Let us add that by a slight modification of the construction of the compactum Y_0 one can obtain a plane compactum Y_0' of dimension 1 such that every plane compactum X has the same shape as a retract of Y_0'.

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References

[1] K. Borsuk, *On the concept of shape for metrizable spaces*, Bull. de l'Ac. Pol. des Sc. 18 (1970), pp. 127–132.
[2] — *Concerning homotopy properties of compacta*, Fund. Math. 62 (1968), pp. 223–254.
[3] — and W. Holsztyński, *Concerning the ordering of shapes of compacta*, Fund. Math. 68 (1970), pp. 107–115.

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On transfinite sequences of B-measurable functions

by

Tibor Šalát (Bratislava)

Abstract. The notion of the convergence of transfinite sequences of real numbers and functions was introduced by Professor W. Sierpiński (Fund. Math. 1 (1920), pp. 132–141). In this paper that notion is extended for metric spaces. A part of results of the paper generalizes some earlier results of W. Sierpiński and H. Malchair, further the transfinite sequences of functions with closed graphs are investigated.

In paper [10] the notion of the limit of the transfinite sequence of real numbers and the notion of the limit function of the transfinite sequence of real functions were introduced. The idea and some results of paper [10] were developed in some further papers by H. Malchair and M. M. Lavrentieff (see e.g. [3]–[7]).

We shall generalize these notions and some results of the above-mentioned papers to metric spaces and we shall prove one theorem on limit functions of transfinite sequences of functions with closed graphs (see Theorem 4).

The following definitions generalize the above-mentioned notions.

DEFINITION 1. Let (X, ϱ) be a metric space and let Ω denote the first uncountable ordinal number. The transfinite sequence

$$\{a_\xi\}_{\xi<\Omega} \tag{1}$$

of elements of the space X is said to be *convergent* and have a limit $a \in X$ if for each $\varepsilon > 0$ there exist an ordinal number $\alpha < \Omega$ such that for each ξ, $\alpha \leqslant \xi < \Omega$ the inequality $\varrho(a_\xi, a) < \varepsilon$ holds. If (1) has the limit a, we write $\lim\limits_{\xi\to\Omega} a_\xi = a$ (or briefly $a_\xi \to a$).

DEFINITION 2. Let X be a set and let (Y, ϱ_1) be a metric space. The transfinite sequence

$$\{f_\xi\}_{\xi<\Omega} \tag{2}$$

of functions $f_\xi\colon X \to Y$ is said to be *convergent* and have a limit function $f\colon X \to Y$ if for each $x \in X$ we have $\lim\limits_{\xi\to\Omega} f_\xi(x) = f(x)$. If (2) has the limit function f, we write $\lim\limits_{\xi\to\Omega} f_\xi = f$ (or briefly $f_\xi \to f$).

It is easy to see that each sequence (1) has at most one limit and each sequence (2) has at most one limit function.

In paper [10] Professor W. Sierpiński studied some properties of limit functions of transfinite sequences of real continuous functions and functions of the first and second Baire class which are defined on $E_1 = (-\infty, +\infty)$. It is proved in [10] that the limit function of any convergent transfinite sequence $\{f_\xi\}_{\xi<\Omega}$ of continuous functions $f_\xi\colon E_1 \to E_1$ is again a continuous function. The proof of this fact is based in [10] on the separability of the space E_1. We shall extend this result to arbitrary metric spaces.

THEOREM 1. *Let X, Y be two metric spaces, let $f_\xi\colon X \to Y$ $(\xi < \Omega)$ be continuous functions. Let $f_\xi \to f$. Then f is also a continuous function on X.*

We shall use in the whole paper the following simple auxiliary result.

LEMMA 1. *Let (Z, τ) be a metric space, $a_\xi \in Z$ $(\xi < \Omega)$ and $a_\xi \to a$. Then there exists an ordinal number $\alpha < \Omega$ such that $a_\xi = a$ for each ξ, $\alpha \leqslant \xi < \Omega$.*

Proof. From the assumption of Lemma 1 we get for $\varepsilon = 1/n$ an ordinal number $\alpha_n < \Omega$ such that $\tau(a_\xi, a) < 1/n$ for each ξ, $\alpha_n \leqslant \xi < \Omega$. Let α denote the first ordinal number which is greater than any α_n $(n = 1, 2, \ldots)$. Then, as is known, we have $\alpha < \Omega$. For each ξ, $\alpha \leqslant \xi < \Omega$ we obtain $\tau(a_\xi, a) < 1/n$ $(n = 1, 2, \ldots)$. Hence $a_\xi = a$ for ξ, $\alpha \leqslant \xi < \Omega$.

Proof of Theorem 1. Let $x_0 \in X$. We prove that f is continuous at x_0. It suffices to prove that if $x_n \in X$ $(n = 1, 2, \ldots)$, $\lim_{n\to\infty} x_n = x_0$, then we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.

We construct the following transfinite sequences:

$$\{f_\xi(x_l)\}_{\xi<\Omega} \quad (l = 0, 1, 2, \ldots).$$

Since $f_\xi \to f$, we have $\lim_{\xi\to\Omega} f_\xi(x_l) = f(x_l)$ $(l = 0, 1, 2, \ldots)$. On account of Lemma 1 we obtain for each l $(l = 0, 1, 2, \ldots)$ an ordinal number $\alpha_l < \Omega$ such that $f_\xi(x_l) = f(x_l)$ for each ξ, $\alpha_l \leqslant \xi < \Omega$. Let α denote the first ordinal number which is greater than any α_l $(l = 0, 1, 2, \ldots)$. Then $\alpha < \Omega$ and

$$f_\alpha(x_l) = f(x_l) \quad (l = 0, 1, 2, \ldots). \tag{3}$$

Since f_α is a continuous function on X, we have

$$\lim_{n\to\infty} f_\alpha(x_n) = f_\alpha(x_0)$$

and this together with (3) yields

$$\lim_{n\to\infty} f(x_n) = f(x_0).$$

This ends the proof.

Let X, Y, Z be metric spaces. The function $f\colon X \times Y \to Z$ is said to be *linearly* (or *separately*) continuous on $X \times Y$ if for each $x \in X$ the function $g_x(y) = f(x, y)$ is continuous on Y and for each $y \in Y$ the function $h_y(x) = f(x, y)$ is continuous on X.

The following theorem is an easy consequence of Theorem 1 (*).

THEOREM 1′. *Let $f_\xi(x, y)$ $(\xi < \Omega)$ be linearly continuous functions on $X \times Y$, $f_\xi\colon X \times Y \to Z$ $(\xi < \Omega)$. If $f_\xi \to f$, then the function f, $f\colon X \times Y \to Z$, is also linearly continuous on $X \times Y$.*

Analogously to Theorem 1 we can prove the following theorem, which is an extension of a result of paper [4].

THEOREM 1″. *Let X be a metric space, $f_\xi\colon X \to E_1$ $(\xi < \Omega)$ be lower (upper) semi-continuous functions. Let $f_\xi \to f$. Then f is a lower (upper) semi-continuous function on X.*

We shall generalize the above-mentioned result of W. Sierpiński also in another direction. Let X, Y be two sets and let S be a class of functions $f\colon X \to Y$. The set $D \subset X$ is said to be a *determining set for S* if any two functions in S that agree on the set D are identical on X. So any dense subset of a metric space X is a determining set for the class of all real continuous functions on X or any infinite subset $M \subset E_1$ is a determining set for the class of all polynomials (cf. [2], [8]).

THEOREM 2. *Let X be a set and Y a metric space. Let S be a class of functions $f\colon X \to Y$. Let us suppose that there exists a countable set $D \subset X$ such that D is a determining set for S. Then the limit function of each convergent transfinite sequence of functions belonging to S is again a function from S.*

Proof. Let $D = \{d_1, d_2, \ldots\}$, $f_\xi \in S$ $(\xi < \Omega)$ and $f_\xi \to f$. Since $\lim_{\xi\to\Omega} f_\xi(d_k) = f(d_k)$, there exists in view of Lemma 1 an ordinal number $\alpha_k < \Omega$ such that

$$f_\xi(d_k) = f(d_k) \tag{4}$$

for each ξ, $\alpha_k \leqslant \xi < \Omega$. Let α denote the first ordinal number which is greater than any α_k $(k = 1, 2, \ldots)$. Then for each ξ, η; $\alpha \leqslant \xi$, $\eta < \Omega$ we obtain on account of (4)

$$f_\xi(d_k) = f_\eta(d_k) \ (= f(d_k)) \quad (k = 1, 2, \ldots).$$

Since D is a determining set for S, we have $f_\xi = f_\eta$ for $\alpha \leqslant \xi$, $\eta < \Omega$. From this it can easily be deduced that $f = f_\alpha$. Hence $f \in S$.

(*) The author is thankful to the reviewer for improving the original version of Theorem 1′.

We shall show an application of Theorem 2.

F_1 denotes the class of all approximately differentiable functions $f\colon \langle 0,1\rangle \to E_1$, F_0 denotes the uniform closure of F_1, i.e. F_0 is the set of all functions $f\colon \langle 0,1\rangle\to E_1$ which are limit functions of uniformly convergent sequences of functions from F_1. It is proved in [9] that F_0 is a proper subset of the class of all approximately continuous functions on $\langle 0,1\rangle$ and that the set of all continuous functions on $\langle 0,1\rangle$ is a proper subset of the class F_0.

THEOREM 3. *Let $f_\xi \in F_0$ $(\xi<\Omega)$ and $f_\xi\to f$. Then $f\in F_0$.*

Proof. Each dense subset $D\subset\langle 0,1\rangle$ is a determining set for the class F_0 (cf. [9]). For D we can choose the set of all rational numbers of the interval $\langle 0,1\rangle$. Theorem now follows at once from the Theorem 2.

Remark. In connection with Theorem 3 the question arises whether the limit function of any convergent transfinite sequence of approximately continuous functions f; $\langle 0,1\rangle\to E_1$ is again an approximately continuous function on $\langle 0,1\rangle$.

In paper [10] it is proved that the limit function of a convergent transfinite sequence $\{f_\xi\}_{\xi<\Omega}$ of functions $f_\xi\colon E_1\to E_1$ of the first Baire class is again a function of the first Baire class. The same idea can be used to prove the following more general result.

THEOREM 4. *Let X be a complete and separable metric space and Y a metric space. Let $\{f_\xi\}_{\xi<\Omega}$ be a transfinite sequence of B-measurable functions $f_\xi\colon X\to Y$ of the first class. Let $f_\xi\to f$. Then also f is a B-measurable function of the first class.*

If X and Y are metric spaces with the metric ϱ and ϱ_1, respectively, then $C(X,Y)$, $B_1(X,Y)$, $U(X,Y)$ denote the class of all continuous functions $f\colon X\to Y$, the set of all B-measurable functions of the first class, and the set of all functions $f\colon X\to Y$ whose graphs are closed subsets of the metric space $X\times Y$ (with the metric $\varrho^*=\sqrt{\varrho^2+\varrho_1^2}$), respectively. In paper [1] it is proved that

$$C(X,E_1)\subset U(X,E_1)\subset B_1(X,E_1) \tag{5}$$

holds.

The class S of functions $f\colon X\to Y$ (Y is a metric space, X is a set) is said to be *closed with respect to the transfinite convergence* (S is c.t.c.) if the limit function of each convergent transfinite sequence $\{f_\xi\}_{\xi<\Omega}$ of functions $f_\xi\in S$ is again a function from S. It follows from the previous results (see Theorems 1 and 4) that

1) the class $C(X,Y)$ is c.t.c. for each metric space X,

2) $B_1(X,Y)$ is c.t.c. if X is a complete and separable metric space.

From Theorem 4 it follows in view of (5) that the limit function of each transfinite sequence of functions from $U(X,E_1)$ is a B-measurable function of the first class if X is a complete and separable metric space. In connection with this fact the question arises whether the class $U(X,Y)$ is also c.t.c. A positive answer to this question is given by the following

THEOREM 5. *Let X, Y be two metric spaces. Let $f_\xi\in U(X,Y)$ $(\xi<\Omega)$ and $f_\xi\to f$. Then $f\in U(X,Y)$.*

Proof. Let G_h denote the graph of the function $h\colon X\to Y$. In order to prove that $f\in U(X,Y)$ it suffices to show that $\bar G_f\subset G_f$ ($\bar G_f$ denotes the closure of G_f in $X\times Y$).

Let $(x_0,y_0)\in\bar G_f$. Then there exists a sequence $\{(x_k,f(x_k))\}_{k=1}^\infty$, $(x_k,f(x_k))\in G_f$ $(k=1,2,\ldots)$ such that

$$\lim_{k\to\infty}(x_k,f(x_k))=(x_0,y_0)\quad\text{ in } X\times Y. \tag{6}$$

Let us construct the transfinite sequences

$$\{f_\xi(x_k)\}_{\xi<\Omega}\qquad(k=1,2,\ldots).$$

Since for a fixed k we have $\lim_{\xi\to\Omega}f_\xi(x_k)=f(x_k)$, there exists in view of Lemma 1 an ordinal number $\alpha_k<\Omega$ such that $f_\xi(x_k)=f(x_k)$ for each ξ, $\alpha_k\leqslant\xi<\Omega$. Denote by α the first ordinal number which is greater than any α_k $(k=1,2,\ldots)$. Then $\alpha<\Omega$ and $f_\xi(x_k)=f(x_k)$ $(k=1,2,\ldots)$ for each ξ, $\alpha\leqslant\xi<\Omega$. Hence

$$(x_k,f(x_k))=(x_k,f_\xi(x_k))\qquad(k=1,2,\ldots) \tag{7}$$

for ξ, $\alpha\leqslant\xi<\Omega$. For each ξ, $\alpha\leqslant\xi<\Omega$ we get from (6), (7)

$$\lim_{k\to\infty}(x_k,f_\xi(x_k))=(x_0,y_0)\quad\text{ in }\quad X\times Y.$$

Since $f_\xi\in U(X,Y)$, the point (x_0,y_0) must belong to the graph of the function f_ξ. So we have $y_0=f_\xi(x_0)$ for ξ, $\alpha\leqslant\xi<\Omega$. From this fact it is easy to see that y_0 is the limit of the transfinite sequence $\{f_\xi(x_0)\}_{\xi<\Omega}$, and since $f_\xi\to f$, we get $y_0=f(x_0)$. Hence $(x_0,y_0)\in G_f$. This ends the proof.

In paper [10] also the notion of the convergence and the sum of transfinite series of real numbers and functions was defined. It is proved in [10] that functions $u_\xi\in C(E_1,E_1)$ $(\xi<\Omega)$ for which the sum of the transfinite series $\sum_{\xi<\Omega}u_\xi$ does not belong to $C(E_1,E_1)$ exist. For the functions u_ξ we may choose suitable polynomials. An analogous result may be proved also for the system $U(E_1,E_1)$.

THEOREM 6. *There exist functions $u_\xi\in U(E_1,E_1)$ $(\xi<\Omega)$ such that the transfinite series $\sum_{\xi<\Omega}u_\xi$ converges and its sum does not belong to $U(E_1,E_1)$.*

Proof. Put $u(0)=1$ and $u(x)=0$ for $x\neq 0$. Then $u\in B_1(E_1,E_1)$ and therefore there exists a sequence $\{f_n\}_{n=0}^\infty$ of continuous functions

$f_n\colon E_1 \to E_1$ such that $\{f_n\}_{n=0}^{\infty}$ converges pointwise to u. Put $u_0 = f_0$, $u_n = f_n - (f_0 + f_1 + \ldots + f_{n-1})$ for $1 \leqslant n < \omega$ and $u_\xi = 0$ for $\omega \leqslant \xi < \Omega$. Then

$$\sum_{\xi<\Omega} u_\xi = u, \quad u_\xi \in U(E_1, E_1) \text{ and } u \notin U(E_1, E_1).$$

References

[1] P. Kostyrko and T. Šalát, *О функциях графы которых являются замкнутыми множествами*, Čas. pěst. mat. 89 (1964), pp. 426–432.

[2] M. Kulbacka, *Sur les ensembles stationaires et déterminants pour certain classes de dérivées symétriques*, Coll. Math. 19 (1968), pp. 255–259.

[3] M. M. Lavrentieff, *Sur la représentation des fonctions measurables par les séries transfinies de polynomes*, Fund. Math. 5 (1924), pp. 123–129.

[4] H. Malchair, *Sur les suites et séries transfinies*, Bull. Soc. Royale des Sci. de Liége 1 (1932), pp. 47–49.

[5] — *Sur les suites et séries transfinies de fonctions non décroissantes*, Bull. Soc. Royale des Sci. de Liége 1 (1932), pp. 75–77.

[6] — *Un théorème sur les suites transfinies de fonctions*, Bull. Soc. Royale des Sci. de Liége 1 (1932), pp. 137–139.

[7] — *Quelques nouvelles considérations aux suites et séries transfinies*, Bull. Soc. Royale des Sci. de Liége 3 (1934), pp. 133–140.

[8] S. Marcus, *Sur les ensembles déterminants des dérivées approximatives*, Compt. Rendus 255 (1962), pp. 1685–1687.

[9] C. J. Neugebauer, *A class of functions determined by dense sets*, Arch. Math. 12 (1961), pp. 206–209.

[10] W. Sierpiński, *Sur les suites transfinies convergentes de fonctions de Baire*, Fund. Math. 1 (1920), pp. 132–141.

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Property Z and Property Y sets in F-manifolds

by

William H. Cutler (*) (Baton Rouge, La.)

Abstract. Let M be a manifold modelled on a Fréchet space F such that $F \cong F^\omega$. K, a closed subset of M, will have *Property Y* if given an open neighborhood U of K and an open cover of M, there exists a set N which is a closed neighborhood of K contained in U and a homeomorphism $h\colon N \Rightarrow \mathrm{Bd}(N) \times [0, 1)$ such that for $x \in \mathrm{Bd}(N)$, $h(x) = (x, 0)$ and $h^{-1}(\{x\} \times [0, 1))$ is contained in some element of the cover. It is shown that (1) Property Y implies infinite deficiency, and (2) Property Z implies Property Y for separable M. The combination gives an alternative proof to the proof of Anderson's that Property Z implies infinite deficiency.

Key words and phrases. Infinite-dimensional manifold, F-manifold, deficiency, Property Z, negligibility, variable product.

1. Introduction. An *F-manifold* is a manifold modelled on a Fréchet space F such that $F \cong F^\omega$. A closed subset K of an F-manifold M has *Property Z* if for every open, non-empty, homotopically trivial set U in M, then $U - K$ is non-empty and homotopically trivial. K has *F-deficiency* if there is a homeomorphism $h\colon M \to M \times F$ such that for $x \in K$, $h(x) = (x, 0)$.

Anderson was the first to show that Property Z implies F-deficiency for separable F-manifolds [1]. More recent results due to Chapman [2] have established this for non-separable F-manifolds. This paper gives a new approach to the problem, one which avoids use of the Hilbert cube, the useful compactification of l_2 (separable Hilbert space), which has no good generalization for other Fréchet spaces. We will also define a new type of deficient subset, which will be used as a stepping stone in the proof that Property Z implies F-deficiency.

Let K be a closed subset of a space X. Then K has *Property Y* if given an open neighborhood U of K and an open cover of X, there exists

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