# Properties of some generalizations of the notion of continuity of a function

by

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**Abstract.** Let $(X, T)$ be a compact topological space with the topology determined by some metric and let $T_1$ be a topology in $X$ stronger than $T$.

In this work the necessary and sufficient condition is given under which the family $F_T$ of all real functions, which are continuous with respect to the topology $T$ is equal to the family $F_{T_1}$ consisting of all real functions, which are continuous with respect to the topology $T_1$.

The main theorem is the following one:

*If $(X, T)$ is a topological space of considered type and $T_1$ is a topology stronger than $T$, then $F_T = F_{T_1}$ if and only if for every pair of families of sets $\{A_y\}$, $\{B_y\}$ indexed by real numbers $y \geqslant y_0$ such that*

1° *for every $y \geqslant y_0$ the set $A_y$ is closed while the set $B_y$ is open,*

2° *for every pair $y_1 < y_2$ $A_{y_1} \supset B_{y_1} \supset A_{y_2} \supset B_{y_2}$ holds,*

3° $A_y \neq \emptyset \neq B_y$

*we have* $\bigcap A_y = \bigcap B_y \neq \emptyset$ (such a topological space is called *$*$-compact*).

**Introduction.** A generalized notion of the limit of a function and also that of the continuity of a function have recently been considered in a number of papers.

A detailed study of these generalizations can be found in papers by G. H. Sindałowski [2], [3] and in papers by T. Świątkowski [4], [5].

In general, the study of a generalized notion of continuity of a function consists in the introduction of some topology, stronger than the natural, in the set of real numbers.

In some cases, the class of continuous functions in the sense of a stronger topology than the natural and the class of continuous functions in the usual sense appear to be identical. Some conditions concerning the topology under which a generalized continuity at every point implies the usual continuity are known. In particular, a sufficient condition has been given in paper [5] mentioned above. Preserving the sufficiency, one can make this condition weaker.

Now we shall give the weaker form of the above-mentioned condition from paper [5].

THEOREM 1. *Let $T$ be a topology such that the conditions*

$1^\circ$ $x_n \to x_0$ *in the usual sense,*

$2^\circ$ $x_n \in E_n$, *where* $E_n$ *is a* $T$*-open set, imply* $x_0 \in (\bigcup E_n)'_T$.

*Then every* $T$*-continuous function defined in the interval* $(a, b)$ *is continuous in* $(a, b)$.

The proof of this theorem is identical with the proof of Lemma 4 in paper [5].

Thus the question is what are the sufficient and the necessary conditions concerning the topology for the $T$-continuity at every point of some set to be equivalent to the usual continuity.

This paper aims at finding these conditions.

**§ 1.** In our further investigations we shall be concerned with topological spaces $(X, T)$ with a fixed set $X$ and various topologies $T$. A real function mapping $X$ onto $R$ ($R$ stands for the set of real numbers) will be called *$T$-continuous* if and only if it is continuous as the mapping $(X, T) \to R$.

Assume now that $(X, T_0)$ is a compact space and that $T \supset T_0$. It is clear that every $T_0$-continuous function is $T$-continuous. We shall seek the conditions for $T$ under which the class of the $T$-continuous functions (we denote it by $F_T$) is identical with the class of the $T_0$-continuous functions (denoted by $F_{T_0}$). Obviously we have $F_{T_0} \subset F_T$.

Since $(X, T_0)$ is a compact space, it follows from $f \in F_{T_0}$ that $f$ is a bounded function.

Thus $F_{T_0} = F_T$; then every function $f \in F_T$ is bounded. It turns out that if $(X, T_0)$ is a metric space, then the necessary condition which has been found is also sufficient. The following theorem explains it more carefully.

THEOREM 2. *Let $(X, T_0)$ be a compact metrizable space with the topology determined by some metric $\varrho$. Let $T$ be some topology stronger than $T_0$ ($T \supset T_0$). Then the necessary and sufficient condition of $F_T = F_{T_0}$ is that every $T$-continuous function be bounded.*

Proof. We shall prove the sufficiency of the given condition by demonstrating that if a $T$-continuous function which is not $T_0$-cont[illegible]ous existed, then a $T$ continuous and non-bounded function (poss[illegible] another one) would exist.

Let $x_0$ be a point of $T_0$-discontinuity of a function $f$. This mea[illegible] that in every $T_0$-neighbourhood of the point $x_0$ there are points such that

$$|f(x) - f(x_0)| \geqslant \varepsilon_0$$

holds, $\varepsilon_0$ being some fixed positive number. Put

$$f_1(x) = \frac{f(x) - f(x_0)}{\varepsilon_0} \cdot \frac{\pi}{2}.$$

Clearly $f_1$ is $T$-continuous.



Since the minimum and the maximum of two $T$-continuous functions is a $T$-continuous function, putting

$$f_2(x) = \min\left(f_1(x), \frac{\pi}{2[1+\varrho^2(x, x_0)]}\right)$$

and

$$f_3(x) = \max\left(f_2(x), -\frac{\pi}{2[1+\varrho^2(x, x_0)]}\right),$$

we easily see that $f_3$ is a $T$-continuous function. We also have $|f_3(x)| < \frac{1}{2}\pi$, and $\sup\limits_{x \in X} |f_3(x)| = \frac{1}{2}\pi$. Moreover, put

$$f_4(x) = \operatorname{tg} f_3(x).$$

The function $f_4$ is a $T$-continuous but non-bounded function, which ends the proof of sufficiency, and by the remarks preceding the theorem, it proves also the theorem.

Because of Theorem 2 in the sequel, we aim at finding out what the structure of the topological space should be in order that the continuity of a function should imply its boundedness. Since in compact spaces continuous functions are bounded, we shall try to weaken the assumption of compactness.

DEFINITION 1. A space $X$ is called *$\sigma$-absolutely closed* if for its arbitrary covering consisting of enumerably many open sets $\{G_n\}$ there exists a finite sequence $\{G_1, G_2, \ldots, G_{n_0}\}$ such that

$$\bigcup_{k=1}^{n_0} \overline{G_k} = X.$$

THEOREM 3. *If $f$ is a continuous function defined in a $\sigma$-absolutely closed space $X$, then $f$ is a bounded function.*

Proof. Let

$$G_n = \{x\colon -n < f(x) < n\}.$$

Hence $\bigcup\limits_{n=1}^{\infty} G_n = X$ and the sets $G_n$ are open. Thus there exists an $n_0$ such that

$$\bigcup_{n=1}^{n_0} \overline{G_n} = X.$$

Then

$$X = \overline{G}_{n_0} \subset \{x\colon |f(x)| \leqslant n_0\} \subset X,$$

i.e.,

$$X = \{x\colon |f(x)| \leqslant n_0\},$$

which means that the function $f$ is bounded.

Remark. If in Definition 1 we replace the enumerable covering by an arbitrary one, we obtain the absolutely closed space considered by Aleksandroff and Uryson in [1]. Every absolutely closed space is clearly $\sigma$-absolutely closed.

Now we shall give a necessary and sufficient condition that every continuous function defined in a topological space be bounded.

DEFINITION 2. A pair of families of sets $\{A_y\}$, $\{B_y\}$ with $y \geqslant y_0$ is said to satisfy condition $(*)$ if

1° for every $y \geqslant y_0$ the set $A_y$ is closed while the set $B_y$ is open,

2° for every pair $y_1 < y_2$ $A_{y_1} \supset B_{y_1} \supset A_{y_2} \supset B_{y_2}$ holds,

3° $A_y \neq \emptyset \neq B_y$.

DEFINITION 3. A topological space will be called *$*$-compact* if for every pair of families satisfying condition $(*)$

$$\bigcap A_y = \bigcap B_y \neq \emptyset$$

holds.

THEOREM 4. *The sufficient and necessary condition that every real and continuous function defined in some topological space be bounded is the $*$-compactness of this space.*

Proof of sufficiency. Suppose there exists in the space $X$ a continuous and unbounded function. Then the pair of families of sets

$$\begin{aligned} A_y &= \{x\colon\ |f(x)| \geqslant y\}, \\ B_y &= \{x\colon\ |f(x)| > y\}, \end{aligned} \qquad y \geqslant y_0$$

satisfies condition $(*)$, but because of the finiteness of the function $f$

$$\bigcap A_y = \bigcap B_y = \emptyset$$

holds, whence $X$ cannot be $*$-compact.

Proof of necessity. We shall prove that if the space $X$ is not $*$-compact, then a finite and continuous function $f$ defined in $X$ can be found which is not bounded. Thus suppose $X$ is not a $*$-compact space. Then there exists a pair of families of sets $\{A_y\}$, $\{B_y\}$ which satisfies condition $(*)$ and is such that

$$\bigcap A_y = \bigcap B_y = \emptyset .$$

Without diminishing the generality of our argument we can assume that $B_y = A_y = X$ for $y \leqslant y_0$. Put

$$f(x) = \sup\{y\colon\ x \in A_y\} .$$

The function $f$ has been defined correctly since, in view of $A_{y_0} = X$,

$$\bigwedge_{x \in X} \{y\colon\ x \in A_y\} \neq \emptyset .$$



Moreover, the function $f$ is finite because, by $\bigcap A_y = \emptyset$,

$$\bigwedge_{x \in X} \bigvee_{x \in R} (x \notin A_y) .$$

Next we observe that $f$ is an unbounded function because by assumption all the sets $A_y$ are non-empty. Now we shall prove that $f$ is a continuous function. To this end we investigate sets of the form

$$\{x\colon\ f(x) \geqslant y\}, \tag{1}$$

$$\{x\colon\ f(x) > y\}. \tag{2}$$

For $y \leqslant y_0$ the sets (1) and (2) coincide with the whole space $X$ and thus are close — open. Let $y > y_0$. First we have

$$\{x\colon\ f(x) > y\} = \bigcup_{y' > y} A_{y'} .$$

By Condition 2° we have $\bigcup_{y'>y} A_{y'} = \bigcup_{y'>y} B_{y'}$ and thus by 1° the set

$$\{x\colon\ f(x) > y\} = \bigcup_{y'>y} B_{y'} \quad \text{is open} .$$

Hence it follows that sets of form (2) are open. Next we have

$$\{x\colon\ f(x) \geqslant y\} = \bigcap_{y'<y} A_{y'} .$$

Hence sets of form (1) are closed.

From the fact that the sets (1) are closed and sets (2) are open it follows that the function $f$ is continuous, which ends the proof.

Remark. If $X$ is a connected space and if is a continuous and unbounded function, e.g. from above, then for some number $y_0$ the inequality $y_0 \leqslant y_1 < y_2$ implies not only

$$\{x\colon\ f(x) \geqslant y_1\} \supset \{x\colon\ f(x) > y_1\} \supset \{x\colon\ f(x) \geqslant y_2\}$$

but also

$$\{x\colon\ f(x) \geqslant y_1\} \neq \{x\colon\ f(x) > y_1\} \neq \{x\colon\ f(x) \geqslant y_2\} .$$

Thus under the assumption of connectedness of the space $X$ in Theorem 4 one may, without essential changes in the proof, replace the $*$-compactness by a weaker condition.

A pair of families of sets $\{A_y\}$, $\{B_y\}$ is said to satisfy condition $(**)$ if it satisfies condition $(*)$ and if $A_{y_1} \neq B_{y_1} \neq A_{y_2} \neq B_{y_2}$.

DEFINITION 4. A topological space will be called *$**$-compact* if, for every pair of families which satisfies the condition $(**)$ we have

$$\bigcap A_y = \bigcap B_y \neq \emptyset .$$

THEOREM 5. *The sufficient and necessary condition that every real and continuous function defined in a topological connected space $X$ be bounded is the $**$-compactness of the space.*

We omit the proof.

§ 2. Theorems 2–4 of the foregoing paragraph enable us to solve the problem stated in the introduction.

THEOREM 6. *Suppose we are given a compact space* $(X, T)$ *with the topology* $T$ *determined by some metric. Let* $T_1$ *be a topology stronger than* $T$. *Then every* $T$*-continuous function is* $T$*-continuous if and only if the space* $(X, T_1)$ *is* $*$*-compact.*

Proof. If $(X, T_1)$ is $*$-compact, then by Theorem 4 it follows that every $T_1$-continuous function is bounded. If follows from Theorem 2 that every $T_1$-continuous function is also $T$-continuous. Let $T_1$-continuity imply $T$-continuity. Thus by Theorem 2 all $T_1$-continuous functions are bounded.

Hence by Theorem 4 the space $(X, T_1)$ is $*$-compact. For a $**$-compact space (by Theorems 2 and 5) we may assert the corresponding theorem.

THEOREM 7. *Suppose we are given a compact and connected topological space* $(X, T)$ *with the topology determined by some metric and let* $T_1$ *be a stronger topology than* $T$. *Then every* $T_1$*-continuous function is* $T$*-continuous if and only if the space* $(X, T_1)$ *is* $**$*-compact.*

The proof of this theorem is analogous to that of Theorem 6.

**Final remarks.**

Remark 1. Spaces in which continuous function are bounded, i.e. pseudo-compact spaces, have been considered in papers dealing with topology.

It follows from Theorem 4 that a Tikhonov space is pseudo-compact if and only if it is $*$-compact.

Remark 2. The characterization of $*$-compact spaces given in Theorem 4 is similar to the following characterization of compact spaces.

THEOREM 8. *A topological space* $X$ *is compact if and only if every real upper (lower) semi-continuous function defined in* $X$ *is upper (lower) bounded.*

Remark 3. If the resulting space is an interval $\langle a, b\rangle$ with the natural topology $T_0$, then, if in some topology $T \supset T_0$ the space $(\langle a, b\rangle, T$ is $*$-compact, two conditions must be satisfied

(1) Every interval $\langle c, d\rangle \subset \langle a, b\rangle$ is $T$-connected (connected in topology $T$).

(2) Every $T$-neighbourhood of an arbitrary point $x_0 \in \langle a, b\rangle$ is a $T_0$-dense set in some $T_0$-neighbourhood of the point $x_0$.

The problem. Do conditions (1) and (2) suffice for the space $(\langle a, b\rangle, T)$ to be $*$-compact, i.e. for

$$F_{T_0}(\langle a, b\rangle, T_0) = F_T(\langle a, b\rangle, T) \quad \text{(notation of p. 2)}.$$



This problem may be considered for an arbitrary compact and connected space $(X, T_0)$.

Remark 4. If a space $(X, T)$ is $*$-compact, then every space $(X, T_1)$ where $T_1$ is a topology weaker than $T$, is also $*$-compact.

**References**


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