Table des matières du tome LXXVIII, fascicule 3

Les FUNDAMENTA MATHEMATICAE publient, en langues des congrès internationaux, des travaux consacrés à la *Théorie des Ensembles*, *Topologie*, *Fondements de Mathématiques*, *Fonctions Réelles*, *Algèbre Abstraite*
Chaque volume paraît en 3 fascicules

Adresse de la Rédaction et de l'Échange:
FUNDAMENTA MATHEMATICAE, Śniadeckich 8, 00-950 Warszawa 1 (Pologne)

Tous les volumes sont à obtenir par l'intermédiaire de
ARS POLONA – RUCH, Krakowskie Przedmieście 7, 00-068 Warszawa 1 (Pologne)

DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

A proof (involving Martin's axiom) of a partition relation

by

J. Baumgartner (Hanover) and **A. Hajnal** (Calgary)

Abstract. Let Φ be an order type such that $\Phi\to(\omega)^1_\omega$. Then $\Phi\to(\alpha)^2_k$ for $\alpha<\omega_1$, $k<\omega$.

This solves a conjecture of Erdös and Rado as generalized by F. Galvin.

§ 0. Statements of the results and outline of the proof. The aim of this paper is to prove (in *ZFC*)

THEOREM 1. *Let $\Phi\to(\omega)^1_\omega$ for a type Φ. Then*

$$\Phi\to(\alpha)^2_k \quad \text{for} \quad \alpha<\omega_1,\ k<\omega .$$

This solves problems 10/A *and* 11 *of* [1].

For the convenience of the reader we state here the definition of the partition symbol for types.

DEFINITION. Let Φ; Φ_ν, $\nu<\gamma$; be types, $r<\omega$. $\Phi\to(\Phi_\nu)^r_{\nu<\gamma}$ denotes that the following statement is true. Whenever A, $\preceq$ is an ordered set and $f\colon [A]^r\to\gamma$ then there are $B\subset A$, $\nu<\gamma$ such that

$$tpB=\Phi_\nu \quad \text{and} \quad f(X)=\nu \quad \text{for} \quad X\in[B]^r.$$

As to further conventions and notations concerning the partition symbols we refer to [1].

The following is a *brief history of the problem.*

The partition symbol was introduced by P. Erdös and R. Rado. In [2] the following results were proved

$$\Phi\to(\omega+n,\ \omega\cdot m)^2, \quad \Phi\to(\omega+n)^2_3, \quad \Phi\to(\omega+1)^2_k \quad \text{for} \quad |\Phi|>\omega;$$
$$\omega_1,\ \omega_1^*\not\leq\Phi, \quad n,m,k<\omega \quad \text{and} \quad \omega_1\to(\omega+1,\ \omega_1)^2;$$
$$\omega_1\to(\omega+n)^2_2 \quad \text{for} \quad n<\omega .$$

Later in [3] the following results were proved

$$\left.\begin{matrix}\Phi\to(\alpha\vee\alpha^*,\ \eta)^2\\ \Phi\to(\omega\cdot n,\ \alpha)^2\end{matrix}\right\} \quad \alpha<\omega_1,\ n<\omega,\ \omega_1,\omega_1^*\not\leq\Phi,\ |\Phi|>\omega$$

(where η is the type of the set of rational numbers) and

$$\omega_1 \to (\omega\cdot 2, \omega\cdot n)^2 \quad \text{for} \quad n < \omega .$$

(0) It was also proved in [3] that $CH \Rightarrow \omega_1 \not\to (\omega+2, \omega_1)^2$. Galvin (unpublished) proved the following results

$$\Phi \to (\alpha)^2_2 \quad \text{for} \quad \alpha < \omega_1, \quad \Phi \to (\eta)^1_\omega \quad \text{and} \quad \omega_1 \to (\omega\cdot 2, \omega^2)^2 .$$

Prikry proved recently (unpublished) $\omega_1 \to (\omega^2+1, \alpha)^2$ for $\alpha < \omega_1$. Galvin also showed that the assumption $\Phi \to (\omega)^1_\omega$ of Theorem 1 is necessary since

$$\Phi \not\to (\omega)^1_\omega \quad \text{implies} \quad \Phi \not\to (\omega, \omega+1)^2 .$$

Outline of the proof. We will first show that the statement is true if we assume Martin's axiom (see [5] and [9]) and $|\Phi| < 2^{\aleph_0}$. We do not state Martin's axiom because we only use Lemmas 1, 2 ([6], [5]), which are consequences of it. More precisely in § 1 we prove

THEOREM 2. *Let Φ be a type such that $\Phi \to (\omega)^1_\omega$. Assume $|\Phi| = \beta$ and MA_β holds* (1). *Then $\Phi \to (\alpha)^2_k$ holds for $\alpha < \omega_1$, $k < \omega$.*

Then we will obtain our Theorem 1 by carrying out some "absoluteness" proofs. The first of these given in § 2 is

THEOREM 3. *Assume that either of the following conditions hold*

(i) *$\mathfrak{M}$ and $\mathfrak{N}$ are transitive models of ZFC, $\mathfrak{N}$ is an extension of $\mathfrak{M}$ and $\omega_1^{\mathfrak{M}} = \omega_1^{\mathfrak{N}}$.*

(ii) *$\mathfrak{M}$ is the universe V of set theory, $\mathfrak{N}$ is the Boolean universe V^B for some complete Boolean algebra B and the sentence "$\check{\omega}_1 = \omega_1$" is Boolean valid in V^B ($\check{\omega}_1$ is the canonical image of ω_1 in V^B).*

Assume $A, \precsim$ is an ordered set in $\mathfrak{M}$, and "for $\alpha < \omega_1$, $k < \omega$, $tpA \to (\alpha)^2_k$" is true in $\mathfrak{N}$ (Boolean valid in $\mathfrak{N}$) then the same is true in $\mathfrak{M}$. The proof of Theorem 3 employs an argument essentially due to Shoenfield and is contained in Silver's paper [8].

In § 3 we prove the following

THEOREM 4. *Assume that either of the following conditions hold*

(i) *$\mathfrak{M}$ is a countable transitive model of ZFC, B is an $\mathfrak{M}$ complete Boolean algebra with the countable chain condition lying in $\mathfrak{M}$. P is the partial order consisting of all non zero members of B (with the ordering inherited from B) G is P-generic over $\mathfrak{M}$ and $\mathfrak{N} = \mathfrak{M}[G]$.*

(ii) *$\mathfrak{M}$ is the universe V of set theory, B is a complete Boolean algebra satisfying the countable chain condition and $\mathfrak{N}$ is the Boolean universe V^B.*

Assume C is an ordered set in $\mathfrak{M}$ such that $tpC \to (\omega)^1_\omega$ is true in $\mathfrak{M}$ then $tpC \to (\omega)^1_\omega$ is true in $\mathfrak{N}$ (Boolean valid in $\mathfrak{N}$).

(1) MA_β is the same as A_β in [6]; $MA(\beta^+)$ in [5]; $M_{\beta+}$ in [9].

We will prove both absoluteness theorems under the respective assumptions (i) because these proofs are easier to follow. It should be clear for the readers familiar with Boolean valued models (see [7], [9]) how to convert them into proofs appropriate for cases (ii).

Using the above Theorems 2, 3, 4 we can prove Theorem 1 as follows.

Proof of Theorem 1. Let $A, \precsim$ be given, $\Phi = tpA$ and assume $\Phi \to (\omega)^1_\omega$. Let $|\Phi| = \beta$. It is proved in [9] (1) that there is a complete Boolean algebra B with the countable chain condition such that MA_β and $\check{\omega}_1 = \omega_1$ are Boolean valid in $V^B = \mathfrak{N}$. By Theorem 4 then, $tpA \to (\omega)^1_\omega$ is Boolean valid in $\mathfrak{N}$. By Theorem 2, then $tpA \to (\alpha)^2_k$ is Boolean valid for $\alpha < \omega_1$, $k < \omega$ in $\mathfrak{N}$. But then, by Theorem 3, $\Phi \to (\alpha)^2_k$, $\alpha < \omega_1$, $k < \omega$ is true in the universe of set theory.

We want to point out that first we only obtained proofs of Theorem 2, Theorem 3 and the proof of the following assertions weaker than Theorem 4.

(1) Let $\mathfrak{M}$ and $\mathfrak{N}$ be related as in Theorem 3 (i) or (ii). Then if $\langle A, \precsim\rangle$ is an ordered set in $\mathfrak{M}$ such that (in $\mathfrak{M}$) (α) $\omega_1 \leqslant tpA$ or (β) A contains a denumerable dense subset, $|A| \geqslant \omega_1$ then (α) or (β) are true in $\mathfrak{N}$ (Boolean valid in $\mathfrak{N}$) respectively. This already implied that both $\omega_1 \to (\alpha)^2_k$ and $\lambda \to (\alpha)^2_k$ hold for $\alpha < \omega_1$, $k < \omega$ where λ is the type of the set of real numbers. After that F. Rowbottom obtained a proof of the following

(2) Let $\mathfrak{M}$ and $\mathfrak{N}$ be related as in Theorem 3 (i) or (ii). Let $A, \precsim$ be an ordered set in $\mathfrak{M}$ such that $\omega_1^* \not\leqslant \Phi = tpA$ in $\mathfrak{M}$ then $\omega_1 \leqslant tpA \vee \vee \omega_1^* \not\leqslant tpA$ is true in $\mathfrak{N}$ (Boolean valid in $\mathfrak{N}$) as well.

(2) together with our results Theorem 2, Theorem 3 furnished a proof for the more general statement.

$$\Phi \to (\alpha)^2_k \quad \text{for} \quad \alpha < \omega_1,\ k < \omega \quad \text{provided} \quad \omega_1^* \not\leqslant \Phi, \quad |\Phi| \geqslant \omega_1 .$$

However this statement is still weaker than Theorem 1 as is shown by the following example obtained by the first author.

(3) There is a Φ, $\Phi \to (\omega)^1_\omega$ such that $\omega_1^* \leqslant \Psi$, holds for every $\Psi \leqslant \Phi$, $|\Psi| \geqslant \omega_1$.

The proof of (3) will be published elsewhere. The problem if (3) was false or true is due to F. Galvin. We mention that one of the lemmas used for the proof of our Theorem 4 relies heavily on the idea of (2) due to F. Rowbottom. In § 4 we state some other corollaries of our method, mostly without proofs. Finally we state the following

PROBLEM. Does MA_{ω_1} imply

$$\omega_1 \to (\omega+2, \omega_1)^2 ?$$

This should be compared with (0).

(1) The proof in [9] is actually given for $\beta = \omega_1$ (see Theorem 7.15 of [9]) although the more general theorem is obvious from the argument.

We would be glad to see a direct combinatorial proof which we were unable to obtain [1].

§ 1. Proof of Theorem 2. Let $A, \precsim$ be an ordered set $tpA = \Phi$, $\Phi \to (\omega)^1_\omega$, $|\Phi| = \beta$. It is obviously sufficient to show that $\Psi \to (\alpha)^2_k$ holds for a suitable $\Psi \leqslant \Phi$. Thus we may assume

(1) $\Psi \not\to (\omega)^1_\omega$ for $\Psi \leqslant \Phi$, $|\Psi| < \beta$; hence $cf(\beta) > \omega$.

Put $J = \{B \subset A\colon tpB \not\to (\omega)^1_\omega\}$. Then

(2) J is an ω_1-complete ideal; $[A]^{<\beta} \subset J$, $A \notin J$.

For $B, C \subset A$ we write $B \precsim C$ iff $x \in B$, $y \in C$ imply $x \precsim y$. We put $B | \succsim x = \{y \in B\colon x \precsim y\}$ for $B \subset A$.

We need the following well-known facts.

(3) Assume $B \subset A$, $B | \succsim x \in J$ for $x \in B$ then $B \in J$.

(4) Assume $B \subset A$, $\omega_1^* \not\leqslant tpB$, $B | \precsim x \in J$ for $x \in B$ then $B \in J$.

The proof of (3) and (4) is an easy exercise and is left to the reader. As a corollary of (3) and (4) we have

(5) Assume $B \subset A$, $B \notin J$ then either $\omega_1 \leqslant tpB$ or there are $C, D \notin J$; $C, D \subset B$ such that $D \precsim C$.

Proof. If (5) is false then $B = B_0 \cup B_1$ where $B_0 = \{x \in B\colon B | \precsim x \in J\}$, $B_1 = \{x \in B\colon B | \succsim x \in J\}$, and $\omega_1^* \not\leqslant tpB$. By (3) and (4) both B_0 and B_1 are in J hence $B \in J$ as well.

Now we state two lemmas due to Solovay and Kunen respectively. Both are consequences of MA_β and Martin's axiom will only be used at this point in the proof.

Lemma 1 (Solovay [6]). *Assume MA_β. Let $k < \omega$ and assume $\omega = \bigcup_{i<k} A(i, \xi)$ for $\xi < \beta$. Then there are $X \subset \omega$, $|X| = \omega$ and $f \in {}^\beta k$ such that*

$$|X - A(f(\xi), \xi)| < \omega \quad \text{for} \quad \xi < \beta$$

i.e. *$A(f(\xi), \xi)$ contains an endsection of X for every $\xi < \beta$.*

Lemma 2 (Kunen [5]). *Assume MA_β. For $f, g \in {}^\omega\omega$ put $f \ll g$ iff there is $n < \omega$ such that $f(m) < g(m)$ for $n \leqslant m < \omega$. Let $\mathcal{F} \subset {}^\omega\omega$, $|\mathcal{F}| \leqslant \beta$. Then there is $g \in {}^\omega\omega$ such that $f \ll g$ for every $f \in \mathcal{F}$.*

Let now $[A]^2 = \bigcup_{i<k} T_i$ be a 2-partition of length k of A. For every $0 < \varrho < \omega_1$ we choose an ascending sequence $\varrho_n < \varrho$, $n < \omega$ such that $\varrho_0 = 0$ and

(6) $$\omega^\varrho = \sum_{n<\omega} \omega^{\varrho_n}.$$

---

[1] Added in proof: F. Galvin obtained a combinatorial proof of Theorem 1.

We agree that in this section ξ^η denotes ordinal power, and $\sum$ denotes ordinal addition. For $C, D \subset A$, $[C, D] = \{\{x, y\}\colon x \neq y \wedge x \in C \wedge y \in D\}$. For each $\varrho < \omega_1$ we define a family $\mathcal{F}^\varrho$ of subsets of A by transfinite induction on ϱ as follows.

(7) $\mathcal{F}^0 = [A]^1$. Assume $0 < \varrho < \omega_1$ and $\mathcal{F}^\sigma$ has already been defined for $\sigma < \varrho$.

Let $\mathcal{F}^\varrho$ be the set of those $X \subset A$ for which there are $N \subset \omega$, $|N| = \omega$, $i < k$ and $X_m \subset A$ for $m \in N$ such that

(i) $X_n \precsim X_m$ for $n < m$; $n, m \in N$,

(ii) $X_m \in \mathcal{F}^{\varrho_m}$, $m \in N$,

(iii) $[X_n, X_m] \subset T_i$ for $n < m$; $n, m \in N$,

(iv) $X = \bigcup_{m \in N} X_m$.

We now state some properties of $\mathcal{F}^\varrho$ which follow immediately from the definitions

(8) (i) $X \in \mathcal{F}^\varrho$ implies $tpX = \omega^\varrho$ for $\varrho < \omega_1$,

(ii) $X \in \mathcal{F}^\varrho$, Y is an endsection of X (i.e., $Y = X | \succsim x$ for $x \in X$) imply $Y \in \mathcal{F}^\varrho$ for $\varrho < \omega_1$,

(iii) $X \in \mathcal{F}^\varrho$, $M \subset N$; $|M| = \omega$; $Y_m \subset X_m$, $Y_m \in \mathcal{F}^{\varrho_m}$ for $m \in M$ imply $Y = \bigcup_{m \in M} Y_m \in \mathcal{F}^\varrho$ for $0 < \varrho < \omega_1$.

Now using Lemmas 1, 2 we generalize Lemma 1 to the following

Lemma 3. *Assume that the statements of Lemmas 1, 2 are true. Let $\varrho < \omega_1$, $X \in \mathcal{F}^\varrho$, $X = \bigcup_{i<k} X(i, \xi)$ for $\xi < \beta$. Then there are $Y \subset X$, $Y \in \mathcal{F}^\varrho$ and $f \in {}^\beta k$ such that $X(f(\xi), \xi)$ contains an endsection of Y for $\xi < \beta$.*

Proof. By induction on ϱ. For $\varrho = 0$ the statement is obvious. Assume $0 < \varrho < \omega_1$ and that Lemma 3 is true for $0 \leqslant \sigma < \varrho$. Let X_m, $m \in N$ satisfy the conditions of (7). Put $X(i, \xi, m) = X_m \cap X(i, \xi)$ for $m \in N$. By the induction hypothesis there are $Y_m \subset X_m$, $f_m \in {}^\beta k$ for $m \in N$ satisfying the following conditions

(9) $$Y_m \in \mathcal{F}^{\varrho_m}, \quad \text{for} \quad m \in N$$

and $X(f_m(\xi), \xi, m)$ contains an endsection of Y_m for every $m \in N$. Let $A(i, \xi) = \{m \in N\colon f_m(\xi) = i\}$ for $\xi < \beta$, $i < k$. Then $N = \bigcup_{i<k} A(i, \xi)$ for $\xi < \beta$. By Lemma 1, there are $M \subset N$, $|M| = \omega$ and $f \in {}^\beta k$ such that $A(f(\xi), \xi)$ contains an endsection of M for $\xi < \beta$. Let $Y_{m,t}$ $(t < \omega)$ be a sequence of subsets of Y_m such that $Y_{m,t}$ is an endsection of Y_m, and every endsection of Y_m contains a $Y_{m,t}$. We have

(10) For every $\xi < \beta$ there are $n < \omega$ and $\Psi_\xi \in {}^M\omega$ such that

$$Y_{m, \Psi_\xi(m)} \subset X(f(\xi), \xi) \quad \text{for} \quad n \leqslant m \in M.$$

Put $\mathcal{F} = \{\Psi_\xi\colon \xi < \beta\}$. By Lemma 2, there is $\Psi \in {}^M\omega$ such that $\Psi_\xi \ll \Psi$ for $\xi < \beta$. Put $Z_m = Y_{m,\Psi(m)}$ for $m \in M$, $Y = \bigcup_{m \in M} Z_m$. Then by (8), $Y \in \mathcal{F}^\varrho$ and, by (10), $X(f(\xi), \xi)$ contains an endsection of Y for $\xi < \beta$. This proves Lemma 3. We now prove

LEMMA 4. *Assume $B \notin J$, $B \subset A$, $\varrho < \omega_1$. Then there is $X \in \mathcal{F}^\varrho$, $X \subset B$.*

Proof. For $x \in A$, $i < k$ put $T_i(x) = \{y \in A\colon \{x, y\} \in T_i\}$. We proceed by induction on $\varrho < \omega_1$. The statement is obvious for $\varrho = 0$. Assume $0 < \varrho < \omega_1$ and Lemma 4 is true for $\sigma < \varrho$.

We now define the sequences Y_n, B_n; $n < \omega$ of subsets of B by induction on $n < \omega$. By (3), there is $x \in B$ with $B | \succ x \notin J$. Put $Y_0 = \{x\}$ for such an x. By (2), there is $i_0 < k$ such that $T_{i_0}(x) \cap B | \succ x \notin J$. Put $B_0 = T_{i_0}(x) \cap B | \succ x$. Assume that $i_n < \omega$, Y_n, and B_n are already defined so that $Y_n \succ B_n \subset B$, $Y_n \in \mathcal{F}^{\varrho_n}$, $B_n \notin J$. We now claim that there are $Z, C \subset B_n$ such that

(11) $$Z \in \mathcal{F}^{\varrho_{n+1}}, \quad C \notin J \quad \text{and} \quad Z \precsim C.$$

We distinguish two cases (i) $\omega_1 \leqslant tpB_n$ (ii) $\omega_1 \not\leqslant tpB_n$.

Case (i). By (1) and by $\omega_1 \to (\omega)^1_\omega$ we have $\beta = \omega_1$. Let $D \subset B_n$, $tpD = \omega_1$. Then $D \notin J$. By induction there is $Z \subset D$, $Z \in \mathcal{F}^{\varrho_{n+1}}$. Put $C = \{x \in D\colon Z \precsim \{x\}\}$. Then Z, C satisfy (11).

Case (ii). By (5), there are $D, C \subset B_n$; $D, C \notin J$, $D \precsim C$. Applying the induction hypothesis for D we get $Y \subset D$, $Y \in \mathcal{F}^{\varrho_{n+1}}$ which satisfies (11) with C.

Put now $Z(i, u) = T_i(u) \cap Z$ for $u \in C$, $i < k$. Then, by Lemma 3, there are $V \subset Z$ and $f \in {}^C k$ such that $V \in \mathcal{F}^{\varrho_{m+1}}$ and $Z(f(u), u)$ contains an endsection of V for $u \in C$. Let now V_t, $t < \omega$ be a cofinal sequence of endsections of V. Put $C(i, t) = \{u \in C\colon V_t \subset Z(f(u), u) \wedge f(u) = i\}$ for $i < k$, $t < \omega$. $C = \bigcup_{i<k} \bigcup_{t<\omega} C(i, t)$. By (2) and (11) there are $i_{n+1} < k$ and $t < \omega$ such that $C(i_{n+1}, t) \notin J$. Put $Y_{n+1} = V_t$, $B_{n+1} = C(i_{n+1}, t)$. Then $Y_{n+1} \precsim B_{n+1} \subset B$, $Y_{n+1} \in \mathcal{F}^{\varrho_{n+1}}$, $B_{n+1} \notin J$. Thus the sequences are defined and we also know that $i_n < k$, $Y_n \precsim Y_{n+1}$, $Y_n \subset B$ for $n < \omega$; $[Y_n, Y_m] \subset T_{i_n}$ for $n < m < \omega$.

Put $M_i = \{n < \omega\colon i_n = i\}$ for $i < k$. There is $i < k$ with $|M_i| = \omega$. Put $M_i = N$, $X = \bigcup_{m \in N} Y_m$. Then by (7) and (8) $X \in \mathcal{F}^\varrho$ and $X \subset B$.

In view of Lemma 4 and $A \notin J$ to conclude the proof of our Theorem 2 it is sufficient to prove the following fairly easy

LEMMA 5. *Let $\alpha < \omega_1$, $k < \omega$. Then there is $\varrho = \varrho(\alpha, k) < \omega_1$ such that $X \in \mathcal{F}^\varrho$ implies the existence of $Y \subset X$, $i < \varkappa$ with $tpY = \alpha$, $[Y]^2 \subset T_i$.*

Proof. Let $2 \leqslant \beta_i < \omega_1$ for $i < k$

(12) $\varrho \Rightarrow (\beta_0, \ldots, \beta_{k-1})^2$ denotes the following statement.

For every $X \in \mathcal{F}^\varrho$ there are $Y \subset X$, $i < k$ with $tpY = \beta_i$ and $[Y]^2 \subset T_i$. We prove the existence of $\varrho(\beta_0, \ldots, \beta_{k-1}) = \varrho < \omega_1$ satisfying (12).

First we proceed by induction on k. For $k = 0, 1$ the statement is trivial. Assume $k \geqslant 2$ and the statement is true for $l < k$. Then it is true provided $\beta_i = 2$ for some $i < k$. Now it is obviously sufficient to prove the existence of ϱ, $\varrho \Rightarrow (\beta_0, \ldots, \beta_{k-1})^2$ under the condition that $\varrho(\alpha_0, \ldots, \alpha_{k-1})$ exists for every sequence with at least one $\alpha_i < \beta_i$; $\beta_j \geqslant \alpha_j \geqslant 2$, $\beta_j > 2$ for $j < k$. Let $\varrho = \varrho(\beta_0, \ldots, \beta_{k-1})$ be the minimal ordinal satisfying the following condition:

There is $n < \omega$ such that

$$\varrho_n \geqslant \sup\{\varrho(\beta_0, \ldots, \alpha_i, \ldots, \beta_{k-1})\colon \alpha_i < \beta_i, i < k\}.$$

We claim that $\varrho \Rightarrow (\beta_0, \ldots, \beta_{k-1})^2$. Let X_m, $m \in N$, $i < k$ satisfy the conditions of (7) for $X \in \mathcal{F}^\varrho$. We may assume $n \leqslant m$ for $m \in N$. By $\beta_i > 1$ there is a sequence γ_t, $t < \omega$ of ordinals $< \beta_i$ such that $\sum_{t \in T} \gamma_t \geqslant \beta_i$ holds for every $T \subset \omega$, $|T| = \omega$. By the definitions either there are $m \in N$, $Y \subset X_m$, $j < k$, $j \neq i$ such that $tpY = \beta_j$, $[Y]^2 \subset T_j$ or for every $m \in N$ there is $Y_m \subset X_m$, $tpY_m = \gamma_m$, $[Y_m]^2 \subset T_i$. Then $Y = \bigcup_{m \in N} Y_m \subset X$ and $tpY \geqslant \beta_i$, $[Y]^2 \subset T_i$.

§ 2. Proof of Theorem 3. First we prove a lemma in *ZFC*.

LEMMA 6. *Let $\alpha < \omega_1$, $k < \omega$ and $A, \precsim$ be an ordered set. Let $g\colon \omega \to \alpha$ be onto and one-to-one. Let f satisfy*

(1) $$f\colon [A]^2 \to k.$$

$X \subset A$ is homogeneous for f if $x, y \in [X]^2$ implies $f(x) = f(y)$.

We define

(2) $P(f, g) = \{s\colon s$ *is a function* $\wedge$ $Do(s) \in \omega$, $\wedge$ $Ra(s) \subset A$ *is homogeneous for* $f \wedge$ *for all* $m, n \in Do(s)$, $s(m) \precsim s(n)$ *iff* $g(m) < g(n)\}$.

We also define a partial order by letting $s \leqslant t$ iff $s \supset t$. Then $tpA \to (\alpha)^2_k$ holds iff for every f satisfying (1), $P(f, g)$ is not well founded.

Proof. If s_n: $n < \omega$ is a descending sequence in $P(f, g)$ then $X = \bigcup_{n<\omega} Ra(s)$ is homogeneous for f and $tpX = \alpha$. If X is a homogeneous set for α, let h: $\alpha \to X$ be an order isomorphism. Then $(hg)|n$; $n < \omega$ is a descending sequence in $P(f, g)$.

For the proof of Theorem 3 let $A, \precsim \in \mathfrak{M}$, $f \in \mathfrak{M}$ satisfying (1), and $\alpha < \omega_1^{\mathfrak{M}}$ be given. Let $g \in \mathfrak{M}$ be as above. It is easy to see that $P(f, g)^{\mathfrak{M}} = P(f, g)^{\mathfrak{N}} = P(f, g)$. Since it is well-known that any relational system lying in $\mathfrak{M}$ is well founded in $\mathfrak{M}$ iff it is well founded in $\mathfrak{N}$, Theorem 3 now follows from Lemma 6.

Remarks. The assumption $\omega_1^{\mathfrak{M}} = \omega_1^{\mathfrak{N}}$ is not used in this proof. We stated (i) in this stronger form because (i) is needed to prove Rowbottom's result § 0, (2). Of course Theorem 2 is true for any denumerable type lying in $\mathfrak{M}$ in place of a.

§ 3. Proof of Theorem 4. It will be more convenient to prove the theorem $\Phi \to (\omega)^1_\omega$ replaced by $\Phi \to (\omega^*)^1_\omega$. We assume that $\mathfrak{M}$ and $\mathfrak{N}$ are related as in (i) (1).

We denote by $\bar{C}$ the completion of C. If $c \in \bar{C}$, then the *left* (*right*) *character* of c denoted by $l(c)$ $(r(c))$ is the least cardinal β for which there is an increasing (decreasing) sequence $\langle c_\alpha\colon \alpha < \beta\rangle$ of members of C with $c = \lim_{\alpha<\beta} c_\alpha$. We start with a strong version of Rowbottom's result § 0, (2).

Lemma 7. *Let $C, \leqslant\, \in \mathfrak{M}$ be an ordered set. If $c \in \bar{C}^{\mathfrak{N}} - \bar{C}^{\mathfrak{M}}$ then $l(c) = r(c)$ (in $\mathfrak{N}$).*

Proof. We work in $\mathfrak{N}$. Put $\beta = |C|$. There are two cases

a) Either $l(c) = \beta$ or $r(c) = \beta$.

b) Otherwise.

Case a). Clearly β is regular. Let $\langle c_\xi\colon \xi < \beta\rangle$ be a well-ordering of C lying in $\mathfrak{M}$. For $c \in \bar{C}$ and $\xi < \beta$ let

$$L_\xi(c) = \{c_\eta\colon \eta < \xi \text{ and } c_\eta \precsim c\}\,, \qquad R_\xi(c) = \{c_\eta\colon \eta < \xi \text{ and } c \precsim c_\eta\}\,.$$

Now fix $c \in \bar{C}^{\mathfrak{N}} - \bar{C}^{\mathfrak{M}}$. By symmetry it is sufficient to see that $r(c) = \beta$ implies $l(c) = \beta$. If $l(c) < \beta$ then for some $\xi < \beta$, $c = \operatorname{lub} L_\xi(c)$. Since $r(c) = \beta$ it follows that there is $c' \in C$ such that $L_\xi(c) \precsim \{c'\} \precsim R_\xi(c)$. Clearly $L_\xi(c') = L_\xi(c)$. But of course, $L_\xi(c')^{\mathfrak{M}} = L_\xi(c')^{\mathfrak{N}}$ (since $\langle c_\xi\colon \xi < \beta\rangle \in \mathfrak{M}$), so $c = \operatorname{lub} L_\xi(c') \in \bar{C}^{\mathfrak{M}}$ a contradiction.

Case b). Say $l(c) \leqslant r(c) < \beta$. Choose a decreasing sequence $\langle x_\alpha\colon \alpha < r(c)\rangle$ and an increasing sequence $\langle y_\alpha\colon \alpha < l(c)\rangle$ of members of C with $\lim x_\alpha = \lim y_\alpha = c$. By a well-known property of countable chain condition extensions there is $D \in \mathfrak{M}$, $|D| = r(c)$ such that $\{x_\alpha\colon \alpha < r(c)\} \cup \cup \{y_\alpha\colon y_\alpha < l(c)\} \subset D$. But now it is clear that $c \in \bar{D}^{\mathfrak{N}} - \bar{D}^{\mathfrak{M}}$ and the left and right character of c with respect to $\bar{D}$ are just $l(c)$ and $r(c)$ respectively, so we are done by case a).

Let now $C, \precsim$ be an ordered set (fixed for the remainder of the proof) and suppose that $C = \bigcup_{i<\omega} A_i$ in $\mathfrak{N}$ where each A_i is well-ordered. We will show that C is the union of countably many well-orderings in $\mathfrak{M}$ as well.

Lemma 8. *If $c \in \bar{C}^{\mathfrak{N}} - \bar{C}^{\mathfrak{M}}$ then $l(c) = r(c) = \omega$ (in $\mathfrak{N}$).*

(1) We use without mention the well-known fact that cardinals and cofinalit[ies] are preserved in the passage from $\mathfrak{M}$ to $\mathfrak{N}$.

Proof. By Lemma 7, $l(c) = r(c)$. If $l(c) = r(c) > \omega$ then $\omega_1^* \leqslant tpC$ and C cannot be the union of countably many well-orderings.

Let $\dot{A}$ be a term of language of forcing which denotes $\langle A_i\colon i < \omega\rangle$. Then $\dot{A}_i$ will denote A_i. Let $b^0 = [\![C = \bigcup_{i<\omega} \dot{A}_i \wedge \text{ each } \dot{A}_i \text{ is a well-ordering}]\!]$. We are assuming that $b^0 \in G$ hence in particular $b^0 \neq 0$ (the zero element of B).

We work now in $\mathfrak{M}$. For each $i < \omega$ let S_i be the set of all α such that $b_\alpha = [\![\dot{A}_i \text{ has order type } \alpha]\!] \wedge b^0 \neq 0$ (here $\wedge$ is the meet operation in the Boolean algebra). If $\alpha_1 \neq \alpha_2$ then $b_{\alpha_1} \wedge b_{\alpha_2} = 0$ so by the countable chain condition for B, each S_i is countable.

For each i and $\alpha \in S_i$ let $C_{i\alpha} = \{c \in C\colon \exists b \in P,\ b \leqslant b_\alpha \text{ and } b \Vdash c \in \dot{A}_i\}$. It is clear that $C = \bigcup_{i<\omega} \bigcup_{\alpha \in S_i} C_{i\alpha}$, so it will suffice to show that each $C_{i\alpha}$ is the union of countably many well-orderings. Fix $i < \omega$ and $\alpha_0 \in S_i$. For each $\beta \leqslant \alpha_0$ let $D_\beta = \{c \in C\colon \exists b \in P\ \exists \gamma < \beta,\ b \leqslant b_{\alpha_0} \text{ and } b \Vdash c \text{ is the } \gamma\text{th member of } \dot{A}_i\}$.

Lemma 9. *For all $\alpha \leqslant \alpha_0$, D_α is the union of countably many well-orderings. Since $D_{\alpha_0} = C_{i,\alpha_0}$, this will complete the proof.*

Proof of Lemma 9. By induction on $\alpha \leqslant \alpha_0$. For $\alpha = 0$ this is clear. Suppose $\alpha > 0$.

Case i. $\alpha = \beta + 1$. Let $D = \{c \in C\colon \exists b \in P,\ b \leqslant b_{\alpha_0} \text{ and } b \Vdash c \text{ is the } \beta\text{th member of } \dot{A}_i\}$. Then $D_\alpha = D_\beta \cup D$. By the countable chain condition, D is countable. Thus D_α is the union of countably many well-orderings.

Case ii. $cf(\alpha) = \omega$. This is trivial by induction hypothesis since if $\alpha = \sup_{n<\omega} \alpha_n$ then $D_\alpha = \bigcup_{n<\omega} D_{\alpha_n}$.

Case iii. $cf(\alpha) > \omega$. Let $\dot{c}$ be the term denoting the limit of the first α members of A_i. By Lemma 8 we know that $b_{\alpha_0} \Vdash \dot{c} \in \bar{C}^{\mathfrak{M}}$.

Let $C^* = \{c \in \bar{C}^{\mathfrak{M}}\colon \exists b \leqslant b_{\alpha_0}\ b \Vdash \dot{c} = c\}$. By the countable chain condition C^* is countable. For each $c \in C^*$ let $b_c = [\![\dot{c} = c]\!] \wedge b_{\alpha_0}$, and let $D_c = \{c' \in C\colon \exists b \leqslant b_c\ \exists \beta < \alpha\ b \Vdash c' \text{ is the } \beta\text{th member of } \dot{A}_i\}$. Since $b_{\alpha_0} = \sum_{c \in C^*} b_c$ it follows that $D_\alpha = \bigcup_{c \in C^*} D_c$ so we need only show that each D_c is the union of countably many well-orderings. Fix $c \in C^*$, by (the inverted version) of § 1, (3), it will suffice to show that $D_c | \precsim c'$ is the countable union of well-orderings for all $c' \in D_c$.

Fix $c' \in D_c$. Let $X = \{(d, \beta)\colon d \in C \text{ and } \exists b \leqslant b_c\ b \Vdash d \text{ is the } \beta\text{th member of } \dot{A}_i \wedge d \succsim c' \wedge (\forall \gamma < \beta) \text{ the } \gamma\text{th member of } A_i \text{ is } \precsim c'\}$. Clearly if $(d, \beta) \in X$ then $d \precsim c$ and $\beta < \alpha$. By the countable chain condition, X is countable. Hence $\beta_0 = \sup\{\beta\colon \exists d (d, \beta) \in X\} < \alpha$. Finally we claim that $D_c | \precsim c' \subset D_{\beta_0}$ which is the union of countably many well-orderings by inductive hypothesis. This will complete the proof.

Let $d \in D_c$, $d \precsim c'$. Choose $b \leqslant b_c$ and $\beta < \alpha$ so that $b \Vdash \dot{d}$ is the βth member of $\dot{A}_i$. It will suffice to show $\beta < \beta_0$. Since $b \Vdash \dot{c} = c$ we also have $b \Vdash (\exists d' \in C)(\exists j < \alpha)[\beta < \gamma \wedge c' \leqslant d' \wedge d'$ is the γth member of $\dot{A}_i \wedge (\forall \delta < \gamma)$ the δth member of $\dot{A}_i$ is $\precsim c']$. Hence there are $d' \in C$, $\gamma < \alpha$ and $b' \leqslant b$ so that $b' \Vdash (\beta < \gamma \wedge c' \precsim d' \wedge d$ is the γth member of $A_i \wedge (\forall \delta < \gamma)$ the δth member of A_i is $\precsim c')$, i.e., $(d', \gamma) \in X$. Hence $\beta < \gamma \leqslant \beta_0$ and the claim is proved.

§ 4. Some further results. As to the definition of polarized partition relation used below see [1]. In § 1 we implicitly proved the following result

COROLLARY 1. *Let A, $\precsim$ be an ordered set, $tpA = \Phi$, $\Phi \to (\omega)^1_\omega$. Assume $MA_{|\Phi|}$ holds, $\varrho < \omega_1$, $k < \omega$, $f\colon \omega^\varrho \times A \to k$. Then there are $X \subset \omega^\varrho$, $B \subset A$ such that $tpX(<) = \omega^\varrho$, $tpB(\precsim) = \Psi$, $\Psi \to (\omega)^1_\omega$ and $X \times B$ is homogeneous for f. In an informal notation this means*

$$\binom{\Phi}{\omega^\varrho} \to \binom{\exists \Psi(\Psi \to (\omega)^1_\omega)}{\omega^\varrho}^{1,1}$$

and certainly yields the following.

COROLLARY 2. *Assume $MA_{\aleph_1}$. Then*

$$\binom{\omega_1}{\omega^\varrho} \to \binom{\omega_1}{\omega^\varrho}^{1,1}_k \quad \text{for} \quad \varrho < \omega_1,\ k < \omega .$$

From Corollary 1 using the "absoluteness" arguments described in this paper one can get

COROLLARY 3. *Assume $\Phi \to (\omega)^1_\omega$; ϱ, $\alpha < \omega_1$, $k < \omega$. Then*

$$\binom{\Phi}{\omega^\varrho} \to \binom{\alpha}{\omega^\varrho}^{1,1}_k .$$

We omit the details. We mention that we have a direct proof of this result as well.

Following the argument given in [4] we easily get

COROLLARY 4. *$MA_{\aleph_1}$ and $r, k < \omega$ implies*

$$\binom{\omega_1}{\omega} \to \binom{\omega_1}{\omega}^{1,r}_k .$$

And using the absoluteness arguments we obtain the following.

COROLLARY 5. *Assume $\alpha < \omega_1$, $r, k < \omega$. Then*

$$\binom{\omega_1}{\omega} \to \binom{\alpha}{\omega}^{1,r}_k .$$

We omit the proof. This yields a new proof of an unpublished result of F. Galvin.

Finally we mention without proof one more result:

THEOREM 5. *Assume $MA_{\aleph_1}$, $\alpha < \omega_1$. Then*

$$\omega_1 \to \left(\omega_1, \binom{\omega_1}{\alpha}\right)^2 .$$

By a theorem of [3], this is false if CH holds and is relevant to the problem stated in the introduction.

References

[1] P. Erdös and A. Hajnal, *Unsolved Problems in Set Theory, Axiomatic Set Theory*, Proceedings of Symposia in pure mathematics, 13 part 1.

[2] — and R. Rado, *A partition calculus in set theory*, Bull. Amer. Math. Soc. 62 (1956), pp. 427–489.

[3] A. Hajnal, *Some results and problems in set theory*, Acta Math. Acad. Sci. Hung. 11 (1960), pp. 277–298.

[4] — *On some combinatorial problems involving large cardinals*, Fund. Math. 69 (1970), pp. 39–53.

[5] K. Kunen, *Inaccessibility properties of cardinals*, Ph. D. dissertation, Stanford University, 1968.

[6] A. Martin and R. M. Solovay, *Internal Cŏhen extensions*, Annals of Math. Logic 2 (1970), pp. 143–178.

[7] D. Scott, *Lectures on Boolean valued models of set theory*, Lecture notes of the UCLA Summer Institute on Set Theory, 1967.

[8] J. Silver, *A large cardinal in the constructible universe*, Fund. Math. 69 (1970), pp. 93–100.

[9] R. M. Solovay and S. Tennenbaum, *Iterated Cohen extensions and Souslin's problem*, Annals of Math. 94 (1971), pp. 201–245.

Reçu par la Rédaction le 10. 5. 1971