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Real maximal round filters in proximity spaces

by

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Abstract. Given a proximity space (X, δ), where $P(X)$ is the collection of real-valued proximity functions on (X, δ), a maximal round filter is called *real* whenever the corresponding maximal p-ideal is real. The maximal p-ideals in $P(X)$ which are not real are characterized in terms of their corresponding maximal round filters. From this follow results concerning the realcompletion of (X, δ). The realcompletion is distinguished from the completion of X relative to the total structure associated with δ and from the completion by local clusters.

If (X, δ) is a dense (topological) subspace of T, conditions are obtained which characterize when every member of $P(X)$ can be continuously extended to T. Examples concerning these results are also provided.

1. Introduction. Let (X, δ) be a proximity space with Smirnov compactification δX. The points x of δX may then serve as indices which make explicit the one-one correspondence between the maximal round filters $\mathcal{F}^x$ on (X, δ) and the maximal "p-ideals" I^x in the collection $P(X)$ of real-valued proximity functions on (X, δ). A maximal round filter $\mathcal{F}^x$ is called *real* if the corresponding maximal p-ideal I^x is real. In this paper we characterize the maximal p-ideals I^x which are not real in terms of maximal round filters. It then follows that the realcompletion of (X, δ) is the completion of the generalized uniform space $(X, \mathfrak{U})$, where $\mathfrak{U}$ is the weak generalized uniformity determined by $P(X)$. It is also shown that the realcompletion of (X, δ) is not, in general, coincidental with the completion of X relative to the total structure $\mathfrak{U}_\alpha$ associated with δ, nor with the completion of (X, δ) by clusters.

When (X, δ) is a dense (topological) subspace of T, conditions are obtained which characterize the property that every member of $P(X)$ can be continuously extended to T. This supplements the results of [6], [7] and [8]. An example is provided to show that this property can hold when X is not C-embedded in T and when there is no compatible proximity on T for which (X, δ) is a p-subspace.

2. Real maximal round filters. We note that the collection $P(X)$ need not be a group nor a lattice (cf. [2], p. 135). The theory of p-ideals (or p-systems) in $P(X)$ is developed in [8] and [9]. Appropriate definitions and results concerning round filters may be found in [10]. The proximity relation on the real numbers will be that induced by the standard metric.

Recall that every member f of $P(X)$ can be extended continuously to a mapping $\bar{f}$ of δX into the Smirnov compactification of R. Then for each $x \in \delta X$, the set $I^x = \{f \in P(X)\colon \bar{f}(x) = 0\}$ is a maximal p-ideal of $P(X)$, and each maximal p-ideal has this form. (See Lemma, p. 416 of [8].) By $\mathcal{F}^x$ we denote the unique maximal round filter on (X, δ) which converges to x.

The following result, implicit in [8], is stated for completeness.

LEMMA 2.1. *For each $x \in \delta X$, the following are equivalent*:

(i) *I^x is real,*

(ii) *$\bar{f}(x) \in R$, for all $f \in P(X)$.*

Proof. (i) implies (ii). For $f \in P(X)$, there exists $r \in R$ such that $(f-r) \in I^x$. Then $\overline{(f-r)}(x) = 0$, and since $\bar{f}-\bar{r} = \overline{f-r}$ and $\bar{r}(x) = r$, it follows that $\bar{f}(x) = r$.

(ii) implies (i). If $\bar{f}(x) \in R$ for all $f \in P(X)$, then the extension to δX of $f-\bar{f}(x)$ vanishes at x, so that $(f-\bar{f}(x)) \in I^x$. Thus I^x is real and the proof is complete.

We next characterize the maximal p-ideals in $P(X)$ which are not real.

THEOREM 2.2. *For a point $x \in \delta X$, the following are equivalent*:

(i) *$\bar{f}(x)$ is not real, for some $f \in P(X)$.*

(ii) *There exists $f \in P(X)$ such that f is unbounded on every member of $\mathcal{F}^x$.*

(iii) *For some $f \in P(X)$ and for each positive integer n, the sets $F_n = \{y \in X\colon |f(y)| \geqslant n\}$ belong to $\mathcal{F}^x$.*

(iv) *I^x is not real.*

Proof. (i) implies (ii). Let f satisfy (i). If $f[F]$ is bounded in R, for some $F \in \mathcal{F}^x$, then since $x \in \mathrm{Cl}_{\delta X}F$, it follows that $f(x) \in \mathrm{Cl}_R f[F]$. Thus $\bar{f}(x) \in R$, contradicting (i).

(ii) implies (iii). If f satisfies (ii), then no F_n is empty. Now F_n is a p-neighborhood of F_{n+1}, for all n, and each F_n meets every member of $\mathcal{F}^x$. Since $\mathcal{F}^x$ is maximal, each F_n is a member of $\mathcal{F}^x$.

(iii) implies (iv). Suppose I^x is real when $\mathcal{F}^x$ satisfies (iii). By Lemma 2.1, $\bar{f}(x) \in R$. Choose $n > |\bar{f}(x)|$. Then $\bar{f}[F_n]$ is remote from $\bar{f}(x)$, contradicting $x \in \mathrm{Cl}_{\delta X}F_n$.

(iv) implies (i). Immediate from Lemma 2.1. This completes the proof.

COROLLARY 2.3. *There exists an unbounded member f of $P(X)$ if and only $P(X)$ contains a maximal p-ideal which is not real.*

Proof. Necessity. Take $f \in P(X)$, where f is unbounded. Then the sets $F_n = \{y \in X\colon |f(y)| \geqslant n\}$ form a base for a round filter on (X, δ) which can be embedded in a maximal round filter $\mathcal{F}^x$. By Theorem 2.2, the corresponding maximal p-ideal I^x is not real.

Sufficiency. Since X is a member of $\mathcal{F}^x$, $P(X)$ contains an unbounded member by (ii) of Theorem 2.2, and the proof is complete.

Let $v_\delta X$ be the minimal realcomplete extension of (X, δ), (see [8], p. 414.) From Corollary 2.3 it follows that if $P(X) = P^*(X)$, where $P^*(X)$ is the algebra of bounded, real-valued functions on (X, δ), and if X is not compact, then (X, δ) is not realcomplete. Clearly, any noncompact pseudocompact space cannot be realcomplete relative to any compatible proximity. We note that we may also have $P(X) = P^*(X)$ where $C(X) \neq C^*(X)$.

Now each member f of $P(X)$ determines a pseudometric σ_f, compatible with δ, by $\sigma_f(x, y) = |f(x)-f(y)|$. In this manner, $P^*(X)$ determines the unique totally bounded uniform structure $\mathfrak{T}^*$ in the proximity class of δ. Thus if $\mathfrak{T} = \{\sigma_f\colon f \in P(X)\} \cup \mathfrak{T}^*$, and if $\mathfrak{G}$ is the collection of all pseudometrics on X which are uniformly continuous with respet- to $\mathfrak{T}$, then $\mathfrak{G}$ is a gage for X in the sense of [5]. The generalized unc formity $\mathfrak{U}_P$ (see [1]) associated with $\mathfrak{G}$ by Leader's theorem of [5] is the "weak generalized uniform structure" (see [8], p. 417) determined by $P(X)$i

The following theorem now provides a characterization of the points of $v_\delta X$.

THEOREM 2.4. *A point x of δX is in $v_\delta X$ if and only if $\mathcal{F}^x$ is a Cauchy filter relative to $\mathfrak{U}_P$.*

Proof. Necessity. Take $x \in v_\delta X$, so that $\bar{f}(x)$ is real, for every $f \in P(X)$. Given $\varepsilon > 0$, the set $N_x = \{y \in v_\delta X\colon \sigma_{\bar{f}}(x, y) < \varepsilon\}$ is a neighborhood of x in $v_\delta X$, and since $\mathcal{F}^x$ converges to x, some member F of $\mathcal{F}^x$ is contained in N_x. Thus $\sigma_f[F] \leqslant 2\varepsilon$. Since $\mathcal{F}^x$ contains small sets relative to the gage $\mathfrak{G}$ of $\mathfrak{U}_P$, $\mathcal{F}^x$ is a $\mathfrak{U}_P$-Cauchy filter.

Sufficiency. If $x \in \delta X - v_\delta X$, then I^x is not real by Theorem 2.2. It follows from (ii) of Theorem 2.2 that there is a member f of $P(X)$ which is unbounded on every member of $\mathcal{F}^x$. Hence, σ_f is unbounded on every member of $\mathcal{F}^x$, and $\mathcal{F}^x$ is not a $\mathfrak{U}_P$-Cauchy filter.

This completes the proof.

Thus, the real maximal round filters on (X, δ) are precisely the Cauchy round filters relative to $(X, \mathfrak{U}_P)$. Let $\mathfrak{V}_P$ denote the weak generalized uniformity on $v_\delta X$ generated by $P(v_\delta X)$.

COROLLARY 2.5. *The completion of $(X, \mathfrak{U}_P)$ is $(v_\delta X, \mathfrak{V}_P)$.*

Proof. Clearly, the canonical injection of $(X, \mathfrak{U}_P)$ into $(v_\delta X, \mathfrak{V}_P)$ is a uniform isomorphism. Since $v_\delta X$ is realcomplete, $(v_\delta X, \mathfrak{V}_P)$ is complete by Theorem 2.4, and the proof is accomplished.

O. Njåstad has shown that the realcompletions of (X, δ) are exactly the p-subspaces of δX which are determined by completions of weak generalized uniformities in the p-class of δ. (See Theorem 3 of [8].) In [4] Leader has defined a proximity space to be "complete" if every local cluster contains a point. The completion of (X, δ) by clusters is then taken to be the set X^* of all points of δX which are close to small subsets of (X, δ). Now every compatible pseudometric σ on (X, δ) can be extended to a compatible pseudometric σ^* on X^*. Thus, if $x \in X^*$, $\mathcal{F}^x$ contains small sets relative to the "total" gage (see [1]) on (X, δ). In particular, $\mathcal{F}^x$ has small sets relative to the gage $\mathcal{G}$. Hence, by Theorem 2.4, $X^* \subseteq v_\delta X$. We now provide an example to show that, in general, $X^* \neq v_\delta X$, and that not every completion of $(X, \mathcal{U})$, where $\mathcal{U}$ is a generalized uniformity in the p-class of δ, is a realcompletion of (X, δ).

EXAMPLE 2.6. Let X be the unit ball in l_2, the space of square summable real sequences, and let δ be the proximity relation on X induced by the standard metric d. Thus, $P(X)$ is just the class of all uniformly continuous real-valued functions on X. Now d is not totally bounded, so (X, δ) is not precompact. (See Theorem 12 of [4].) But every member of $P(X)$ is bounded, i.e. $P(X) = P^*(X)$, so that $v_\delta X = \delta X$ but $X^* \neq \delta X$. Moreover, X is complete relative to the uniformity $\mathcal{U}_a$ associated with d, which is the total structure in the p-class of δ. (See Theorem 5 of [1].) Since $v_\delta X$ is the minimal realcompletion of (X, δ), $\mathcal{U}_a$ is not a weak generalized uniformity for any subcollection of $P(X)$. We also note that while X is realcompact, X is not realcomplete relative to δ.

3. Extensions of $P(X)$. Given a proximity space (X, δ), we now suppose that X is a dense (topological) subspace of a topological space T. From [6] it is known that every member of $P^*(X)$ has an extension to a member of $C^*(T)$ if and only if each point of T is a cluster point of a unique maximal round filter on (X, δ). Here we show that every member of $P(X)$ can be extended to a member of $C(T)$ if and only if every point of T is a cluster point of a unique real maximal round filter on (X, δ). We note that if β is the proximity relation on X induced by the Stone–Čech compactification βX of X, then for $\delta = \beta$ Theorem 3.2 is a result characterizing when X is C-embedded in T.

LEMMA 3.1. *If each point x in T is a cluster point of a unique real maximal round filter $\mathcal{F}^x$ on (X, δ), then every pseudometric σ in the gage $\mathcal{G}$ has an extension to a continuous pseudometric $\bar\sigma$ on T.*

Proof. If $v_\delta X$ is regarded as a p-subspace of δX, then (X, δ) is a p-subspace of $v_\delta X$, and by Theorem 1 of [4] and Theorem 2.4, every pseudometric σ in $\mathcal{G}$ has a unique extension to a compatible pseudometric σ_1 on $v_\delta X$. For each $x \in T$, let x_1 be the unique limit point in $v_\delta X$ of the real maximal round filter $\mathcal{F}^x$. Define $\bar\sigma(x, y) = \sigma_1(x_1, y_1)$. Since each $\mathcal{F}^x$ contains small sets relative to $\bar\sigma$, the proof of Lemma 2 of [6] may be applied here to $\bar\sigma$. Thus, for $x \in T$ and $\varepsilon > 0$, the ε-ball about x determined by $\bar\sigma$ is a T-neighborhood of x, and the proof is complete.

We now proceed with our main theorem on extensions.

THEOREM 3.2. *Let (X, δ) be a proximity space, where X is a dense (topological) subspace of T. Then the following are equivalent:*

(i) *Every point x in T is a cluster point of a unique real maximal round filter $\mathcal{F}^x$ on (X, δ).*

(ii) *Every pseudometric in the gage $\mathcal{G}$ associated with the weak generalized uniformity on X determined by $P(X)$ has a unique continuous extension to T.*

(iii) *The canonical injection of (X, δ) into its real-completion $v_\delta X$ can be extended to a continuous mapping of T into $v_\delta X$.*

(iv) *Every member f of $P(X)$ has an extension to a member of $C(T)$.*

Proof. (i) implies (ii). Immediate from Lemma 3.1.

(ii) implies (iii). The collection $\mathcal{G}_1 = \{\bar\sigma\colon \sigma \in \mathcal{G}\}$ is a gage for T (not necessarily compatible with the topology for T). If $\mathcal{U}_1$ is the generalized uniformity for T associated with the gage $\mathcal{G}_1$, then $(X, \mathcal{U})$ is a uniform subspace of $(T, \mathcal{U}_1)$. By Corollary 2.5, $(v_\delta X, \mathcal{V})$ is a completion of $(X, \mathcal{U})$, hence it follows that the canonical injection τ_0 of $(X, \mathcal{U})$ into $(v_\delta X, \mathcal{V})$ has an extension τ to a uniformly continuous mapping of $(T, \mathcal{U}_1)$ into $(v_\delta X, \mathcal{V})$. Since every pseudometric in $\mathcal{G}_1$ is continuous (relative to the original topology for T), τ is a continuous mapping of T into $v_\delta X$.

(iii) implies (iv). Let $f \in P(X)$ and let τ be the continuous extension of the canonical injection of (X, δ) into $v_\delta X$. Now f has an extension to member f^* of $P(v_\delta X)$, so that $f_1 = f^* \circ \tau$ is the unique continuous extension of f to T.

(iv) implies (i). Since (iv) of the extension theorem of [6] is satisfied, each point x in T is a cluster point of a unique maximal round filter $\mathcal{F}^x$ on (X, δ). If $\mathcal{F}^x$ is not real, it follows from Theorem 2.2 that there exists some f in $P(X)$ which cannot have a continuous real-valued extension at x, contradicting (iv). Hence $\mathcal{F}^x$ is real, and the proof is complete.

EXAMPLE 3.3. Let T be the subset $\{(x, y)\colon y \geqslant 0\}$ of the plane. The topology for T is determined by the usual neighborhoods of points in T together with the following neighborhoods of the points $(x, 0)$.

For $\varepsilon > 0$, $N_\varepsilon(x, 0) = \{(x, 0)\} \cup \{(u, v) \in T\colon\ (u-x)^2 + (v-\varepsilon)^2 < \varepsilon^2\}$. Then T is a completely regular, Hausdorff space. (See Example 3.K of [3].)

Let X be the subspace $\{(x, y)\colon y > 0\}$ of T and let δ be the proximity on X generated by the usual metric d in the plane. Now each point of T is a cluster point of a unique maximal round filter $\mathcal{F}^x$ in (X, δ), where $\mathcal{F}^x$ is a Cauchy filter relative to d. Thus, $\mathcal{F}^x$ is real. Now $P(X) \neq P^*(X)$, and by Theorem 3.2, every member of $P(X)$ may be extended to a member of $C(T)$. However, the function $f(x, y) = \sin(y^{-1})$ belongs to $C^*(X)$, but clearly has no continuous extension to T. Thus X is not C^*-embedded in T. We also observe that there is no compatible proximity on T for which (X, δ) is a p-subspace of T. (See Example 1 of [7].)

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A remark on the independence of a basis hypothesis

by

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Abstract. In the paper we prove the independence of a basis hypothesis used by Enderton and Friedman in the proof of the existence of a minimal β_n-model for analysis. The main result is the consistency of ZFC with the axiom

$$(a)_{P(\omega)}(ER)_{P(P(\omega))}[R \in \Pi^1_2[a] \;\&\; R \cap \mathrm{HOD}[a] = 0].$$

The aim of this paper is to prove the independence of a basis hypothesis used by Enderton and Friedman [1] in the proof of the existence of a minimal β_n-model for analysis.

The hypothesis is as follows:

(BH_n): Let $a \subseteq \omega$ and R be a class of subsets of ω, defined by a Σ^1_n formula with parameter a. Then there exists a subset x of ω, defined simultaneously by the formulae Σ^1_n and Π^1_n, such that $x \in R$.

This is exactly the formulation of the fact that $\Delta^1_n[a]$ is a basis for $\Sigma^1_n[a]$. It is well known that (BH_2) is a theorem of ZF (Zermelo–Fraenkel set theory). Addison proved that the axiom of constructibility implies (BH_n) for every natural $n \geqslant 2$. Using the axiom of projective determinateness, Martin and Solovay proved that for an odd n, (BH_n) does not hold. Their conjecture is that under the same assumption (BH_n) holds for even n. Silver proved that (BH_n) is consistent with the existense of a measurable cardinal. For references see [1].

In the present paper we prove that assuming the consistency of ZF, the theory ZF with an additional axiom "(BH_3) does not hold" is consistent. Namely, our theorem is

THEOREM 1. *If M is a countable standard model for $\mathrm{ZF}+V = L$, then there exists a model $N \supsetneqq M$ for ZFC, satisfying the following sentence: for every $a \subseteq \omega$ there exists a class R_a of subsets of ω, $R_a \in \Pi^1_2[a]$ such that no element of R_a is ordinal definable from a.*

In the proof we use the method of forcing, so by the well known reasoning one can obtain the following consistency results:

COROLLARY 2.

$$\mathrm{Con}(\mathrm{ZF}) \to \mathrm{Con}\big(\mathrm{ZFC} + (a)_{P(\omega)}(ER)_{P(P(\omega))}[R \in \Pi^1_2[a] \;\&\; R \cap \mathrm{HOD}[a] = 0]\big).$$