If the domain of a strongly additive function is a ring, then the condition of (σ, δ)-additivity reduces to σ-additivity.

2.11. THEOREM. *Let λ be a σ-additive function on a ring $\mathfrak{R}$ of subsets of a set T, with values in a complete metric Abelian group G. Then λ is uniquely extendable to a σ-additive function λ' on the σ-ring $\mathfrak{R}'$ generated by $\mathfrak{R}$ if only if λ is monotonely convergent.*

Proof. Let $\mathfrak{L}'$ $(\mathfrak{R}')$ be the (σ, δ)-lattice (σ-ring) generated by $\mathfrak{R}$. We note that $\mathfrak{R}'$ is the monotone class generated by $\mathfrak{R}$ ([1], p. 12). Since $\mathfrak{L}'$ is a monotone class containing $\mathfrak{R}$, $\mathfrak{L}' \supseteq \mathfrak{R}'$. On the other hand, $\mathfrak{R}'$ is a (σ, δ)-lattice containing $\mathfrak{R}$, hence also $\mathfrak{L}'$, so we have $\mathfrak{R}' = \mathfrak{L}'$. Now the theorem will follow, as a corollary of 2.10, if we show that the monotone convergence of λ implies the λ_δ-lower regularity of λ_σ. Let $E \in \mathfrak{R}_\sigma$. Since λ_δ is monotonely convergent (hypothesis and 2.6), the argument (b) $\Rightarrow$ (a) of the proof of 2.4 shows that $\lim\limits_{F \subseteq E, F \in \mathfrak{R}_\delta} \lambda_\delta(F) = \mu(E)$ exists. Let $\varepsilon > 0$ be arbitrary. There exists an $\mathfrak{R}_\delta$-set K contained in E such that $K \subseteq K' \subseteq E$ and $K' \in \mathfrak{R}_\delta$ implies $|\lambda_\delta(K') - \mu(E)|$, $|\lambda_\delta(K')| - \lambda_\delta(K)| < \varepsilon$. Let $R_n \uparrow R$, $R_n \in \mathfrak{R}$. Then

$$|\lambda(R_n) - \mu(E)| \leqslant |\lambda(R_n) - \lambda_\delta(K)| + |\lambda_\delta(K) - \mu(E)|$$
$$< |\lambda_\delta(R_n \cup K) - \lambda_\delta(K)| + |\bar{\lambda}_\delta(K - R_n)| + \varepsilon$$

and $|\lambda_\delta(R_n \cup K) - \lambda_\delta(K)| < \varepsilon$. Because $\mathfrak{R}$ is a ring, $K - R_n \in \mathfrak{R}_\delta$, so $\bar{\lambda}_\delta(K - R_n) = \lambda_\delta(K - R_n) \to 0$ (2.6). Therefore $\lambda(R_n) \to \mu(E)$. On the other hand, $\lambda(R_n) \to \lambda_\sigma(E)$, therefore $\mu(E) = \lambda_\sigma(E)$.

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Various approaches to the fundamental groups

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Abstract. The notion of the fundamental group introduced by K. Borsuk in [1] is useful in the Borsuk approach to the theory of shapes. However, if one is concerned with the Mardešić and Segal approach (see [3], [4]), then some other notions seem to be more convenient. One of them, the notion of the limit homotopy group, has been defined by the author in [7]. Another one, the notion of the shape group, is defined here in § 4. As regards compact metric spaces, these three approaches turn out to be in some sense equivalent (§ 6). To explain these connections we start with preliminaries concerning the category theory (§ 1).

For the convenience of the reader, definitions of the two categories $\mathfrak{R}^*, \hat{\mathfrak{R}}^*_{\sim}$ (of ANR($\mathfrak{R}$)-systems) and of the categories $\mathfrak{G}^*, \hat{\mathfrak{G}}^*$ (of inverse systems of groups) are recalled in the Appendix.

1. Isomorphism and quasi-isomorphism of functors. One of the basic concepts in category theory is the notion of natural transformation and of natural equivalence of functors (see [6], p. 59). The notion of natural equivalence enables us to identify two functors Π, Π': $\mathfrak{K} \to \mathfrak{L}$, which, from the intuitive point of view, coincide. Here, the natural transformations are treated as morphisms in some category of functors; then the natural equivalence is simply an isomorphism in this category. In turn, this notion of isomorphism of functors from $\mathfrak{K}$ to $\mathfrak{L}$, where the categories $\mathfrak{K}$, $\mathfrak{L}$ are both fixed, is extended to the notion of quasi-isomorphism of functors. It enables us to study the connection between two functors Π: $\mathfrak{K} \to \mathfrak{L}$ and Π': $\mathfrak{K}' \to \mathfrak{L}$.

Given two categories $\mathfrak{K}$, $\mathfrak{L}$, we are concerned with covariant functors from $\mathfrak{K}$ to $\mathfrak{L}$. Let us consider the category $\mathcal{M}^{\mathfrak{K}\mathfrak{L}}$ (1) with all those functors as objects and with morphisms defined as follows:

for $\Pi, \Pi' \in \mathrm{Ob}_{\mathcal{M}}$

$$\Lambda \in \mathrm{Mor}_{\mathcal{M}}(\Pi, \Pi') \quad \text{whenever } \Lambda = \{\lambda_X\}_{X \in \mathrm{Ob}_{\mathfrak{K}}}$$

where

$$\lambda_X \in \mathrm{Mor}_{\mathfrak{L}}(\Pi(X), \Pi'(X))$$

(1) We shall often write $\mathcal{M}$ instead of $\mathcal{M}^{\mathfrak{K}\mathfrak{L}}$.

and for any $f \in \mathrm{Mor}_{\mathcal{K}}(X, Y)$ the diagram

$$\begin{tikzcd} \Pi(X) \arrow[r, "\Pi(f)"] \arrow[d, "\lambda_X"'] & \Pi(Y) \arrow[d, "\lambda_Y"] \\ \Pi'(X) \arrow[r, "\Pi'(f)"'] & \Pi'(Y) \end{tikzcd} \quad \text{is commutative.}$$

Let us refer to this diagram as the Λ*-diagram.*

For any functor $\Pi \in \mathrm{Ob}_{\mathcal{M}}$, the identity 1_Π is defined as $\{1_{\Pi(X)}\}$.

The composition of morphisms in $\mathcal{M}^{\mathcal{K}\mathfrak{L}}$ is defined in the natural way: if $\Lambda = \{\lambda_X\}$, $\Lambda' = \{\lambda'_X\}$, then

$$\Lambda'\Lambda \underset{\mathrm{Df}}{=} \{\lambda'_X \lambda_X\}.$$

Let us notice that

1.1. *For any Π, Π': $\mathcal{K} \to \mathfrak{L}$, $\Lambda = \{\lambda_x\}$ is an isomorphism in $\mathcal{M}^{\mathcal{K}\mathfrak{L}}$ if and only if all λ_x are isomorphisms in $\mathfrak{L}$.* ■

1.2. *For any covariant functor Ψ: $\mathcal{K}_1 \to \mathcal{K}_2$ and every pair of functors Π, Π': $\mathcal{K}_2 \to \mathfrak{L}$, if Π, Π' are isomorphic in $\mathcal{M}^{\mathcal{K}_2\mathfrak{L}}$, then $\Pi\Psi$, $\Pi'\Psi$ are isomorphic in $\mathcal{M}^{\mathcal{K}_1\mathfrak{L}}$.* ■

In particular, if Ψ is an inclusion, we get

1.3. *Let $\mathcal{K}_1$ be a subcategory of $\mathcal{K}_2$. If Π, Π': $\mathcal{K}_2 \to \mathfrak{L}$ are isomorphic in $M^{\mathcal{K}_2\mathfrak{L}}$, then the partial functors $\Pi|\mathcal{K}_1$, $\Pi'|\mathcal{K}_1$ are isomorphic in $\mathcal{M}^{\mathcal{K}_1\mathfrak{L}}$.* ■

Now let us take two functors Π: $\mathcal{K} \to \mathfrak{L}$, Π': $\mathcal{K}' \to \mathfrak{L}$. The functors Π, Π' are said to be *quasi-isomorphic* whenever there exists an isomorphism Φ: $\mathcal{K} \to \mathcal{K}'$ such that the diagram

$$\begin{tikzcd} \mathcal{K} \arrow[r, "\Pi"] \arrow[d, "\Phi"'] & \mathfrak{L} \\ \mathcal{K}' \arrow[ur, "\Pi'"'] & \end{tikzcd} \quad \text{commutes up to isomorphism in } \mathcal{M}^{\mathcal{K}\mathfrak{L}},$$

i.e. the functors Π and $\Pi'\Phi$ are isomorphic in $\mathcal{M}^{\mathcal{K}\mathfrak{L}}$ (2).

(2) The quasi-isomorphism of functors could be defined in another way as an isomorphism in the category $\mathcal{M}^{\mathfrak{L}}$ with all covariant functors with values in $\mathfrak{L}$ as objects and morphisms understood as follows: for Π: $\mathcal{K} \to \mathfrak{L}$, Π': $\mathcal{K}' \to \mathfrak{L}$

$$\Lambda \in \mathrm{Mor}_{\mathcal{M}^{\mathfrak{L}}}(\Pi, \Pi') \quad \text{whenever } \Lambda = (\Phi, \{\lambda_x\}),$$

where Φ: $\mathcal{K} \to \mathcal{K}'$ is a covariant functor, $\lambda_X \in \mathrm{Mor}_{\mathfrak{L}}(\Pi(X), \Pi'\Phi(X))$ and for any $f \in \mathrm{Mor}_{\mathcal{K}}(X, Y)$, the diagram

$$\begin{tikzcd} \Pi(X) \arrow[r, "\Pi(f)"] \arrow[d, "\lambda_x"'] & \Pi(Y) \arrow[d, "\lambda_Y"] \\ \Pi\Phi(X) \arrow[r, "\Pi'\Phi(f)"'] & \Pi'\Phi(Y) \end{tikzcd} \quad \text{is commutative.}$$

2. Fundamental category and shape category. We are concerned with two approaches to the theory of shapes, the first one due to Borsuk ([1]), the second one due to Mardešić and Segal ([3], [5]). In both approaches the shapes are understood as isomorphic types of objects in some category — in the fundamental category $\mathcal{F}$ in the first case (see [1]) and in the shape category $\mathcal{S}$ in the second case (see [5]).

Let us briefly recall the definitions.

1) Pointed fundamental category $\mathcal{F}$. The objects of $\mathcal{F}$ are pointed compact subsets of the Hilbert cube Q, i.e. the pairs of the form (X, x_0), where $X = \overline{X} \subset Q$, $x_0 \in X$.

The sequence of maps f^k: $(Q, x_0) \to (Q, y_0)$ is said to be a *fundamental sequence* from (X, x_0) to (Y, y_0) (in symbols $\underline{f} = \{f^k, (X, x_0), (Y, y_0)\}$) whenever for every neighbourhood V of Y in Q there is a neighbourhood U of X in Q and a natural number k_0 such that $f^k|U \simeq f^{k+1}|U$ in (V, y_0) for $k \geqslant k_0$.

Two fundamental sequences

$$\underline{f} = \{f^k, (X, x_0), (Y, y_0)\}, \quad \underline{f}' = \{f'^k, (X, x_0), (Y, y_0)\}$$

are said to be *homotopic* (in symbols $\underline{f} \simeq \underline{f}'$) whenever for every neighbourhood V of Y there is a neighbourhood U of X and a k_0 such that $f^k|U \simeq f'^k|U$ in (V, y_0) for $k \geqslant k_0$.

Morphisms in $\mathcal{F}$ are defined as fundamental classes, i.e. the equivalence classes of fundamental sequences with respect to the homotopy relation.

For any $(X, x_0) \in \mathrm{Ob}_{\mathcal{F}}$, $\underline{1}_{(X,x_0)}$ is defined as the fundamental class of the sequence $\{1_Q, (X, x_0), (X, x_0)\}$.

The composition of fundamental sequences $\underline{f} = \{f^k, (X, x_0), (Y, y_0)\}$ and $\underline{g} = \{g^k, (Y, y_0), (Z, z_0)\}$:

$$\underline{g}\underline{f} \underset{\mathrm{Df}}{=} \{g^k f^k, (X, x_0), (Z, z_0)\}.$$

2) Pointed shape category $\mathcal{S}$. The objects of $\mathcal{S}$ are pointed compact Hausdorff spaces.

Morphisms are defined as follows.

Take two objects (X, x_0), (Y, y_0) and consider the two classes of all $\mathrm{ANR}(\mathfrak{R})$-systems (i.e. inverse systems of absolute neighbourhood retracts for compact Hausdorff spaces) associated with (X, x_0) and (Y, y_0), respectively. For any pair $(\boldsymbol{X}, \boldsymbol{x}_0)$, $(\boldsymbol{X}', \boldsymbol{x}'_0)$ associated with (X, x_0) $\big((\boldsymbol{Y}, \boldsymbol{y}_0)$, $(\boldsymbol{Y}', \boldsymbol{y}'_0)$ associated with $(Y, y_0)\big)$ there exists a map $\boldsymbol{i}$: $(\boldsymbol{X}, \boldsymbol{x}_0) \to (\boldsymbol{X}', \boldsymbol{x}'_0)$ $\big(\boldsymbol{j}$: $(\boldsymbol{Y}, \boldsymbol{y}_0) \to (\boldsymbol{Y}', \boldsymbol{y}'_0)\big)$ associated with $1_{(X,x_0)}$ (with $1_{(Y,y_0)}$) (see [3], Th. 10). By Th. 11 of [3], any two such maps $\boldsymbol{i}, \boldsymbol{i}'$ associated with $1_{(Y,y_0)}$ are homotopic. Thus the following equivalence relation between arbitrary two maps $\boldsymbol{f}$: $(\boldsymbol{X}, \boldsymbol{x}_0) \to (\boldsymbol{Y}, \boldsymbol{y}_0)$, $\boldsymbol{f}'$: $(\boldsymbol{X}', \boldsymbol{x}'_0) \to (\boldsymbol{Y}', \boldsymbol{y}'_0)$ can be defined:

$f \simeq f'$ whenever the diagram

$$\begin{tikzcd} (X, x_0) \arrow[r, "f"] \arrow[d, "i"'] & (Y, y_0) \arrow[d, "j"] \\ (X', x_0') \arrow[r, "f'"'] & (Y', y_0') \end{tikzcd}$$

commutes up to homotopy in the sense of [3].

A morphism from (X, x_0) to (Y, y_0) in $\mathcal{S}$ is defined as the equivalence class of a map $f\colon (X, x_0) \to (Y, y_0)$ (for (X, x_0), (Y, y_0) associated with (X, x_0), (Y, y_0)) with respect to $\simeq$; it is called a *shape map* (in symbols $[\![f]\!]$).

For any pair (X, x_0), the identity in $\mathcal{S}$ is defined as the class $[\![i]\!]$ of a map $i\colon (X, x_0) \to (X, x_0)$ which is associated with $1_{(X, x_0)}$.

The composition of shape maps is defined in the natural way.

Now, let $\mathcal{S}'$ be the subcategory of the shape category $\mathcal{S}$, with metric compact pairs as objects. Since all the statements of [4] are also valid in the case of pointed compacta, the main result of [4] can be expressed in the following form:

2.1. *The subcategory $\mathcal{S}'$ of the pointed shape category is isomorphic to the pointed fundamental category $\mathcal{F}$* ([3]).

More precisely

2.2. *The functor $\Phi\colon \mathcal{F} \to \mathcal{S}$ defined by the formulae*

$$\Phi(X, x_0) = (X, x_0) \quad \textit{for any } (X, x_0) \in \mathrm{Ob}_{\mathcal{F}},$$

$$\Phi[\underline{f}] = [\![f]\!], \ \textit{the map } f \textit{ being related to the fundamental sequence } \underline{f},$$

is an isomorphism.

3. Fundamental groups. Let us recall briefly the Borsuk definition of the fundamental groups (see [1]). It is based on the notion of approximative sequence.

Take two compact subsets S, X of the Hilbert cube Q; let $s_0 \in S$, $x_0 \in X$. The sequence of maps $f^k\colon (S, s_0) \to (Q, x_0)$ is said to be an *approximative sequence* whenever for every neighbourhood U of X in Q there is a natural number k_0 such that

$$f^{k+1} \simeq f^k \quad \text{in } (U, x_0) \quad \text{for every } k \geqslant k_0;$$

in symbols $\underline{f} = \{f^k, (S, s_0) \to (X, x_0)\}$.

([3]) As a matter of fact, the objects of the shape category $\mathcal{S}$ ($\mathcal{S}'$) are not the Hausdorff compact (metric) spaces themselves, but their topological types. Thus, the statement 2.1 should be formulated precisely as follows. The category $\mathcal{S}'$ is isomorphic to the quotient category $\mathcal{F}_{\equiv}$ with topological types of pointed compacta as objects and the corresponding sets of fundamental classes as morphisms. However, from the standpoint of topology, any two homeomorphic spaces used to be identified; therefore, we do not distinguish between a space and its topological type.

Two approximative sequences, $\underline{f} = \{f^k, (S, s_0) \to (X, x_0)\}$ and $\underline{f}' = \{f'^k, (S, s_0) \to (X, x_0)\}$ are said to be *homotopic* whenever for every neighbourhood U of X in Q there is a k_0 such that

$$f^k \simeq f'^k \quad \text{in } (U, x_0) \quad \text{for every } k \geqslant k_0.$$

Now, put S^n for S (S^n being the n-dimensional sphere). Take $s_0 \in S^n$.

The set of all homotopy classes of pointed approximative sequences $(S^n, s_0) \to (X, x_0)$ is proved to be a group with respect to the operation of joining (see [1]), the neutral element being the homotopy class of the constant sequence $\{f^k, (S^n, s_0) \to (X, x_0)\}$, $f^k(s) = x_0$ for all $s \in S^n$. This group is called the *n-th fundamental group* of (X, x_0); in symbols $\underline{\pi}_n(X, x_0)$.

Since any approximative sequence can be composed with a fundamental sequence, for any fundamental sequence $\underline{g} = \{g^k, (X, x_0), (Y, y_0)\}$ the induced homomorphism $\underline{g}_n\colon \underline{\pi}_n(X, x_0) \to \underline{\pi}_n(Y, y_0)$ can be defined by the formula

$$\underline{g}_n[\underline{f}] \underset{\mathrm{Df}}{=} [\underline{g}\underline{f}].$$

By Theorem 15.4 of [1] one obtains the covariant functor $\underline{\pi}_n\colon \mathcal{F} \to \mathcal{G}$ from the fundamental category $\mathcal{F}$ to the category of groups, $\mathcal{G}$.

4. Shape groups. Now we are concerned with the pointed shape category $\mathcal{S}$ (see § 2).

Let us consider two pointed compact Hausdorff spaces, (S, s_0) and (X, x_0). The pair (S, s_0) can be treated as a constant inverse system (for A consisting of a single element). In a similar way to that followed by S. Mardešić in [5] (here § 2), we define the equivalence relation $\simeq$ in the set of all maps of (S, s_0) into ANR$(\mathfrak{R})$-systems associated with (X, x_0). By Th. 10 of [3], for any two ANR$(\mathfrak{R})$-systems (X, x_0), (X_0', x_0') associated with (X, x_0) there exists a map $i\colon (X, x_0) \to (X_0', x_0')$ associated with $1_{(X, x_0)}$; by Th. 11 of [3], any two such maps i, i' are homotopic. For any two maps $f\colon (S, s_0) \to (X, x_0)$, $f'\colon (S, s_0) \to (X', x_0')$, let

$f \simeq f'$ whenever the diagram

$$\begin{tikzcd} (S, s_0) \arrow[r, "f"] \arrow[dr, "f'"'] & (X, x_0) \arrow[d, "i"] \\ & (X', x_0') \end{tikzcd}$$

commutes up to homotopy (in the sense of [3]) for any map i associated with $1_{(X, x_0)}$.

The equivalence class of the map f with respect to the relation $\simeq$ will be referred to as a *simple shape map*, in symbols $[\![f]\!]$.

The set of all simple shape maps from (S, s_0) to (X, x_0) will be denoted by $[X, x_0]^{(S, s_0)}$.

Take a shape map $[\![g]\!] \in \mathrm{Mor}_{\mathcal{S}}((X, x_0), (Y, y_0))$ and a simple shape map $[\![f]\!] \in [X, x_0]^{(S, s_0)}$; the composition $[\![g]\!][\![f]\!]$ is defined as the simple shape map in $[\![Y, y_0]\!]^{(S,s_0)}$ given by the formula

$$[\![g]\!][\![f]\!] \underset{\mathrm{Df}}{=} [\![g'f']\!],$$

where $f'\colon (S, s_0)\to(X, x_0)$, $g'\colon (X, x_0)\to(Y, y_0)$ are representatives of $[\![g]\!]$ and $[\![f]\!]$, respectively.

Now let us fix a natural number n and put S^n for S (S^n being the n-dimensional sphere). For any compact Hausdorff pair (X, x_0) we are going to define a group operation in $[\![X, x_0]\!]^{(S^n,s_0)}$. For this purpose take an $\mathrm{ANR}(\mathfrak{R})$-system $(X, x_0) = \{(X_a, x_{0a}), p_a^{a'}, A\}$ associated with (X, x_0) and consider two maps $f', f''\colon (S^n, s_0)\to(X, x_0)$, $f' = (\varphi, f'_a)$, $f'' = (\varphi, f''_a)$, $\varphi=\mathrm{const}$. The maps f', f'' are said to be *separated* whenever, for any $a\in A$, the maps f'_a, f''_a are separated in the sense of Borsuk (see [2], p. 44). To any pair of separated maps f', f'', the following map $f'\circ f''\colon (S^n, s_0)\to(X, x_0)$ can be assigned:

$$f'\circ f'' \underset{\mathrm{Df}}{=} (\varphi, f'_a\circ f''_a),$$

the map $f'_a\circ f''_a$ being the join of the maps f'_a, f''_a in the sense of Borsuk (see [1] or [2], p. 44).

By the definition of a simple shape map and by the arguments used in [2], it follows that

4.1. *For every pair of simple shape maps, $[\![f']\!], [\![f'']\!]\in[\![X, x_0]\!]^{(S^n,s_0)}$, there exists a pair of separated maps, $\hat f', \hat f''\colon (S^n, s_0)\to(X, x_0)$, such that $\hat f'\in[\![f']\!]$ and $\hat f''\in[\![f'']\!]$.* ■

4.2. *The simple shape map $[\![f'\circ f'']\!]$ does not depend on the choice of separated representatives of $[\![f']\!]$ and $[\![f'']\!]$.* ■

The statements 4.1 and 4.2 make us possible to define the join of simple shape maps by the formula

$$[\![f']\!]\circ[\![f'']\!] \underset{\mathrm{Df}}{=} [\![\hat f'\circ\hat f'']\!],$$

where $\hat f', \hat f''\colon (S^n, s_0)\to(X, x_0)$ are two separated representatives of $[\![f']\!]$ and $[\![f'']\!]$, respectively.

As the neutral element $[\![0]\!]$ of the set $[\![X, x_0]\!]^{(S^n,s_0)}$ we take the simple shape map represented by the constant map.

One can easily prove that

4.3. *The system* $([\![X, x_0]\!], \circ, [\![0]\!])$ *is a group.* ■

This group will be referred to as the *n-th shape group* of the pair (X, x_0); in symbols $\tilde\pi_n(X, x_0)$.

Any shape map $[\![g]\!]\in\mathrm{Mor}_{\mathcal{S}}((X, x_0), (Y, y_0))$ induces a homomorphism $\tilde g_n\colon \tilde\pi_n(X, x_0)\to\tilde\pi_n(Y, y_0)$ defined by the formula

$$\tilde g_n[\![f]\!] \underset{\mathrm{Df}}{=} [\![g]\!][\![f]\!].$$

Setting

$$\tilde\pi_n[\![g]\!] \underset{\mathrm{Df}}{=} \tilde g_n,$$

we get a covariant functor $\tilde\pi_n\colon \mathcal{S}\to\mathfrak{G}$.

5. Limit homotopy groups. Let $\mathfrak{R}$ be the category of pointed $\mathrm{ANR}(\mathfrak{R})$ spaces. First, we are concerned with the category $\hat{\mathfrak{R}}^*_{\simeq}$ (see the Appendix or [7]). Let us consider the functor $\pi_n\colon \hat{\mathfrak{R}}^*_{\simeq}\to\hat{\mathfrak{G}}^*$, which has been defined in [7] as follows: For any pointed $\mathrm{ANR}(\mathfrak{R})$-system $(X, x_0) = \{(X_a, x_{0a}), p_a^{a'}, A\}$, put

$$\pi_n(X, x_0) \underset{\mathrm{Df}}{=} \{\pi_n(X_a, x_{0a}), (p_a^{a'})_n, A\}.$$

This inverse system of homotopy groups is referred to as *n-th homotopy system* of (X, x_0). Any map of inverse systems, $g = (\psi, g_\beta)\colon (X, x_0)\to(Y, y_0)$, induces morphism of nth homotopy systems,

$$g_n = (\psi, (g_\beta)_n)\colon \pi_n(X, x_0)\to\pi_n(Y, y_0).$$

If g and g' are homotopic, $g\simeq g'$, then $g_n\cong g'_n$ (see the Appendix); so the homotopy class $[g]$ induces the morphism $[g_n]$ in $\hat{\mathfrak{G}}^*$. Setting

$$\pi_n([g]) \underset{\mathrm{Df}}{=} [g_n],$$

we get the functor $\pi_n\colon \hat{\mathfrak{R}}^*_{\simeq}\to\hat{\mathfrak{G}}^*$.

Since this functor is covariant, it carries the set of isomorphisms in $\hat{\mathfrak{R}}^*_{\simeq}$ into the set of isomorphisms in $\hat{\mathfrak{G}}^*$; thus

5.1. *If $g\colon (X, x_0)\to(Y, y_0)$ is a homotopy equivalence, then the induced morphism $g_n\colon \pi_n(X, x_0)\to\pi_n(Y, y_0)$ is an isomorphism* in $\hat{\mathfrak{G}}^*$. ■

Now, let us consider the pointed shape category $\mathcal{S}$. As was shown in [7], for any compact Hausdorff pair (X, x_0) the inverse limit of $\pi_n(X, x_0)$ is independent of the choice of the $\mathrm{ANR}(\mathfrak{R})$-system (X, x_0) associated with (X, x_0). Therefore, the limit homotopy group of (X, x_0) has been defined by the formula

$$\pi_n^*(X, x_0) = \varprojlim \pi_n(X, x_0), \tag{1}$$

(X, x_0) being an $\mathrm{ANR}(\mathfrak{R})$-system associated with (X, x_0). Take any shape map $[\![g]\!]\in\mathrm{Mor}_{\mathcal{S}}((X, x_0), (Y, y_0))$, and let $g\colon (X, x_0)\to(Y, y_0)$, $g'\colon (X', x'_0)\to(Y', y'_0)$ be two representatives of $[\![g]\!]$. Then, there exist maps i, j associated with the identities $1_{(X,x_0)}, 1_{(Y,y_0)}$ and such that the diagram

$$\begin{array}{ccc} (X, x_0) & \xrightarrow{g} & (Y, y_0) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ (X', x'_0) & \xrightarrow[g']{} & (Y', y'_0) \end{array}$$

commutes up to $\simeq$.

By 5.1, since i, j are homotopy equivalences, they induce isomorphisms of the homotopy systems. Then, by 2.1 of [7], the inverse limits $\varprojlim \boldsymbol{g}_n$, $\varprojlim \boldsymbol{g}'_n$ coincide up to isomorphism. It enables us to define the induced homomorphism $\pi_n^*[\![\boldsymbol{g}]\!]$ by the formula

$$\pi_n^*[\![\boldsymbol{g}]\!] \underset{\mathrm{Df}}{=} \varprojlim \boldsymbol{g}_n . \tag{2}$$

The functor of nth limit homotopy group, $\pi_n^*\colon \mathcal{S} \to \mathcal{G}$, is defined by the formulae (1) and (2).

6. A connection between the fundamental groups, shape groups and limit homotopy groups. Given a natural number n, let us consider the pair of functors $\tilde{\pi}_n, \pi_n^*$ from the shape category $\mathcal{S}$ to the category of groups, $\mathcal{G}$. We are going to prove

6.1. THEOREM. *The functors $\tilde{\pi}_n$ and π_n^* are isomorphic.*

Proof. According to 1.1, we have to find a family of isomorphisms in $\mathcal{G}$, $\Lambda = \{\lambda_{(X,x_0)}\}_{(X,x_0)\in \mathrm{Ob}_{\mathcal{S}}}$ such that for every $[\![g]\!] \in \mathrm{Mor}_{\mathcal{S}}((X,x_0),(Y,y_0))$ the Λ-diagram

$$\begin{tikzcd}
\tilde{\pi}_n(X,x_0) \arrow[r, "\tilde{\pi}_n[\![g]\!]"] \arrow[d, "\lambda_{(X,x_0)}"'] & \tilde{\pi}_n(Y,y_0) \arrow[d, "\lambda_{(Y,y_0)}"] \\
\pi_n^*(X,x_0) \arrow[r, "\pi_n^*[\![g]\!]"'] & \pi_n^*(Y,y_0)
\end{tikzcd} \quad \text{is commutative.}$$

Take a compact Hausdorff pair (X, x_0) and let $(\boldsymbol{X}, \boldsymbol{x}_0) = \{(X_\alpha, x_{0\alpha}), p_\alpha^{\alpha'}, A\}$ be an ANR$(\mathfrak{R})$-system associated with (X, x_0). Any element $[\![f]\!]$ of the shape group $\tilde{\pi}_n(X, x_0)$ has a representative $\hat{\boldsymbol{f}} = (\varphi, \hat{f}_a)\colon (S^n, s_0) \to (\boldsymbol{X}, \boldsymbol{x}_0)$, which is determined uniquely up to homotopy. The maps $\hat{f}_a$, $\alpha \in A$, satisfy the following condition:

$$\bigwedge_{\alpha'\geqslant\alpha} p_\alpha^{\alpha'} \hat{f}_{\alpha'} \simeq \hat{f}_\alpha \quad \text{in} \quad (X_\alpha, x_{0\alpha}) .$$

Thus

$$\bigwedge_{\alpha'\geqslant\alpha} (p_\alpha^{\alpha'})_n [\hat{f}_{\alpha'}] = [\hat{f}_\alpha] ,$$

i.e. the system $\{[\hat{f}_a]\}_{\alpha\in A}$ belongs to $\varinjlim \pi_n(\boldsymbol{X}, \boldsymbol{x}_0)$.

Setting

$$\lambda_{(X,x_0)}[\![f]\!] \underset{\mathrm{Df}}{=} \{[\hat{f}_a]\}_{\alpha\in A} ,$$

we obtain a homomorphism $\lambda_{(X,x_0)}\colon \tilde{\pi}_n(X,x_0) \to \varprojlim \pi_n(\boldsymbol{X},\boldsymbol{x}_0)$. Obviously, the homomorphism $\lambda^{-1}_{(X,x_0)}$ defined by the formula

$$\lambda^{-1}_{(X,x_0)}(\{[\hat{f}_a]\}) \underset{\mathrm{Df}}{=} [\![\hat{f}]\!]$$

is an inverse of $\lambda_{(X,x_0)}$; therefore $\lambda_{(X,x_0)}$ is an isomorphism. Now let us take any shape map $[\![g]\!] \in \mathrm{Mor}_{\mathcal{S}}((X,x_0),(Y,y_0))$ and verify the commutativity of the Λ-diagram. Take ANR$(\mathfrak{R})$-systems $(\boldsymbol{X}, \boldsymbol{x}_0)$, $(\boldsymbol{Y}, \boldsymbol{y}_0)$ (over A, B) associated with (X, x_0), (Y, y_0) respectively, and let $\boldsymbol{g} = (\psi, g_\beta)\colon (\boldsymbol{X}, \boldsymbol{x}_0) \to (\boldsymbol{Y}, \boldsymbol{y}_0)$ be a representative of $[\![g]\!]$. Then the Λ-diagram can be written in the form

$$\begin{tikzcd}
\tilde{\pi}_n(X,x_0) \arrow[r, "\tilde{g}_n"] \arrow[d, "\lambda_{(X,x_0)}"'] & \tilde{\pi}_n(Y,y_0) \arrow[d, "\lambda_{(Y,y_0)}"] \\
\varprojlim \pi_n(\boldsymbol{X},\boldsymbol{x}_0) \arrow[r, "\varprojlim \boldsymbol{g}_n"'] & \varprojlim \pi_n(\boldsymbol{Y},\boldsymbol{y}_0) .
\end{tikzcd}$$

Take a simple shape map $[\![f]\!] \in \tilde{\pi}_n(X, x_0)$ and its representative $\hat{\boldsymbol{f}} = (\varphi, f_a)\colon (S^n, s_0) \to (\boldsymbol{X}, \boldsymbol{x}_0)$. We have

$$\lambda_{(Y,y_0)}\tilde{g}_n[\![f]\!] = \lambda_{(Y,y_0)}[\![g]\!][\![f]\!] = \lambda_{(Y,y_0)}[\![g\hat{f}]\!] = \{[g_\beta \hat{f}_{\psi(\beta)}]\}_{\beta\in B} ;$$

on the other hand,

$$\varprojlim \boldsymbol{g}_n \lambda_{(X,x_0)}[\![f]\!] = \varprojlim \boldsymbol{g}_n \{[\hat{f}_a]\}_{\alpha\in A} = \{[g_\beta \hat{f}_{\psi(\beta)}]\}_{\beta\in B} .$$

Thus the Λ-diagram is commutative, which completes the proof. ∎

Let $\mathcal{S}'$ be the subcategory of $\mathcal{S}$ with pointed compact metric spaces as objects. By 1.3 and 6.1 we get the following

6.2. COROLLARY. *The functors $\tilde{\pi}_n | \mathcal{S}'$ and $\pi_n^* | \mathcal{S}'$ are isomorphic.* ∎

Now we are going to establish a connection between the two functors, $\underline{\pi}_n\colon \mathcal{F} \to \mathcal{G}$ and $\tilde{\pi}_n | \mathcal{S}'\colon \mathcal{S}' \to \mathcal{G}$ (Theorem 6.4). For this purpose, given an approximative sequence, let us define the related map into ANR-sequence.

Consider two arbitrary pointed compacta (S, s_0) and (X, x_0), $X \subset Q$, and take an approximative sequence $\underline{f} = \{f^k, (S, s_0) \to (X, x_0)\}$. Let $(\boldsymbol{X}, \boldsymbol{x}_0) = \{(X_k, x_0), p_k^{k'}, N\}$ be an inclusion ANR-sequence associated with (X, x_0). Then there is an increasing function $\varphi\colon N \to N$ such that $\varphi(k) \geqslant k$ for every k and

$$k' \geqslant k \Rightarrow f^{\varphi(k)} \simeq f^{\varphi(k')} \text{ in } (X_k, x_0) ;$$

hence the maps $f_k\colon (S, s_0) \to (X_k, x_0)$ given by the formula

$$f_k(s) \underset{\mathrm{Df}}{=} f^{\varphi(k)}(s) \quad \text{for } s \in S ,$$

form a map of ANR-systems, $\boldsymbol{f} = (\mathrm{const}, f_k)\colon (S, s_0) \to (\boldsymbol{X}, \boldsymbol{x}_0)$. This map $\boldsymbol{f}$ is said to be *related* to $\underline{f}$.

Applying Lemmas 5 and 6 of [4], one can easily prove

6.3. *There is a biunique correspondence between the set of homotopy classes of all approximative sequences from (S, s_0) to (X, x_0) and the set of simple shape maps, $[\![X, x_0]\!]^{(S,s_0)}$. For any fundamental sequence $\underline{g}$ from*

(X, x_0) *to* (Y, y_0), *the homotopy class* $[\underline{gf}]$ *corresponds to the simple shape map* $[\![\boldsymbol{g}]\!][\![\boldsymbol{f}]\!]$, *where* $\boldsymbol{g}$ *and* $\boldsymbol{f}$ *are related to* $\underline{g}$ *and* $\underline{f}$ *respectively.*

Now let us prove

6.4. THEOREM. *The functors* $\underline{\pi}_n$ *and* $\tilde{\pi}_n|\mathcal{S}'$ *are quasi-isomorphic.*

Proof. Let $\Phi\colon \mathcal{F}\to\mathcal{S}'$ be the isomorphism of the pointed fundamental category $\mathcal{F}$ onto $\mathcal{S}'$ (see 2.2). We have to show that the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[d, "\Phi"'] \arrow[dr, "\underline{\pi}_n"] & \\ \mathcal{S}' \arrow[r, "\tilde{\pi}_n"'] & \mathcal{G} \end{tikzcd}$$

commutes up to isomorphism,

i.e. that the functors $\underline{\pi}_n$ and $\tilde{\pi}_n\Phi$ are isomorphic.

Take any compact metric pair (X, x_0) and define a homomorphism

$$\lambda_{(X,x_0)}\colon \underline{\pi}_n(X, x_0)\to\tilde{\pi}_n(X, x_0)$$

by the formula

$$\lambda_{(X,x_0)}[\underline{f}] \underset{\mathrm{Df}}{=} [\![\boldsymbol{f}]\!],$$

where $\underline{f}$ is a pointed approximative sequence and $\boldsymbol{f}$ — a related map of ANR-sequences. It is easy to see that $\lambda_{(X,x_0)}$ is a homomorphism. By 6.3 it is an isomorphism.

It remains to verify the commutativity of the Δ-diagram:

$$\begin{tikzcd} \underline{\pi}_n(X, x_0) \arrow[r, "\underline{\pi}_n{[\underline{g}]}"] \arrow[d, "\lambda_{(X,x_0)}"'] & \underline{\pi}_n(Y, y_0) \arrow[d, "\lambda_{(Y,y_0)}"] \\ \tilde{\pi}_n(X, x_0) \arrow[r, "\tilde{\pi}_n\Phi{[\underline{g}]}"'] & \tilde{\pi}_n(Y, y_0) \end{tikzcd}$$

for any pointed fundamental sequence $\underline{g} = \{g^k, (X, x_0), (Y, y_0)\}$. This diagram can be written as follows:

$$\begin{tikzcd} \underline{\pi}_n(X, x_0) \arrow[r, "\underline{g}_n"] \arrow[d, "\lambda_{(X,x_0)}"'] & \underline{\pi}_n(Y, y_0) \arrow[d, "\lambda_{(Y,y_0)}"] \\ \tilde{\pi}_n(X, x_0) \arrow[r, "\tilde{g}_n"'] & \tilde{\pi}_n(Y, y_0) \end{tikzcd}$$

Take $[\underline{f}]\in\underline{\pi}_n(X, x_0)$; by 6.3, we have

$$\lambda_{(Y,y_0)}\underline{g}_n([\underline{f}]) = \lambda_{(Y,y_0)}[\underline{gf}] = [\![\boldsymbol{g}]\!][\![\boldsymbol{f}]\!];$$

on the other hand,

$$\tilde{\boldsymbol{g}}_n\lambda_{(X,x_0)}[\underline{f}] = \tilde{\boldsymbol{g}}_n[\![\boldsymbol{f}]\!] = [\![\boldsymbol{g}]\!][\![\boldsymbol{f}]\!].$$

Thus the Δ-diagram is commutative, which completes the proof. ■

Since $\Phi|\mathrm{Ob}_F$ is an identity, the statements 6.2, 6.4 imply

6.5. COROLLARY. *For any compact metric pair* (X, x_0) *the groups* $\underline{\pi}_n(X, x_0)$, $\tilde{\pi}_n(X, x_0)$ *and* $\pi_n^*(X, x_0)$ *are isomorphic.* ■

We get also the following

6.6. COROLLARY. *Let* $\underline{g}$ *be a pointed fundamental sequence, and let* $\boldsymbol{g}$ *be a related map of* ANR$(\mathfrak{R})$*-sequences. Then*

$\underline{g}_n$ *is an isomorphism* $\Leftrightarrow$ $\tilde{\boldsymbol{g}}_n$ *is an isomorphism* $\Leftrightarrow$ $\varprojlim \boldsymbol{g}_n$ *is an isomorphism,*

$\underline{g}_n$ *is a monomorphism* $\Leftrightarrow$ $\tilde{\boldsymbol{g}}_n$ *is a monomorphism* $\Leftrightarrow$ $\varprojlim \boldsymbol{g}_n$ *is a monomorphism,*

$\underline{g}_n$ *is an epimorphism* $\Leftrightarrow$ $\tilde{\boldsymbol{g}}_n$ *is an epimorphism* $\Leftrightarrow$ $\varprojlim \boldsymbol{g}_n$ *is an epimorphism.* ■

Appendix. Let us consider a pair $(\mathcal{K}, \sim)$ which consists of a category $\mathcal{K}$ and an equivalence relation $\sim$ in $\mathrm{Mor}_{\mathcal{K}}$ satisfying the condition:

$$f\sim f' \wedge g\sim g' \Rightarrow gf\sim g'f'.$$

In [7], to such a pair $(\mathcal{K}, \sim)$ the following two categories $\mathcal{K}^*_\sim$, $\hat{\mathcal{K}}^*_\sim$ have been assigned. Both $\mathcal{K}^*_\sim$, $\hat{\mathcal{K}}^*_\sim$ have inverse systems in $\mathcal{K}$ (over closure-finite directed sets) as objects. Morphisms of $\mathcal{K}^*_\sim$ are maps of inverse systems, i.e. for $\boldsymbol{X} = \{X_\alpha, p_\alpha^{\alpha'}, A\}$, $\boldsymbol{Y} = \{Y_\beta, q_\beta^{\beta'}, B\}$, $\boldsymbol{f}\in\mathrm{Mor}(\boldsymbol{X}, \boldsymbol{Y})$ in $\mathcal{K}^*_\sim$ whenever $\boldsymbol{f} = (\varphi, f_\beta)$, where $\varphi\colon B\to A$ is an increasing function, $f_\beta\in\mathrm{Mor}_{\mathcal{K}}(X_{\varphi(\beta)}, Y_\beta)$ and all the diagrams

$$\begin{tikzcd} X_{\varphi(\beta)} \arrow[r, "p_{\varphi(\beta)}^{\varphi(\beta')}"] \arrow[d, "f_\beta"'] & X_{\varphi(\beta')} \arrow[d, "f_{\beta'}"] \\ Y_\beta \arrow[r, "q_\beta^{\beta'}"'] & Y_{\beta'} \end{tikzcd}$$

commute up to $\sim$ for $\beta'\geqslant\beta$.

Morphisms in $\hat{\mathcal{K}}^*_\sim$ are equivalence classes of morphisms in $\mathcal{K}^*_\sim$ with respect to the following relation $\approx$: let $\boldsymbol{f}, \boldsymbol{f}'\in\mathrm{Mor}_{\mathcal{K}^*_\sim}(\boldsymbol{X}, \boldsymbol{Y})$, $\boldsymbol{f} = (\varphi, f_\beta)$, $\boldsymbol{f}' = (\varphi', f'_\beta)$, then

$$\boldsymbol{f}\approx\boldsymbol{f}' \underset{\mathrm{Df}}{\Leftrightarrow} \bigwedge_{\beta}\ \bigvee_{\alpha\geqslant\varphi(\beta),\varphi'(\beta)} f_\beta p^\alpha_{\varphi(\beta)} \sim f'_\beta p^\alpha_{\varphi'(\beta)}.$$

In particular, if $\sim$ is an identity relation, then we write $\mathcal{K}^*(\hat{\mathcal{K}}^*)$ instead of $\mathcal{K}^*_\sim(\hat{\mathcal{K}}^*_\sim)$.

Here we are interested in the following two cases:

1) $\mathcal{K} = \mathfrak{R}$, the category of pointed ANR$(\mathfrak{R})$-spaces,

2) $\mathcal{K} = \mathcal{G}$, the category of groups.

In the first case we get the categories $\mathfrak{R}^*_{\simeq}$ and $\hat{\mathfrak{R}}^*_{\simeq}$ introduced by S. Mardešić and J. Segal in [3]. In the second case we get $\mathfrak{G}^*$ and $\hat{\mathfrak{G}}^*$ used here in § 5.

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On free pseudo-complemented and relatively pseudo-complemented semi-lattices (*)

by

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Abstract. The first part of this paper is concerned with properties of the free algebras in the class of pseudo-complemented semi-lattices. In particular, an explicit construction is given for the free pseudo-complemented semi-lattice $P(n)$ with n free generators ($n < \infty$). As a result of this construction, the word problem is solved for the free algebras in this class and it is shown that the number of elements in $P(n)$ is $1+\sum_{k=0}^{n}\binom{n}{k}(2^{2^{n-k}}-1)$. In the last section we develop some properties of free relatively pseudo-complemented semi-lattices with n free generators ($n < \infty$). It is shown that these algebras are all (distributive) lattices and that for $n = 2$ the free algebra is isomorphic with $2\times 3\times 3$.

1. Preliminaries. The notation $\Pi_L S$ (or simply ΠS) will be used to denote the greatest lower bound of a non-empty subset S of a meet semi-lattice L; the greatest element of L, if it exists, is denoted by $1_L(1)$. If $S = \{x, y\}$ then $\Pi S = xy$; it is convenient to define $\Pi\phi = 1$ when L has a greatest element. The symbols ΣS, 0, $x+y$, and $\Sigma\phi$ are defined dually. We will identify each integer $n \geqslant 0$ with the set $\{0, \ldots, n-1\}$. In Sections 2 and 3, the topic is pseudo-complemented semi-lattices and so the terms "homomorphism", "subalgebra", etc. should be regarded in this context. However, the meaning of these terms is suitably altered in Section 4, where we discuss relatively pseudo-complemented semi-lattices.

2. Pseudo-complemented semi-lattices. A pseudo-complemented semi-lattice is an algebra $\langle L; \cdot, 0, *\rangle$ in which $\langle L; \cdot, 0\rangle$ is a meet semi-lattice with 0 and such that for each $x \in L$, there exists a largest y (denoted by x^*) such that $xy = 0$. It is well known that these algebras form an equational class. Some of the elementary properties of these algebras are listed below for easy reference (cf. [4]).

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