model of T of cardinality $\varkappa$ has a set of cardinality $\varkappa$ which is φ-indiscernible for all quantifier-free formulae φ.

It follows for instance that if $\varkappa$ is regular and $\varkappa \not\to (\varkappa)^2_2$, then there is no model of Peano arithmetic of cardinality $\varkappa$ which is embeddable in all models of Peano arithmetic of cardinality $\varkappa$. One presumes the same is true for all uncountable $\varkappa$, but for $\varkappa$ singular or weakly compact the proof must be different.

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On successors in cardinal arithmetic

by

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Abstract. Properties of the three kinds of successor of a cardinal number defined by Tarski (Indagationes Mathematicae 16 (1954), pp. 26–32) are discussed. Let them be 1, 2, 3-successors respectively. A Fraenkel-Mostowski model is given in which the axiom of choice fails, but every cardinal has a unique 1-successor. It is proved that if every cardinal has a 3-successor, then x infinite implies $x = 2x$. Models are given containing cardinals x, y such that $2x$ is a successor of x, and y^2 a successor of y, respectively, and various other properties and characterizations of 3-successors are mentioned. The positive results are based mainly on Tarski's methods in cardinal arithmetic (see Lindenbaum-Tarski, *Communication sur les recherches de la Théorie des Ensembles*, C. R. Soc. Sc. Varsovie, Cl. III 19 (1926), pp. 299–330), together with some cofinality arguments.

§ 1. Introduction (1). In [8] Tarski defined three types of successor of a cardinal number (henceforth called 1, 2, 3-successors respectively) and proved that "for all x (x has a 2-successor)" implies the axiom of choice. (If x has a 2-successor, it is necessarily unique). We show in § 3 that "for all x (x has a unique 1-successor)" does not imply the axiom of choice (at least in a Fraenkel-Mostowski setting) nor even that every Dedekind finite cardinal is finite. In § 4 we show that "for all x (x has a 3-successor)" implies that for all infinite x, $x = 2x$. We feel that probably neither of these assertions, nor even the former with "unique" inserted, implies the axiom of choice, but no proofs of any of these have yet been announced. For completeness we begin § 4 with a proof, pointed out to the author by Prof. A. Levy, that "for all well-ordered x (x has a 2-successor)" implies the axiom of choice, and conclude it with one or two characterizations of cardinals which can or cannot be 3-successors.

§ 5 is devoted to a few special cases. Models are given in which there are cardinals x, y such that $2x$ is a 3-successor of x and y^2 is a 3-successor of y. Of course it is known that 2^x can be a 1-successor of x. We show that whenever this happens, 2^x is also a 3-successor of x. The same is

(1) In a letter, Professor Tarski informed the author that he had proved Theorem 3 independently some time ago. Lemma 2 and Theorem 7 (ii) were first announced in Lindenbaum–Tarski, *Communication sur les Recherches de la Théorie des Ensembles*, C. R. Soc. Sc. Varsovie, Cl. III 19 (1926), pp. 299–330.

shown for $x\cdot\aleph(x)$ and $x+\aleph(x)$, where $\aleph$ is the Hartogs aleph function. We do not know if 2^x a 3-successor of x is possible when x is not well-ordered. For it to be impossible would be a natural, but perhaps unlikely, extension of results of Sierpiński [5] and Specker [7].

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§ 2. Definitions and well-known lemmas.

We say that y is a 1-*successor* of x (written $x\ \mathrm{adj}\,y$) if $x<y$ and whenever $x<z\leqslant y$, $z=y$.

We say that y is a 2-*successor* of x (written $x\ \mathrm{adj}_2 y$) if $x<y$ and whenever $z>x$, $z\geqslant y$.

We say that y is a 3-*successor* of x (written $x\ \mathrm{adj}_3 y$) if $x<y$ and whenever $z<y$, $z\leqslant x$.

For any cardinal x, $\aleph(x)$ is the least well-ordered cardinal $\aleph$ such that $\aleph\nleqslant x$.

For cardinals x and y, $x\leqslant^* y$ means that if $|X|=x$, $|Y|=y$, there is a function from a subset of Y onto X.

LEMMA 1 (Tarski [9], p. 80, Theorem 2). *If $x+y=x+z$ there are p,q,r such that $x=x+p=x+q$, $y=p+r$, $z=q+r$.*

Proof. Let X, Y, Z be disjoint sets of cardinals x, y, z respectively, f a 1-1 mapping from $X\cup Y$ onto $X\cup Z$.

Let

$$P=\{\eta\in Y\colon \text{ for all } n\in\omega,\ f^n(\eta)\in X\cup Y\},$$

$$Q=\{\zeta\in Z\colon \text{ for all } n\in\omega,\ f^{-n}(\zeta)\in X\cup Z\},$$

$$R=\{\eta\in Y\colon \text{ for some } n\in\omega,\ f^n(\eta)\in Z\},$$

$$R_1=\{\zeta\in Z\colon \text{ for some } n\in\omega,\ f^{-n}(\zeta)\in Y\}.$$

Let $p=|P|$, $q=|Q|$, $r=|R|$. Map $R\to R_1$, 1-1 and "onto" by $\eta\to f^n(\eta)$ where n is the least integer such that $f^n(\eta)\in Z$. That this map is 1-1 and "onto" is easily verified. Hence $|R_1|=r$.

Therefore $y=p+r$, $z=q+r$.

Map $P\cup X$ 1-1 onto X thus $\xi\to f(\xi)$ if $\xi\in\bigcup\{f^n(P)\colon n\in\omega\}$, $\xi\to\xi$ otherwise. If, for $\xi\in P\cup X$, $f(\xi)\in Z$, then by definition of P, $f^{-n}(\xi)\notin P$, each $n\in\omega$. Hence the image of the map is contained in X.

Similarly it is 1-1 and "onto". Therefore $x=x+p$ and in a similar fashion, $x=x+q$.

LEMMA 2. *If $x\leqslant y\leqslant x+t$ there is an $s\leqslant t$ such that $y=x+s$.*

Proof. As $x\leqslant y$ we may let $x+a=y$. As $x+a\leqslant x+t$ we may let $x+a+b=x+t$.

By Lemma 1 there are p,q,r such that $x=x+p=x+q$, $a+b=p+r$, $t=q+r$. Thus $a\leqslant p+r$ and so may be written as p_1+r_1 where $p_1\leqslant p$, $r_1\leqslant r$. Since $p_1\leqslant p$ and $x=x+p$, $x=x+p_1$. Therefore $y=x+a=x+p_1+r_1=x+r_1$ and $r_1\leqslant r\leqslant t$.

So r_1 is our choice for s.

LEMMA 3 (Tarski [8], p. 30, Theorem 1). *Any cardinal x has a 1-successor, denoted by x^+, and defined by*

(i) $x^+=x+1$ *if* $x<x+1$ *and* (ii) $x^+=x+\aleph(x)$ *otherwise.*

Proof. (i) is clear.

(ii) Since $x=x+1$, $\aleph_0\leqslant x$. Suppose $y<\aleph(x)$. Then by definition of $\aleph(x)$, $y\leqslant x$. Let $x=y'+z$ where $y'=\max(y,\aleph_0)$, using the fact that $x\geqslant\aleph_0$. Then $x+y=y'+z+y=z+(y+y')=z+y'$ because y, y' are well-ordered, one at least of them is infinite, and $y'=\max(y,y')$. Therefore $x+y=x$. Hence $y<\aleph(x)\to x+y=x$.

Now let $x<y\leqslant x^+=x+\aleph(x)$. Then by Lemma 2, $y=x+t$, some $t\leqslant\aleph(x)$. If $t<\aleph(x)$, $y=x+t=x$, contrary to $x<y$. Thus $t=\aleph(x)$ and $y=x+\aleph(x)$.

§ 3. 1-successors. We consider a Fraenkel-Mostowski model which in fact is one of those defined by Mostowski in [4], though our use for it is rather different from his.

Suppose that $\mathfrak{M}$ is a model of set theory with the axiom of choice, suitably modified to accommodate urelemente (just modify extensionality), in which U, the set of urelemente, has cardinal $\aleph_0$.

Then U may be indexed by $\omega\times 2$, $U=\{u_{ij}\colon i\in\omega, j\in 2\}$.

Let $U_i=\{u_{i0},u_{i1}\}$ for each i, and $V_n=\bigcup\{U_i\colon i\leqslant n\}$.

G is the group of all permutations of U which preserve each U_i. Notice that every member of G has order 1 or 2, and so G is Abelian.

If $\sigma\in G$ and $\xi\in\mathfrak{M}$, the action of σ on ξ is defined by transfinite induction on rank ξ, thus, $\sigma\xi=\{\sigma\eta\colon \eta\in\xi\}$.

If $\xi\in\mathfrak{M}$, let

$$H(\xi)=\{\sigma\in G\colon \sigma\xi=\xi\}\quad\text{and}\quad K(\xi)=\{\sigma\in G\colon \eta\in\xi\to\sigma\eta=\eta\}.$$

$\mathfrak{F}$ is the filter of subgroups of G generated by $\{H(u)\colon u\in U\}$. Thus $H\in\mathfrak{F}\longleftrightarrow H\supset K(V_n)$, some n.

$\mathfrak{N}$ is the Fraenkel-Mostowski model defined by U, G, and $\mathfrak{F}$. That is, $\xi\in\mathfrak{N}\longleftrightarrow\xi\subset\mathfrak{N}$ and $H(\xi)\in\mathfrak{F}$.

(This defines $\xi\in\mathfrak{N}$ by transfinite induction on rank ξ).

That $\mathfrak{N}$ is a model of set theory (modified to include urelemente) except for the axiom of choice is proved by Mostowski in [4], p. 153–157.

It is easily seen that $K(V_n)$ has finite index in G. (In fact its index is 2^{n+1}). Hence $\mathfrak{F}$ is countable and we let $\mathfrak{F}=\{G_0,G_1,G_2,\ldots\}$ where whenever the least m such that $G_i \supset K(V_m)$ is less than the least n such that $G_j \supset K(V_n)$, then $i < j$.

Let $n_i =$ the least n such that $G_i \supset K(V_n)$. Thus $i \leqslant j \to n_i \leqslant n_j$.

(I) Let $P =$ the set of equivalence classes of ω under the relation $\sim$, $i \sim j$ if $n_i = n_j$. P will figure quite prominently later.

Notice that $p \in P \to p$ finite.

Let $\boldsymbol{v}_n$ be the sequence $(u_{00}, u_{01}, u_{10}, u_{11}, u_{20}, u_{21}, \ldots, u_{n0}, u_{n1})$ whose entries are all the members of V_n.

Let $w_i = \{\sigma \boldsymbol{v}_{n_i}\colon \sigma \in G_i\}$. We show that

(II) $$H(w_i) = G_i .$$

Clearly $H(w_i) \supset G_i$. Conversely, suppose that $\sigma \in H(w_i)$. Now $\boldsymbol{v}_{n_i} \in w_i$, so $\sigma \boldsymbol{v}_{n_i} \in \sigma w_i = w_i$. Therefore $\sigma \boldsymbol{v}_{n_i}$ is of the form $\tau \boldsymbol{v}_{n_i}$, some $\tau \in G_i$. Hence $\tau^{-1}\sigma \in H(\boldsymbol{v}_{n_i}) = K(V_{n_i})$ (anything which fixes $\boldsymbol{v}_{n_i}$ fixes each of its entries). Therefore $\tau^{-1}\sigma \in G_i$ by definition of n_i. Therefore $\sigma \in \tau G_i = G_i$.

Let $X_{ni} = \{\sigma w_i\colon \sigma \in K(V_n)\} \times \{(n, i)\}$. The indexing at the end is just designed to ensure that all the X_{ni}'s are disjoint.

Then X_{ni} is in $\mathfrak{N}$ as it is hereditarily finite. X_{ni} is a typical set which is transitive under the action of $K(V_n)$, which is supported by $K(V_n)$, and which contains a member whose symmetry group is precisely G_i.

We are able to express any cardinal of the model quite simply in terms of the X_{ni}'s.

LEMMA 4 (Läuchli [3], p. 34, Lemma 2). *$X \in \mathfrak{N}$ can be well-ordered in* $\mathfrak{N} \leftrightarrow K(X) \in \mathfrak{F}$.

Proof. Suppose $K(X) \in \mathfrak{F}$. Let f be a 1-1 function from X onto an ordinal, in $\mathfrak{M}$ (using the axiom of choice in $\mathfrak{M}$).

Then if $\sigma \in K(X)$,

$$\begin{aligned} \sigma f &= \sigma\{(\xi, f\xi)\colon \xi \in X\} \\ &= \{(\sigma\xi, \sigma f\xi)\colon \xi \in X\} \\ &= \{(\xi, f\xi)\colon \xi \in X\} \quad \text{because } \sigma \in K(X), \text{ and every ordinal is fixed by } \sigma \\ &= f. \end{aligned}$$

Hence $H(f) \supset K(X) \in \mathfrak{F}$ and $f \in \mathfrak{N}$. Therefore X can be well-ordered in $\mathfrak{N}$.

Conversely, if X can be well-ordered in $\mathfrak{N}$, there is a 1-1 function f in $\mathfrak{N}$ from X onto an ordinal.

Since $\sigma \in G$ fixes each ordinal, anything in $H(f)$ fixes each member of X, i.e. $K(X) \in \mathfrak{F}$.

Now let $X \in \mathfrak{N}$. Then for some $n \in \omega$, $H(X) \supset K(V_n)$. Define $\sim$ on X by $\xi \sim \eta$ if there is a $\sigma \in K(V_n)$ mapping ξ onto η. $\sim$ is an equivalence relation on X and the $\sim$-classes are called the *$K(V_n)$-orbits* of X.

Let $\mathfrak{X}$ be the set of all the orbits.

For each $Y \in \mathfrak{X}$, $H(Y) \supset K(V_n)$, so $K(\mathfrak{X}) \supset K(V_n)$. By Lemma 4, $\mathfrak{X}$ can be well-ordered in $\mathfrak{N}$. Let $\mathfrak{X} = \{X_\alpha\colon \alpha < \beta\}$ some ordinal β. Pick $\xi_\alpha \in X_\alpha$, each α, by the axiom of choice in $\mathfrak{M}$. Let i_α be the unique i such that $H(\xi_\alpha) \cap K(V_n) = G_i$.

For each $i \in \omega$, let $A_i = \{\alpha < \beta\colon i_\alpha = i\}$.

Map $X \to \bigcup \{A_i \times X_{ni}\colon i \in \omega\}$ by f as follows. If $\xi \in X$, ξ is in a unique X_α, as $\mathfrak{X}$ is a partition of X. As X_α is a $K(V_n)$-orbit there is a $\sigma \in K(V_n)$ such that $\xi = \sigma\xi_\alpha$. Let $f(\xi) = \big(\alpha, (\sigma w_{i_\alpha}, n, i_\alpha)\big)$.

Then f is well-defined, 1-1, "onto", and $H(f) \supset K(V_n)$.

1. Well-defined and 1-1.
   $$\begin{aligned} &\sigma\xi_\alpha = \tau\xi_\alpha, \ \sigma, \tau \in K(V_n) \\ &\leftrightarrow \tau^{-1}\sigma \in H(\xi_\alpha) \cap K(V_n) = G_{i_\alpha} \\ &\leftrightarrow \tau^{-1}\sigma \in H(w_{i_\alpha}) \text{ by (II)} \\ &\leftrightarrow \sigma w_{i_\alpha} = \tau w_{i_\alpha}. \end{aligned}$$
2. "Onto".
   By definition of X_{ni}, $X_{ni} = \{\sigma w_i\colon \sigma \in K(V_n)\} \times \{n, i)\}$ so $\big(\alpha, (\sigma w_{i_\alpha}, n, i_\alpha)\big) = f(\sigma\xi_\alpha)$.
3. $H(f) \supset K(V_n)$.
   If $\tau \in K(V_n)$, and $\sigma\xi_\alpha \in X_\alpha$,
   $$\tau f(\sigma\xi_\alpha) = \tau\big(\alpha, (\sigma w_{i_\alpha}, n, i_\alpha)\big) = \big(\alpha, (\tau\sigma w_{i_\alpha}, n, i_\alpha)\big),$$
   and $f\tau(\sigma\xi_\alpha) = f(\tau\sigma\xi_\alpha) = \big(\alpha, (\tau\sigma w_{i_\alpha}, n, i_\alpha)\big)$.

Now any map from ordinals to ordinals which is in $\mathfrak{M}$ is also in $\mathfrak{N}$, so we may map each A_i 1-1 onto a well-ordered cardinal $\varkappa_i$, using the axiom of choice in $\mathfrak{M}$.

Hence any set of $\mathfrak{N}$ can be put into 1-1 correspondence with a set of the form $\bigcup \{\varkappa_i \times X_{ni}\colon i \in \omega\}$, some $n \in \omega$.

Thus from the point of view of discussing the cardinals of $\mathfrak{N}$, we need only concern ourselves with sets of this form.

THEOREM 1. *In $\mathfrak{N}$, every cardinal has a unique 1-successor, but U is an infinite set with no countable subset, and hence the axiom of choice fails.*

Proof. By Lemma 4, if C is a countable subset of U, $K(C) \in \mathfrak{F}$, so $K(C) \supset K(V_n)$ some $n \in \omega$.

As C is infinite, C contains some point u_{mj} outside V_n. Then the permutation of U which interchanges u_{m0} and u_{m1}, and leaves everything else fixed, is in $K(V_n)$ but not in $K(C)$, contradicting $K(C) \supset K(V_n)$.

Hence U has no countable subset in $\mathfrak{N}$, and in particular, cannot be well-ordered in $\mathfrak{N}$.

Now to show that every cardinal has a unique 1-successor in $\mathfrak{N}$. By Lemma 3 any cardinal has a 1-successor and we just need to show uniqueness.

Suppose x adj$x+a$. We must show that $x+a=x+b$ where b is a well-ordered cardinal. Then by the comparability of well-ordered cardinals, $b \geqslant \aleph(x)$ ($b \geqslant 1$ if $x^+ = x+1$) and $x+a \geqslant x+\aleph(x)$ (or $x+a \geqslant x+1$) which gives $x+a = x+\aleph(x)$ ($x+a=x+1$) as desired, because x adj$x+a$.

Choose an n large enough to support sets X, A of cardinals x, a respectively. n is now kept fixed. By the remarks before the statement of the theorem, we may suppose that

$$X = \bigcup\{\lambda_i \times X_{ni} \times \{\emptyset\}\colon i \in \omega\} \quad \text{and} \quad A = \bigcup\{\varkappa_i \times X_{ni}\colon i \in \omega\}.$$

The $\emptyset$ inserted as a last co-ordinate of X is just to ensure that $X \cap A = \emptyset$. X, A "supported" by n means that $H(X), H(A) \supset K(V_n)$. In fact any union of X_{ni}'s is in $\mathfrak{N}$, and is also supported by n.

Suppose now that the following holds.

(III) There is an $m \in \omega$ such that there are only finitely many $p \in P$ for which there is no 1-1 mapping g from

$$\bigcup\{\lambda_j \times X_{nj} \times \{\emptyset\}\colon j \in p\} \cup \bigcup\{\varkappa_j \times X_{nj}\colon j \in p\} \text{ into}$$
$$\bigcup\{\lambda_j \times X_{nj} \times \{\emptyset\}\colon j \in p\} \text{ with } H(g) \supset K(V_m).$$

(See (I) for the definition of P).

Then for every other p (outside this finite set) there is such a g, g_p say. Thus for some finite $B \subset \omega$,

$$g = \bigcup\{g_p\colon p \in P \text{ and } p \cap B = \emptyset\} \text{ maps}$$
$$\bigcup\{\lambda_j \times X_{nj} \times \{\emptyset\}\colon j \in \omega\} \cup \bigcup\{\varkappa_j \times X_{nj}\colon j \notin B\} \text{ 1-1 into}$$
$$\bigcup\{\lambda_j \times X_{nj} \times \{\emptyset\}\colon j \in \omega\}, \text{ and } H(g) \supset K(V_m).$$

Thus the union of all the g_p's for p outside a finite set gives an "absorption map" for $\bigcup\{\varkappa_j \times X_{nj}\colon j \notin B\}$ into $\bigcup\{\lambda_j \times X_{nj} \times \{\emptyset\}\colon j \in \omega\}$.

Hence $x+a = x+|\bigcup\{\varkappa_j \times X_{nj}\colon j \in B\}|$. Now as B is finite, and each X_{nj} is finite,

$$\bigcup\{\varkappa_j \times X_{nj}\colon j \in B\} \text{ can be well-ordered.}$$

Thus $x+a = x+b$ where b is a well-ordered cardinal. This is just what was required.

We now show that the alternative assumption to (III) results in a contradiction.

(IV) Suppose that for all $m \in \omega$ there are infinitely many $p \in P$ such that there is no 1-1 mapping g from

$$\bigcup\{\lambda_j \times X_{nj} \times \{\emptyset\}\colon j \in p\} \cup \bigcup\{\varkappa_j \times X_{nj}\colon j \in p\} \text{ into}$$
$$\bigcup\{\lambda_j \times X_{nj} \times \{\emptyset\}\colon j \in p\} \text{ with } H(g) \supset K(V_m).$$

Then we may partition P into two sets P_0, P_1 with precisely the same property. Let q map ω 1-1 onto $\omega^2 \times 2$. If $q(r) = (m, s, k) \in \omega^2 \times 2$, at stage r we put into P_k the least p not already in P_0 or P_1 such that property (IV) is satisfied with respect to m and p. The co-ordinate s is designed to ensure that for each m we get infinitely many p's in each of P_0, P_1. Any p's which are left over are put in P_0.

Let

$$A_0 = \bigcup\{\varkappa_j \times X_{nj}\colon j \in \bigcup P_0\}, \qquad |A_0| = a_0,$$
$$A_1 = \bigcup\{\varkappa_j \times X_{nj}\colon j \in \bigcup P_1\}, \qquad |A_1| = a_1.$$

Then $A = A_0 \cup A_1$ and $A_0 \cap A_1 = \emptyset$.

Suppose that

$$\text{(V)} \qquad x + a_0 = x.$$

Then there is a 1-1 map g from $X \cup A_0$ onto X in $\mathfrak{N}$. Let $H(g) \supset K(V_m)$ where $m \geqslant n$.

By (IV) for P_0 there is a $p \in P_0$ for all of whose members i, $n_i > m$, such that there is no 1-1 mapping g_p from

$$\bigcup\{\lambda_j \times X_{nj} \times \{\emptyset\}\colon j \in p\} \cup \bigcup\{\varkappa_j \times X_{nj}\colon j \in p\} \text{ into}$$
$$\bigcup\{\lambda_j \times X_{nj} \times \{\emptyset\}\colon j \in p\} \text{ with } H(g_p) \supset K(V_m).$$

In particular, the restriction of g to the first set fails to be such a g_p. Hence g maps some point of

$$\bigcup\{\lambda_j \times X_{nj} \times \{\emptyset\}\colon j \in p\} \cup \bigcup\{\varkappa_j \times X_{nj}\colon j \in p\} \text{ into}$$
$$\bigcup\{\lambda_j \times X_{nj} \times \{\emptyset\}\colon j \notin p\}.$$

Let $\big((\alpha, (\sigma w_i, n, i), \emptyset), (\beta, (\tau w_j, n, j), \emptyset)\big) \in g$, where $\alpha \in \lambda_i$, $\beta \in \lambda_j$, $i \in p$, $j \notin p$, $\sigma, \tau \in K(V_n)$, and the first $\emptyset$ may not be there (and in which case $\varkappa \in \alpha_i$) if tl oint lies in A_0, not in X.

Case $n_j < n_i$ ($n_i \neq n_j$ as $j \notin p$, $i \in p$).

Let φ the permutation of U which interchanges u_{n_i0} and u_{n_i1} and leaves everything else fixed. Then $\varphi \in K(V_{n_j})$ as $n_j < n_i$, so $\varphi(\tau w_j) = \tau w_j$. Also $\varphi \in K(V_m)$ because $n_i > m$, so $\varphi \in H(g)$. Applying φ to the assertion that

$$\big((\alpha, (\sigma w_i, n, i), \emptyset), (\beta, (\tau w_j, n, j), \emptyset)\big) \in g$$

we get

$$\big((\alpha, (\varphi\sigma w_i, n, i), \emptyset), (\beta, (\tau w_j, n, j), \emptyset)\big) \in g,$$

and since g is 1-1,

$$\varphi\sigma w_i = \sigma w_i \quad \text{(similarly if } \emptyset \text{ is not present)}.$$

Since G is Abelian, $\varphi w_i = w_i$.

Now $w_i = \{\chi v_{n_i}\colon \chi \in G_i\}$. Hence $\varphi v_{n_i} = \chi v_{n_i}$, some $\chi \in G_i$. Therefore $\chi^{-1}\varphi \in H(v_{n_i}) = K(V_{n_i}) \subset G_i$ by definition of n_i. Therefore $\varphi \in \chi G_i = G_i$. So $G_i \supset K(V_{n_i}) \cup \varphi K(V_{n_i}) = K(V_{n_i-1})$, contradicting the definition of n_i as the least l such that $G_i \supset K(V_l)$.

Case (ii). $n_j > n_i$.

Let φ be the permutation of U which interchanges u_{n_j0} and u_{n_j1} and leaves everything else fixed. As before, $\varphi \in K(V_{n_i})$ so $\varphi(\sigma w_i) = \sigma w_i$ and $\varphi \in H(g)$. This time using the fact that g is a function we derive $\varphi\tau w_j = \tau w_j$ and this leads to a contradiction of the definition of n_j.

Hence our assumption (V) is contradicted, and $x + a_0 > x$. Similarly $x + a_1 > x$. Since x adj $x+a$ and $a_0, a_1 \leqslant a$, we have

$$x + a_0 = x + a_1 = x + a\,.$$

Let g be a 1-1 map from $X \cup A_0$ onto $X \cup A_1$. Let $H(g) \supset K(V_m)$, where $m \geqslant n$.

Then by (IV) for P_0 there is a $p \in P_0$, for all of whose members i, $n_i > m$, such that there is no 1-1 mapping g_p from

$$\bigcup \{\lambda_j \times X_{n_j} \times \{\emptyset\}\colon j \in p\} \cup \bigcup \{\varkappa_j \times X_{n_j}\colon j \in p\} \text{ into}$$

$$\bigcup \{\lambda_j \times X_{n_j} \times \{\emptyset\}\colon j \in p\} \text{ with } H(g_p) \supset K(V_m)\,.$$

As before we get an $\big((\alpha, (\sigma w_i, n, i), \emptyset), (\beta, (\tau w_j, n, j), \emptyset)\big)$ such that $\alpha \in \lambda_i$ or $\varkappa_i$, $\beta \in \lambda_j$ or $\varkappa_j$, $i \in p$, $j \notin p$, $\sigma, \tau \in K(V_n)$, and the two $\emptyset$'s will or will not be present (it makes no difference) depending on whether the points lie in A_0, A_1 or X.

The only extra point to notice here is that $P_0 \cap P_1 = \emptyset$, and so, intuitively, A_0 and A_1 cannot "interfere" with each other. A contradiction is now arrived at in the same way as before.

Assumption (IV) is therefore false, and the theorem is proved.

In order to formulate Theorem 1 as a relative consistency result, let T be the theory formed from ZF by modifying extensionality to allow urelemente. Then Theorem 1 gives

COROLLARY. *If T is consistent, then so is T + there is a cardinal incomparable with $\aleph_0$ (and hence $\neg AC$) + every cardinal has a unique 1-successor.*

§ 4. 2 and 3-successors.

THEOREM 2. *If every well-ordered cardinal has a 2-successor, then the axiom of choice holds.* (Pointed out by A. Levy).

Proof. Let $\varkappa$ be a well-ordered cardinal. Then $\aleph(\varkappa) = \varkappa^+ =$ the least well-ordered cardinal $> \varkappa$ is a 1-successor of $\varkappa$ by Lemma 3. By definition of 2-successor, $\varkappa^+$ is $\geqslant$, and hence $=$ the 2-successor of $\varkappa$ given by our hypothesis. (In fact this is the argument showing, as mentioned in the introduction, that if a cardinal x has a 2-successor it is unique, and is also the only 1-successor of x).

Suppose that x is any cardinal.

If $\aleph(x) < x + \aleph(x)$, $\aleph^+(x) \leqslant x + \aleph(x)$, as $\aleph^+(x)$ is the 2-successor of $\aleph(x)$, as shown above.

Therefore $\aleph^+(x) = \aleph\big(x + \aleph(x)\big) \geqslant \aleph\big(\aleph^+(x)\big)$, contradicting the definition of $\aleph\big(\aleph^+(x)\big)$. So $\aleph(x) = x + \aleph(x)$, giving $x \leqslant \aleph(x)$ and x well-ordered.

LEMMA 6 (Tarski [9], p. 81, Theorem 3). *If $kx+y = (k+1)x+z$, where k is a positive integer ($\neq 0$), then $x+y = 2x+z$.*

Proof. We use induction. If $k = 1$ the conclusion is the same as the premise. Otherwise, suppose $(k+1)x+y = (k+2)x+z$. Then $x + (kx+y) = x + \big((k+1)x+z\big)$.

By Lemma 1 there are p, q, r such that

$$x = x+p = x+q\,, \quad kx+y = p+r\,, \quad (k+1)x+z = q+r\,.$$

Hence

$$\begin{aligned} kx+y &= (k-1)x+x+y \qquad (\text{as } k > 0)\\ &= (k-1)x+(x+q)+y\\ &= kx+q+y = p+q+r\\ &= (k+1)x+z+p = (k+1)x+z\,. \end{aligned}$$

Therefore $x+y = 2x+z$, by induction.

LEMMA 7. *If $x < 2x$ and x adj $x+y$, then $x+y < 2x+y$.*

Proof. Suppose that $x+y = 2x+y$. Then $x < 2x \leqslant 3x \leqslant 3x+y = 2x+y = x+y$.

Since x adj $x+y$, $2x = 3x$. By Lemma 6 with $y = z = 0$, $x = 2x$, contrary to assumption.

LEMMA 8. *If $x < 2x$ and x adj $x+y$, where $y \nleqslant x$, then $x+y$ has no 3-successor.*

Proof. Suppose not, and let $(x+y)'$ be a 3-successor of $x+y$. Then $(x+y)' > x$.

Let $(x+y)' = x+z$. If $z = (x+y)'$, $x+z = z$, so

$$x+y < 2x+y < 3x+y \leqslant 3x+(x+z) = z = (x+y)'\,,$$

by Lemmas 6 and 7, contradicting $x+y$ adj $(x+y)'$.

Therefore $z < (x+y)'$. By definition of 3-successor, $z \leqslant x+y$. Therefore $(x+y)' = x+z \leqslant 2x+y$.

By Lemma 2 there is a $t \leqslant x$ such that $(x+y)' = x+y+t$. Now $x < x+y$. So by Lemma 6, $2x < 2x+y$. Hence $x+t < x+y+t = (x+y)'$. Using again the definition of 3-successor, $x+t \leqslant x+y$. Thus $x \leqslant x+t \leqslant x+y$, and as x adj $x+y$, $x = x+t$ or $x+t = x+y$.

If $x = x+t$, $x+y+t = x+y$, contradicting $x+y < (x+y)'$. If on the other hand $x+t = x+y$, by Lemma 1 there are p, q, r such that $x = x+p = x+q$, $t = p+r$, $y = q+r$.

Therefore $y = q+r \leqslant p+q+r = t+q \leqslant x+q = x$. This contradicts $y \nleqslant x$.

THEOREM 3. *If $x \geqslant \aleph_0$ and $x < 2x$, then $x+\aleph(x)$ has no 3-successor. Hence "every cardinal has a 3-successor" implies "for all infinite x, $x = 2x$".*

Proof. The first part is immediate from Lemma 8, Lemma 3, and the definition of $\aleph(x)$.

The second part follows easily too provided that, given an infinite x such that $x < 2x$, we may find one $\geqslant \aleph_0$.

Replace our original x by $x+\aleph_0$. We must show that $x+\aleph_0 < 2x+\aleph_0$.

If $\aleph_0 \leqslant x$, $x+\aleph_0 = x < 2x = 2x+\aleph_0$ so there is no problem. If $\aleph_0 \nleqslant x$, then $x < x+1$.

Suppose $x+\aleph_0 = 2x+\aleph_0$. By Lemma 1 there are p, q, r such that

$$x = x+p = x+q\,, \quad \aleph_0 = p+r\,, \quad x+\aleph_0 = q+r\,.$$

As $x < x+1$, $p = q = 0$. Therefore $x \leqslant x+\aleph_0 = r \leqslant \aleph_0$, contradicting x not well-ordered.

We conclude § 4 with a simple characterization of some 3-successors, and of some cardinals which cannot be 3-successors. These characterizations coincide when "for all infinite x, $x = 2x$".

THEOREM 4. (i) *If x adj y, $x+y = y$, and y is not the sum of x and an incomparable cardinal, then x adj$_3$ y.*

(ii) *If y is the sum of two incomparable cardinals a and b such that $t \leqslant a$ and $b \to t+x = x$, then y is not a 3-successor of x.*

(iii) *If for every infinite x, $x = 2x$, then y a 3-successor $\leftrightarrow y$ a 1-successor and y is not the sum of two incomparable cardinals.*

(iv) *If x adj$_2$ y and x adj$_3$ y, x and y are Dedekind finite or well-ordered.*

Proof. (i) Let $z < y$. Then $x \leqslant x+z \leqslant x+y = y$. Hence $x+z = x$ or y. If $x+z = x$, $z \leqslant x$ as desired.

If $x+z = y$, z is comparable with x, giving $z \leqslant x$ or $z = y$.

(ii) Suppose x adj$_3$ $y = a+b$. Then $a, b < a+b$, as a and b are incomparable. Therefore $a, b \leqslant x$. Let $A, B \subset X$ be of the appropriate cardinals. Let $P = A \cap B$, $Q = A-B$, $R = B-A$, $p = |P|$, $q = |Q|$, $r = |R|$. Then $a = p+q$, $b = p+r$, $p+q+r \leqslant x$. So $p \leqslant a, b$ and $x+p = x$. Therefore $y = a+b = 2p+q+r \leqslant p+x = x$, contradicting $x < y$.

(iii) Suppose x adj$_3$ y and $y = a+b$, a, b incomparable. Then x adj y certainly. If x is finite there is no problem. If $t \leqslant a, b$, then $t < y$, so $t \leqslant x$, and $t+x = x$ (as $x = 2x$). Now use (ii). Conversely, if x adj y and y is not the sum of two incomparable cardinals, by (i), x is finite or x adj$_3$ y.

(iv) By the remark during the proof of Theorem 2, and Lemma 3, $y = x+\aleph(x)$, or x and y are Dedekind finite.

If $y = x+\aleph(x)$, let $t \leqslant x, \aleph(x)$. Then as $\aleph(x) \geqslant \aleph_1$, $t+x = x$. By (ii) x and $\aleph(x)$ are comparable, so x is well-ordered, and $y = x+\aleph(x)$ is too.

§ 5. Miscellaneous results. Firstly we give a model containing cardinals x, y such that x adj $2x$, y adj $2y$, and such that x adj$_3$ $2x$ but not y adj$_3$ $2y$.

We use the notation for a Fraenkel-Mostowski model introduced at the beginning of § 3. (In this case it is quite easy to construct a Cohen model giving us the result, or else the Jech-Sochor Theorem [2] can be used.)

Let $U = \{u_\alpha \colon \alpha < \omega_1\}$, $G =$ the group of all permutations of U, and $\mathfrak{F}$ be the countably closed filter of sub-groups of G generated by $\{H(u)\colon u \in U\}$. $\mathfrak{N}$ is the resulting model.

LEMMA 9. *If $x+y \leqslant x+z$ and whenever $t \leqslant x$ and y, $z+t = z$, then $y \leqslant z$.*

Proof. Let $x = a+b = a+c$, $y = c+d$, $z \geqslant b+d$. Then $c \leqslant x$ and y, so $z+c = z$. Therefore $y = c+d \leqslant b+c+d \leqslant c+z = z$.

COROLLARY. *If $x+y \leqslant x+z$, x well-ordered and $\aleph(y) \leqslant \aleph(z) > \aleph_0$, then $y \leqslant z$.*

Now let $x = |U|^{\mathfrak{N}}$.

THEOREM 5. *The following hold in $\mathfrak{N}$.*

(i) *$\aleph_0$ adj$_3$ x (so $\aleph_0$ has in fact two distinct 3-successors).*

(ii) *x adj$_3$ $2x$.*

(iii) *$x+\varkappa$ adj $2(x+\varkappa)$ but not $x+\varkappa$ adj$_3$ $2(x+\varkappa)$, any well-ordered $\varkappa \geqslant \aleph_1$.*

Proof. (i) Let V be any subset of U in $\mathfrak{N}$. Then for a countable subset A of U,

$$H(V) \supset K(A)\,.$$

If $V \subset A$, $|V| \leqslant \aleph_0$ as desired. If $u \in V-A$, let v be any member of $U-A$, and σ be the permutation of U fixing everything except u, v and interchanging these two.

Then $u, v \notin A$, so $\sigma \in K(A)$. Therefore $\sigma \in H(V)$. Hence $v \in V$. This shows that $V \supset U-A$.

Now let B be a countable subset of $U-A$. Then

$$x = |U| \leqslant |V-B|+|A|+|B| = |V-B|+\aleph_0 = |V-B|+|B| = |V|\,.$$

(ii) Suppose $y \leqslant 2x$. Then $y = y_1+y_2$ where $y_1, y_2 \leqslant x$. By (i), $y_1, y_2 \leqslant \aleph_0$ or $= x$. If both of y_1, y_2 equal x, $y = 2x$ as desired. If not, $y = y_1+y_2 \leqslant x+\aleph_0 = x$.

We must also show that $x < 2x$. If $x = 2x$, U is a disjoint union of two uncountable sets. But the proof of (i) showed that any $V \subset U$ is countable or has a countable complement.

(iii) If $x+\varkappa \leqslant y \leqslant 2(x+\varkappa) = 2x+\varkappa$, by Lemma 2 y is of the form $x+\varkappa+t$, some $t \leqslant x$. By (i) $t \leqslant \aleph_0$ or $t = x$. Therefore $y = x+\varkappa+t = 2x+\varkappa$ or $\leqslant x+\varkappa+\aleph_0 = x+\varkappa$.

We must also show that $x+\varkappa < 2x+\varkappa$. If $x+\varkappa = 2x+\varkappa$, by the corollary to Lemma 9 and the fact that $\aleph(2x) = \aleph(x) > \aleph_0$, $x = 2x$, contrary to (ii).

Finally to show that $x+\varkappa$ adj$_3$ $2x+\varkappa$ fails, we observe that $2x < 2x+\varkappa$ since $\varkappa \not\leqslant x$. However, if $2x \leqslant x+\varkappa$, by Lemma 1 there are p, q, r such that $x = x+p = x+q = p+r$, $\varkappa \geqslant q+r$. $r \leqslant \varkappa$, so is well-ordered, and being $\leqslant x$, is $\leqslant \aleph_0$. Hence $p = x$, and $x = 2x$, contradicting (ii).

We now do the same for x and x^2 as Theorem 5 (ii) does for x and $2x$. We do not know whether x adj $x^2 \to x$ adj$_3$ x^2. The model used, Solovay's model of [6], also satisfies $\aleph_0$ has two distinct 3-successors, in his case $\aleph_1$ and $2^{\aleph_0}$. He uses an inaccessible cardinal for his construction, but it seems unlikely that we need it in fact for our conclusions.

His model satisfies

(1) $2^{\aleph_0}$ cannot be well-ordered,

(2) any uncountable set of reals contains a non-empty perfect closed subset and

(3) every infinite well-ordered successor cardinal is regular.

From (2) follows immediately (4) $\aleph_0$ adj $2^{\aleph_0}$, since any non-empty perfect closed set of reals has cardinal $2^{\aleph_0}$.

Lemma 10 (Tarski [10], p. 148, Lemma 1). *If $xy \leqslant z+t$, then $x \leqslant z$ or $y \leqslant^* t$. If in addition one at least of x, y, z, t is well-ordered, then $x \leqslant z$ or $y \leqslant t$.*

Proof. Choose disjoint sets X, Y, Z, T of the appropriate cardinals such that $X \times Y \subset Z \cup T$.

We suppose without loss of generality that if any of x, y, z, t is well-ordered, that x or t is well-ordered.

If for some $\eta \in Y$, $X \times \{\eta\} \subset Z$, then $x \leqslant z$. If not, for every $\eta \in Y$, $(X \times \{\eta\}) \cap T \neq \emptyset$.

Let $f(\tau) = \eta$ if $(X \times \{\eta\})$ contains τ. f is clearly well-defined, and maps a subset of T onto Y. Hence $y \leqslant^* t$.

If we know that x or t is well-ordered, for each $\eta \in Y$ we may pick a member $f(\eta)$ of $(X \times \{\eta\}) \cap T$, making f this time a 1-1 map from Y into T, and giving $y \leqslant t$.

We write x adj^n y if there is a sequence

$$z_0 \text{ adj } z_1 \text{ adj } z_2 \ldots \text{ adj } z_n \text{ such that } z_0 = x,\ z_n = y .$$

It can be shown, though we do not need it here, that the n, if it exists, is unique (see [11]).

Before stating Theorem 6 we just remark that if x adj $2x$ then nx adj^n $2(nx)$. The proof is very simple and is omitted.

Theorem 6. *If $\aleph_0$ adj $2^{\aleph_0} \neq \aleph_1$ and every infinite well-ordered successor cardinal is regular, then there are $x_1, x_2, \ldots$ satisfying x_n adj^n x_n^2, each n. In addition, x_1 adj$_3$ x_1^2.*

Proof. We show briefly that $\aleph_0$ adj$_3$ $2^{\aleph_0}$, though this in fact follows at once from (i) or (ii) of Theorem 7.

Suppose $x \leqslant 2^{\aleph_0}$. Then $\aleph_0 \leqslant x+\aleph_0 \leqslant 2^{\aleph_0}+\aleph_0 = 2^{\aleph_0} = 2^{\aleph_0}\cdot 2^{\aleph_0}$. If $\aleph_0 = \aleph_0+x$, $x \leqslant \aleph_0$ as desired. If $\aleph_0+x = 2^{\aleph_0}\cdot 2^{\aleph_0}$, Lemma 10 gives $x \geqslant 2^{\aleph_0}$ or $\aleph_0 \geqslant 2^{\aleph_0}$. Hence $x = 2^{\aleph_0}$, also as desired.

Let $\varkappa$ be the least well-ordered cardinal such that $\varkappa \not\leqslant^* 2^{\aleph_0}$. (In fact $\varkappa = \aleph_2$ in Solovay's model.) Let $\varkappa_0 = \varkappa$, $\varkappa_{n+1} = \varkappa_n^+$.

Let $x_n = \varkappa\cdot 2^{\aleph_0}+\varkappa_n$. To show that x_n adj^n x_n^2, it is enough to show that

$$x_n = \varkappa\cdot 2^{\aleph_0}+\varkappa_n \text{ adj } \varkappa_1\cdot 2^{\aleph_0}+\varkappa_n \ldots \text{ adj } \varkappa_n\cdot 2^{\aleph_0}+\varkappa_n = \varkappa_n\cdot 2^{\aleph_0} = x_n^2,$$

and this is the same as

$$\varkappa_m\cdot 2^{\aleph_0}+\varkappa_n \text{ adj } \varkappa_{m+1}\cdot 2^{\aleph_0}+\varkappa_n, \quad \text{each } m < n .$$

Firstly suppose we have equality.

Then $\varkappa_{m+1}\cdot 2^{\aleph_0} \leqslant \varkappa_m\cdot 2^{\aleph_0}+\varkappa_n$. By Lemma 10, $\varkappa_{m+1} \leqslant \varkappa_m\cdot 2^{\aleph_0}$ or $2^{\aleph_0} \leqslant \varkappa_n$. Each of these is impossible, the first because

$$\aleph(\varkappa_{m+1}) > \varkappa_{m+1} = \aleph(\varkappa_m\cdot 2^{\aleph_0}) \quad (\text{since } \aleph(x)\cdot\aleph(y) = \aleph(xy)),$$

and the second because $2^{\aleph_0}$ is not well-ordered.

Secondly suppose that

$$\varkappa_m\cdot 2^{\aleph_0}+\varkappa_n \leqslant y+\varkappa_n \leqslant \varkappa_{m+1}\cdot 2^{\aleph_0}+\varkappa_n \quad \text{where } \varkappa_m\cdot 2^{\aleph_0} \leqslant y \leqslant \varkappa_{m+1}\cdot 2^{\aleph_0}$$

(using Lemma 2 of course). Let R = the real numbers (any set of cardinal $2^{\aleph_0}$ would do), and let $Y \subset \varkappa_{m+1} \times R$ of cardinal y be such that for each $r \in R$, $Y \cap (\varkappa_{m+1} \times \{r\})$ is an initial segment of $\varkappa_{m+1} \times \{r\}$. Let the order-type of $Y \cap (\varkappa_{m+1} \times \{r\})$ be α_r. If there are $2^{\aleph_0}$ r's in R such that $\alpha_r = \varkappa_{m+1}$, then $y = \varkappa_{m+1}\cdot 2^{\aleph_0}$ as desired. If there are less than $2^{\aleph_0}$ such r's, there are $\leqslant \aleph_0$ of them as $\aleph_0$ adj$_3$ $2^{\aleph_0}$, so

(VI) $$\left|Y \cap \bigcup \{\varkappa_{m+1} \times \{r\}:\ \alpha_r = \varkappa_{m+1}\}\right| \leqslant \aleph_0\cdot\varkappa_{m+1} = \varkappa_{m+1} .$$

Now $\operatorname{cf}\varkappa_{m+1} = \varkappa_{m+1}$ because $\varkappa_{m+1}$ is an infinite well-ordered successor cardinal.

Hence $\operatorname{cf}\varkappa_{m+1} \not\leqslant^* 2^{\aleph_0}$ since $\varkappa \not\leqslant^* 2^{\aleph_0}$. This shows that the values of α_r below $\varkappa_{m+1}$ are bounded below $\varkappa_{m+1}$, and so

(VII) $$\left|Y \cap \bigcup \{\varkappa_{m+1} \times \{r\}:\ \alpha_r < \varkappa_{m+1}\}\right| \leqslant 2^{\aleph_0}\cdot\varkappa_m .$$

Putting together (VI) and (VII),

$$y = |Y| \leqslant \varkappa_m\cdot 2^{\aleph_0}+\varkappa_{m+1} .$$

Therefore $y+\varkappa_n \leqslant \varkappa_m \cdot 2^{\aleph_0}+\varkappa_n$. Hence $\varkappa_m \cdot 2^{\aleph_0}+\varkappa_n$ adj $\varkappa_{m+1} \cdot 2^{\aleph_0}+\varkappa_n$ as desired.

The fact that x_1 adj$_3$ x_1^2 follows from (iii) of the next theorem, since

$$x_1 = \varkappa \cdot 2^{\aleph_0}+\aleph(\varkappa \cdot 2^{\aleph_0}) \quad \text{and} \quad x_1^2 = \varkappa \cdot 2^{\aleph_0} \cdot \aleph(\varkappa \cdot 2^{\aleph_0}).$$

THEOREM 7. (i) *If $\varkappa$ is well-ordered and $\varkappa$ adj $x = x^2$, then $\varkappa$ adj$_3$ x.*

(ii) *If x adj 2^x then x adj$_3$ 2^x.*

(iii) *If $x+\aleph(x)$ adj $x \cdot \aleph(x)$ then $x+\aleph(x)$ adj$_3$ $x \cdot \aleph(x)$.*

Proof. (i) Let $y \leqslant x$. Then $\varkappa \leqslant \varkappa+y \leqslant x+y \leqslant x+x = x$. Therefore $\varkappa+y = \varkappa$ or x.

If $\varkappa+y = \varkappa$, $y \leqslant \varkappa$ as desired. If $\varkappa+y = x = x^2$, Lemma 10 gives $\varkappa \geqslant x$ or $y \geqslant x$. $\varkappa \geqslant x$ contradicts $\varkappa$ adj x, so $x = y$ as desired.

(ii) $x \leqslant x+1 \leqslant x+2 \leqslant 2^x$. Since x adj 2^x, $x = x+1$ or $x+1 = x+2$. Each of these implies $x = x+1$ (if $x+1 = x+2$, we have $1+x = 1+ +(x+1)$, so by Lemma 1 there are p, q, r with $1 = 1+p = 1+q$, $x = p+r$, $x+1 = q+r$, etc.)

$$x \leqslant 2x \leqslant 3x \leqslant 4 \cdot 2^x = 2^{x+2} = 2^x.$$

Again, $x = 2x$ or $2x = 3x$. By Lemma 6 each of these implies $x = 2x$.

Now suppose that $y \leqslant 2^x$. Then $x \leqslant x+y \leqslant x+2^x = 2^x$. If $x = x+y$, $y \leqslant x$ as desired. If $x+y = 2^x = 2^x \cdot 2^x$, Lemma 10 gives us

$$2^x \leqslant^* x \quad \text{or} \quad 2^x \leqslant y.$$

The first is impossible by Cantor's Theorem, so the second holds, which is what was wanted.

(iii) As in (ii) we just need to show that if $x+\aleph(x)+y = x \cdot \aleph(x)$ then

$$y = x \cdot \aleph(x) \quad \text{or} \quad y \leqslant x+\aleph(x).$$

By Lemma 10, $x \cdot \aleph(x) = (x \cdot \aleph(x)) \cdot \aleph(x)$, and the fact that $x \cdot \aleph(x) \nleqslant x+\aleph(x)$, we get $\aleph(x) \leqslant y$. Therefore $x+\aleph(x)+y = x+y = x \cdot \aleph(x)$. Again using Lemma 10, and $x \ngeqslant \aleph(x)$, we get $x \leqslant y$.

As $y \geqslant \aleph(x)$ let $y = z+\aleph(x)$. Then $x+\aleph(x) \leqslant y+\aleph(x) = z+\aleph(x) = y \leqslant x \cdot \aleph(x)$, giving the desired conclusion by $x+\aleph(x)$ adj $x \cdot \aleph(x)$.

Now Theorem 6 showed that Solovay's model contains an x such that $x+\aleph(x)$ adj $x \cdot \aleph(x)$ (namely $\varkappa \cdot 2^{\aleph_0}$). We finish by showing that this holds under slightly more general circumstances, for example in $\mathfrak{N}$ of Theorem 5.

THEOREM 8. *If $\varkappa$ adj$_3$ x and $\mathrm{cf}\,\varkappa^+ \nleqslant^* x$, where $\varkappa$ is well-ordered, then $x \cdot \varkappa+\varkappa^+$ adj $x \cdot \varkappa^+$.*

Proof. Let $|X| = x$ and $Y \subset X \times \varkappa^+$ be a subset of $X \times \varkappa^+$ of cardinal y such that for each $\xi \in X$, $Y \cap (\{\xi\} \times \varkappa^+)$ is an initial segment of $\{\xi\} \times \varkappa^+$.

Let the order-type of $Y \cap (\{\xi\} \times \varkappa^+)$ be α_ξ. Let

$$X_1 = \{\xi \in X\colon\ \alpha_\xi = \varkappa^+\} \quad \text{and} \quad X_2 = \{\xi \in X\colon\ \alpha_\xi < \varkappa^+\}.$$

If $|X_1| = x$, $y = x \cdot \varkappa^+$. If $|X_1| < x$, $|X_1| \leqslant \varkappa$, so $|\bigcup \{\{\xi\} \times \varkappa^+\colon\ \xi \in X_1\}| = |X_1 \times \varkappa^+| \leqslant \varkappa^+$.

Since $\mathrm{cf}\,\varkappa^+ \nleqslant^* x$, the values of α_ξ on X_2 are bounded below $\varkappa^+$. Therefore $|Y| \leqslant \varkappa^+ + x \cdot \varkappa$, so

$$y \leqslant x \cdot \varkappa^+ \rightarrow y = x \cdot \varkappa^+ \quad \text{or} \quad y \leqslant x \cdot \varkappa + \varkappa^+.$$

It remains to show that $x \cdot \varkappa+\varkappa^+ < x \cdot \varkappa^+$. Now $x \ngeqslant \varkappa^+$, since $\mathrm{cf}\,\varkappa^+ \leqslant \varkappa^+$ and $\mathrm{cf}\,\varkappa^+ \nleqslant^* x$. Therefore $\aleph(x) = \varkappa^+$.

If $x \cdot \varkappa+\varkappa^+ = x \cdot \varkappa^+$, by Lemma 10,

$$\varkappa^+ \leqslant x \cdot \varkappa \quad \text{or} \quad x \leqslant \varkappa^+.$$

The first is impossible, as $\aleph(\varkappa^+) > \varkappa^+ = \aleph(x \cdot \varkappa)$, and the second has just been ruled out.

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