambda, 2) \to B(2)$ in the terminology of [1].