O est donc de dimension supérieure à deux; il est de dimension trois car il est intersection des trois ordres totaux suivants (écrits sous forme de permutations):

$$T_1:\ o\,a_1\,a_2\,a_3\,b_2\,b_1\,c_2\,d_2\,c_3\,c_1\,f_2\,d_3\,e_2\,b_3\,u\ ,$$
$$T_2:\ o\,c_1\,e_2\,f_2\,d_3\,b_1\,d_2\,c_3\,c_2\,a_1\,b_2\,b_3\,a_2\,a_3\,u\ ,$$
$$T_3:\ o\,a_1\,b_1\,c_1\,b_2\,c_2\,e_2\,b_3\,a_2\,a_3\,d_2\,c_3\,f_2\,d_3\,u\ .$$

On vérifie que $O = T_1 \cap T_2 \cap T_3$.

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Reçu par la Rédaction le 30. 5. 1971

Retracts of Tychonoff and normal spaces

by

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Abstract. Let $\mathcal{C}$ be a class of topological spaces. A space X is called an *absolute retract for the class* $\mathcal{C}$ (written $X \in \mathrm{AR}(\mathcal{C})$) if

(1) X can be embedded as a closed subset of some element of $\mathcal{C}$ and

(2) if $h\colon X \to Y$, where $Y \in \mathcal{C}$, is an embedding such that $h[X]$ is closed in Y, then $h[X]$ is a retract of Y.

If the word retract in the above definition is replaced by the word neighborhood retract, then X is called an *absolute neighborhood retract for the class* $\mathcal{C}$ (written $X \in \mathrm{ANR}(\mathcal{C})$). A neighborhood retract of a space is always assumed to be a closed subset of that space. Let $\mathcal{T}$ denote the class of Tychonoff spaces and $\mathcal{N}$ the class of normal spaces. $\mathrm{ANR}(\mathcal{T})$ [$\mathrm{AR}(\mathcal{T})$] are precisely the "almost" neighborhood retracts [resp., "almost" retracts] of Tychonoff cubes. In a compact Hausdorff setting, $\mathrm{ANR}(\mathcal{T})$ [$\mathrm{AR}(\mathcal{T})$] are precisely the neighborhood retracts [resp., retracts] of Tychonoff cubes. Similar statements hold for the normal case. In a locally compact Hausdorff setting, $\mathrm{ANR}(\mathcal{T})$ are precisely the retracts of open subsets of Tychonoff cubes. Results are obtained concerning contractibility of $\mathrm{ANR}(\mathcal{T})$ [$\mathrm{AR}(\mathcal{T})$], and subsets and products of $\mathrm{ANR}(\mathcal{T})$ and $\mathrm{ANR}(\mathcal{N})$.

1. Introduction. Let $\mathcal{C}$ be a class of topological spaces. A space X is called an *absolute retract for the class* $\mathcal{C}$ (written $X \in \mathrm{AR}(\mathcal{C})$) if

(1) X can be embedded as a closed subset of some element of $\mathcal{C}$ and

(2) if $h\colon X \to Y$, where $Y \in \mathcal{C}$, is an embedding such that $h[X]$ is closed in Y, then $h[X]$ is a retract of Y.

If the word retract in the above definition is replaced by the word neighborhood retract, then X is called an *absolute neighborhood retract for the class* $\mathcal{C}$ (written $X \in \mathrm{ANR}(\mathcal{C})$). Note that a neighborhood retract of a space is always assumed to be a closed subset of that space. Note also in the above definition that if $\mathcal{C}$ is a weakly hereditary class, then condition (1) is equivalent to $X \in \mathcal{C}$. Finally note that if $\mathcal{C} \subset \mathcal{C}'$, $\mathcal{C}$ is a weakly hereditary class, and $X \in \mathcal{C}$, then $X \in \mathrm{ANR}(\mathcal{C}')$ implies $X \in \mathrm{ANR}(\mathcal{C})$.

In order to determine whether or not a space is an absolute retract, one must check all embeddings of X into each element of $\mathcal{C}$. One of the

(1) The author is indebted to Professor R. Richard Summerhill for bringing this problem to his attention and for helpful discussions relating to this problem.

major problems in the study of retracts is to reduce the number of embeddings one need observe.

There are special cases where one need only consider one embedding, namely when certain embedding and extension theorems are known. Let $\mathfrak{D}$ be a nonempty subclass of $\mathfrak{C}$ and suppose the following theorems are known about the class $\mathfrak{C}$.

EXTENSION THEOREM. *If $X \in \mathfrak{C}$, $Y \in \mathfrak{D}$, and A is a closed subset of X, then any map $f\colon A \to Y$ extends to a map $F\colon X \to Y$.*

EMBEDDING THEOREM. *If $X \in \mathfrak{C}$, then X can be embedded as a closed subset of some element $Y \in \mathfrak{D}$.*

We observe that these powerful facts yield the following useful characterization of $\mathrm{ANR}(\mathfrak{C})$ and $\mathrm{AR}(\mathfrak{C})$.

THEOREM 0. *$X \in \mathrm{ANR}(\mathfrak{C})[X \in \mathrm{AR}(\mathfrak{C})]$ if and only if X can be embedded as a neighborhood retract* [resp., *retract*] *of some element of $\mathfrak{D}$.*

Theorem 0 is a schema of well known results concerning particular classes of spaces, e.g. when $\mathfrak{C}$ is the class of metric spaces and $\mathfrak{D}$ the class of all convex subsets of normed linear spaces [see [1] and [5]), or when $\mathfrak{C}$ is the class of compact Hausdorff spaces and $\mathfrak{D}$ is the class of Tychonoff cubes (see [3]).

It is the goal of this paper to prove a result somewhat similar to Theorem 0 for the class of locally compact Tychonoff spaces (i.e., completely regular, Hausdorff) and, in general, to examine retracts of Tychonoff and normal spaces. Section 2 contains a characterization of AR and ANR for the class $\mathcal{T}$ of Tychonoff spaces, a partial characterization of AR and ANR for the class $\mathcal{N}$ of normal spaces, and a characterization of Tychonoff cubes. In section 3, we prove two contractibility results, in section 4 we study properties of subsets and products of $\mathrm{ANR}(\mathcal{T})$ and $\mathrm{ANR}(\mathcal{N})$, and in section 5 we examine locally compact Tychonoff spaces.

2. Characterization theorems. Let I denote the unit interval. An arbitrary product of unit intervals is called a *Tychonoff cube*. If T is such a cube, we say a set $\mathcal{A}$ *indexes* T if $T = \prod_{a\in\mathcal{A}} I_a$, where $I_a = I$ for each $a \in \mathcal{A}$. The symbol $\{O\}$ denotes the point in T with a zero in each slot.

We now show that retracts of Tychonoff spaces are "almost" retracts of Tychonoff cubes (see [4]). The main obstacle is that such a space need not be a closed subset of a Tychonoff cube. Note that any Tychonoff space can be embedded in some Tychonoff cube T.

THEOREM 1. *Let X be a Tychonoff space. Then $X \in \mathrm{ANR}(\mathcal{T})[X \in \mathrm{AR}(\mathcal{T})]$ if and only if for each pair of cubes (T, T') and each embedding $h\colon X \to T$, the set $h[X]\times\{O\}$ is a neighborhood retract* [resp. *retract*] *of $(h[X]\times\{O\}) \cup \big((T\times T')-(T\times\{O\})\big)$.*

Proof. We prove only the neighborhood retract case. If $X \in \mathrm{ANR}(\mathcal{T})$, then the condition holds because $h[X]\times\{O\}$ is a closed subset of the Tychonoff space $(h[X]\times\{O\}) \cup \big((T\times T')-(T\times\{O\})\big)$. Conversely, suppose X W.L.O.G. is a closed subset of a Tychonoff space Y and let $h\colon Y \to T$ be an embedding for some cube T. For each $x \in Y-X$, let $f_x\colon Y \to I$ be a map such that $f_x[X] = 0$ and $f_x(x) = 1$, and let T' be the cube indexed over $Y-X$. Then define $\varrho\colon Y \to T'$ by $\varrho(y) = \{f_x(y)\}_{x\in Y-X}$. Note that ϱ is a map and $\varrho(x) = \{O\}$ if and only if $x \in X$. Now T, T' and $h|X$ satisfy the hypotheses of the theorem, so there is an open set U' in $(h[X]\times\{O\}) \cup \big((T\times T')-(T\times\{O\})\big)$ and a retraction $r'\colon U' \to h[X]\times\{O\}$. Define $F\colon Y \to (h[X]\times\{O\}) \cup \big((T\times T')-(T\times\{O\})\big)$ by $F(y) = \big(h(y), \varrho(y)\big)$ and let π denote the homeomorphism from $h[X]\times\{O\}$ to X. Then $U = F^{-1}[U']$ is open in Y and contains X and $r = \pi r' F|U\colon U \to X$ is a retraction. Q.E.D.

We now give a partial characterization relating retracts of normal spaces and "almost" retracts of Tychonoff cubes.

THEOREM 2. *Let X be a normal space and let $h\colon X \to T$ be an embedding of X into a Tychonoff cube T. If $X \in \mathrm{ANR}(\mathcal{T})[X \in \mathrm{AR}(\mathcal{T})]$, then $X \in \mathrm{ANR}(\mathcal{N})$* [resp., *$X \in \mathrm{AR}(\mathcal{N})$*] *and $h[X]\times\{O\}$ is a neighborhood retract* [resp., *retract*] *of $(h[X]\times\{O\}) \cup \big((T\times T')-(T\times\{O\})\big)$ for each Tychonoff cube T'. Conversely, if $h[x]\times\{O\}$ is a neighborhood retract* [*retract*] *of $(h[X]\times\{O\}) \cup \big((T\times T')-(T\times\{O\})\big)$ for each Tychonoff cube T', then $X \in \mathrm{ANR}(\mathcal{N})$* [resp., *$X \in \mathrm{AR}(\mathcal{N})$*].

Proof. The first part of the theorem follows from Theorem 1 and the remarks in Section 1. For the converse, we refer ourselves to the second part of the proof of Theorem 1 and note that it is not necessary to have h a homeomorphism on $Y-X$, and hence if h is defined only on X and then extended by Tietze's theorem to a map from Y to T, the proof follows as in the proof of Theorem 1. Q.E.D.

In compact Hausdorff spaces, Theorem 1 yield the following corollary.

COROLLARY 3. *Let $X \in \mathfrak{CH}$ (the class of compact Hausdorff spaces). Then the following are equivalent:*

(1) $X \in \mathrm{ANR}(\mathcal{T})$ [$X \in \mathrm{AR}(\mathcal{T})$];

(2) $X \in \mathrm{ANR}(\mathcal{N})$ [resp., $X \in \mathrm{AR}(\mathcal{N})$];

(3) $X \in \mathrm{ANR}(\mathfrak{CH})$ [resp., $X \in \mathrm{AR}(\mathfrak{CH})$];

(4) *X can be embedded as a neighborhood retract* [resp., *retract*] *of some Tychonoff cube.*

Proof. $(1)\Rightarrow(2)$ and $(2)\Rightarrow(3)$ follow from Section 1 and $(3)\Leftrightarrow(4)$ in Theorem 0 for $\mathfrak{CH}$. For $(3)\Rightarrow(1)$, let (T, T') be any pair of cubes and $h\colon X \to T$ an embedding. By definition of $\mathrm{ANR}(\mathfrak{CH})$, there is an open set U' in $T\times T'$ containing $h[X]\times\{O\}$ and a retraction $r'\colon U' \to h[X]\times\{O\}$. Then $U = U' \cap \big[(h[X]\times\{O\}) \cup \big((T\times T')-(T\times\{O\})\big)\big]$ and $r = r'|U$

are the desired open set and retraction, respectively. By Theorem 1, $X \in \mathrm{ANR}(\mathcal{C})$. Q.E.D.

COROLLARY 4. *Every Tychonoff cube is an* $\mathrm{AR}(\mathcal{C})$.

Open question. Given the hypotheses of Theorem 2, is it true that $X \in \mathrm{ANR}(\mathcal{N})$ $[X \in \mathrm{AR}(\mathcal{N})]$ implies $h[X]\times\{O\}$ is a neighborhood retract [resp., retract] of $(h[X]\times\{O\}) \cup \big((T\times T')-(T\times\{O\})\big)$ for each Tychonoff cube T'?

An affirmative answer would make Theorem 2 a complete characterization. This question is more complex than one might expect. Calling the condition of the above question P, we say that a *topological space* X *has property* Q if there exists a cube T and an embedding $h\colon X\to T$ such that for each cube T', $(h[X]\times\{O\}) \cup \big((T\times T')-(T\times\{O\})\big)$ is normal. Clearly, Theorem 2 is a complete characterization for all spaces having Q. This class of spaces is very restrictive. In fact, many $\mathrm{ANR}(\mathcal{N})$ do not have Q, even though some still satisfy P. We need the following lemma.

LEMMA 5. $I^I-\{O\}$ *is not normal, where* $I^I = \{f\colon I\to I,$ *where f is a function*$\}$ *has the standard product topology.*

Proof ([2]). Let $A = \{x_m^n \mid x_m^n(t) = 0,\ t\neq m,\ x_m^n(m) = 1/n,\ m\in I,$ and n is a positive integer$\}$, and $B = \{y^n \mid y^n(t) = 1/n,\ t\in I,$ and n a positive integer$\}$. Clearly A and B are nonempty, closed, disjoint subsets of $I^I-\{O\}$. Let W be any open set in $I^I-\{O\}$ containing B. Let $y_n\in B$, and let U_n be a basic open set such that $y_n \in U_n \subset W$. Let U_t^n denote the projection of U_n into the t-slot. Now we can choose $\{t_n\}\subset I$ such that if $s\in I-\{t_n\}$, then $U_s^n = I$. Let V be an open set in $I^I-\{O\}$ containing A, and let U_s be a basic open set such that $x_s^1 \in U_s \subset V$. Then U_s^n must intersect U_s. From this it can be verified that V must intersect W. Q.E.D.

THEOREM 6. *Let X be a topological space. Then X has Q if and only if X is a Tychonoff cube.*

Proof. For necessity, suppose X is not a Tychonoff cube, let T be any cube and $h\colon X\to T$ any embedding. Choose $T' = I^I$. Consider $k\in T-h[X]$. Then

$$\{k\}\times(I^I-\{O\}) \subset (T\times I^I)-(T\times\{O\})$$

and

$$\{k\}\times\{O\} \notin (h[X]\times\{O\}) \cup \big((T\times I^I)-(T\times\{O\})\big).$$

Define $C = \{k\}\times A$, $D = \{k\}\times B$, where A and B are the sets constructed in the proof of Lemma 5, and let U, V be arbitrary open sets in $(h[X]\times\{O\}) \cup \big((T\times I^I)-(T\times\{O\})\big)$ containing the nonempty, disjoint, closed sets C and D, respectively. Suppose $U\cap V = \emptyset$. Let U' and V' be the projection of U and V into I^I, respectively. Then U' and V' are disjoint, open sets in $I^I-\{O\}$ separating A from B, which is absurd by Lemma 5. Sufficiency can be verified in the obvious way. Q.E.D.

([2]) This proof is due to Professor Euline Green and is given here with his kind permission.

Thus Theorem 2 is a complete characterization for the class of Tychonoff cubes, which are already known to be $\mathrm{AR}(\mathcal{N})$. However, from Corollary 4, Theorem 6, and Theorem 12 (Section 4), we have

COROLLARY 7. *Every proper, open subset of any countably indexed Tychonoff cube is an* $\mathrm{ANR}(\mathcal{N})$, *but does not have* Q.

Indeed, if we let $X = [0,1)$, $T = I$, $h\colon X\to T$ by $h(x) = x$, and T' be any Tychonoff cube, it can be verified that $h[X]\times\{O\}$ is a retract of $(h[x]\times\{O\}) \cup \big((T\times T')-(T\times\{O\})\big)$. Thus X satisfies P, but $X\in\mathrm{ANR}(\mathcal{N})$ and does not have Q.

3. Contractibility. Suppose $X\in\mathrm{AR}(\mathcal{C})$ and W.L.O.G. let T be a Tychonoff cube containing X. By Theorem 1 there is a retraction $r\colon (X\times\{O\}) \cup \big(T\times(0,1]\big) \to X\times\{O\}$. Let p be a point of T and define $h\colon X\times I\to X$ by $h(x,t) = \pi r\big(tp+(1-t)x, t\big)$, where $\pi\colon T\times\{O\}\to T$ is the projection. Then for each $x\in X$, $h(x,0) = x$, $h(x,1) = \pi r(p,1)$. We have proved

THEOREM 8. *If $X\in\mathrm{AR}(\mathcal{C})$, then X is contractible.*

THEOREM 9. *If $X\in\mathrm{ANR}(\mathcal{C})$, then X is locally contractible.*

Proof. As before, we assume X W.L.O.G. is a subset of a cube T and let U be open in $(X\times\{O\}) \cup \big(T\times(0,1]\big)$ and retracting to $X\times\{O\}$ by r. Let U' be open in $T\times I$ so that $U = U' \cap \big((X\times\{O\}) \cup (T\times(0,1])\big)$. Suppose V is open in T and contains a point x of X. We create a neighborhood of x in X which shrinks in $V\cap X$. W.L.O.G., $V\subset U'$. Then there is a compact neighborhood K of x in T which is contained in V and an $\varepsilon > 0$ such that $K\times[0,\varepsilon]\subset U'$. W.L.O.G., K is a Tychonoff cube, so there is a map $m\colon K\times I\to K$ such that $m(k,0) = k$ and $m(k,1)$ is a point p. Now define $h\colon (K\cap X)\times I\to V\cap X$ by $h(x,t) = \pi r\big(m(x,t), \varepsilon t\big)$, where $\pi\colon X\times\{O\}\to X$ is the projection. Then h is the desired contraction. Q.E.D.

4. Subsets and products. In this section, aided principally by Theorem 2, we determine what subsets of $\mathrm{ANR}(\mathcal{C})$ $[\mathrm{AR}(\mathcal{C})]$ are $\mathrm{ANR}(\mathcal{N})$ [resp., $\mathrm{AR}(\mathcal{N})$]. We then determine when a standard product theorem holds for $\mathrm{ANR}(\mathcal{N})$.

THEOREM 10. *Let $X \to \mathrm{ANR}(\mathcal{C})$ and let A be a normal subset of X. If A is a neighborhood retract of X, then $A\in\mathrm{ANR}(\mathcal{N})$.*

Proof. Let $h\colon X\to T$ be an embedding of X into a Tychonoff cube T and let T' be any Tychonoff cube. Then by Theorem 1 we have an open set V in $(h[X]\times\{O\}) \cup \big((T\times T')-(T\times\{O\})\big)$ containing $h[X]\times\{O\}$ and a retraction $r_1\colon V\to h[X]\times\{O\}$. By hypothesis, we have an open set W

in $h[X]\times\{O\}$ containing $h[A]\times\{O\}$ and a retraction r_2: $W\to h[A]\times\{O\}$. Now A is a normal space, $h[A]\times\{O\}$ is closed in $(h[A]\times\{O\})\cup\big((T\times T')-(T\times\{O\})\big)$, and $U=r_1^{-1}(W)\cap\big|(h[A]\times\{O\})\cup\big((T\times T')-(T\times\{O\})\big)\big|$ and $r=r_2r_1|U$: $U\to h[A]\times\{O\}$ are the required open set and retraction, respectively. By Theorem 2, $A\in \mathrm{ANR}(\mathcal{N})$. Q.E.D.

Using the same technique, one may prove a variation of Theorem 10.

THEOREM 11. *Let $X\in \mathrm{ANR}(\mathcal{T})$ [$X\in \mathrm{AR}(\mathcal{T})$] and let A be a normal subset of X. If A is a retract of X, then $A\in \mathrm{ANR}(\mathcal{N})$* [resp., $A\in \mathrm{AR}(\mathcal{N})$].

In the next theorem we drop the retract restriction on A and obtain a partial analog of a theorem of Hanner for the metric case (see [1]).

THEOREM 12. *Let $X\in \mathrm{ANR}(\mathcal{T})$ and let A be a normal subset of X. If A is open in X, then $A\in \mathrm{ANR}(\mathcal{N})$.*

Proof. Let h: $X\to T$ be an embedding of X into a Tychonoff cube T and an open set U' in $(h[X]\times\{O\})\cup\big((T\times T')-(T\times\{O\})\big)$ retracting to $h[X]\times\{O\}$ by r', where T' is any Tychonoff cube. Now $h[A]\times\{O\}$ is open in $h[X]\times\{O\}$. We have A is a normal space, $h[A]\times\{O\}$ is closed in $(h[A]\times\{O\})\cup\big((T\times T')-(T\times\{O\})\big)$, and $U=(r')^{-1}(h[A]\times\{O\})\cap\big|(h[A]\times\{O\})\cup\big((T\times T')-(T\times\{O\})\big)\big|$ and $r=r'|U$: $U\to h[A]\times\{O\}$ are the required open set and retraction, respectively. By Theorem 2, $A\in \mathrm{ANR}(\mathcal{N})$. Q.E.D.

We now consider a standard product theorem. Let X_α be a topological space for each $\alpha\in\mathcal{A}$, let $\prod_{\alpha\in\mathcal{A}} X_\alpha$ be endowed with the product topology, and let basic open sets be denoted by $\langle U_{\alpha_1},\dots,U_{\alpha_n}\rangle$ (see [2]). Also let π_α be the projection to the α-slot and i_α be the injection of the α-slot. We need the following lemma relating products and local contractibility.

LEMMA 13. *Let X_α be a topological space for each $\alpha\in\mathcal{A}$. Then $\prod_{\alpha\in\mathcal{A}} X_\alpha$ is locally contractible if and only if X_α is locally contractible for each $\alpha\in\mathcal{A}$ and X_α is contractible for all but finitely many $\alpha\in\mathcal{A}$.*

Proof. Necessity. We first show that each X_α is locally contractible. Let $\beta\in\mathcal{A}$, $x_\beta\in X_\beta$, $U(x_\beta)$ be an open set in X_β about x_β, $y\in\prod_{\alpha\neq\beta} X_\alpha$, $x=\{x_\beta\}\times\{y\}$, and $U(x)=U(x_\beta)\times\prod_{\alpha\neq\beta} X_\alpha$. Then there is an open set $V(x)$ about x such that $V(x)\subset U(x)$, and a homotopy h: $V(x)\times I\to U(x)$ such that the identity on $V(x)$ is homotopic to a point $z\in U(x)$ by h. Now $x_\beta\in\pi_\beta[V(x)]\subset U(x_\beta)$. Define h_β: $\pi_\beta[V(x)]\times I\to U(x_\beta)$ by $h_\beta=\pi_\beta hi$, where i: $\pi_\beta[V(x)]\times I\to V(x)\times I$ by $i(s,t)=(i_\beta(s),t)$. The identity on $\pi_\beta[V(x)]$ is homotopic to the point $z_\beta\in U(x_\beta)$ by h_β. To see that all but finitely many of the X_α are contractible, let $x\in\prod_{\alpha\in\mathcal{A}} X_\alpha$, $U(x)$ and $V(x)$ be open such that $x\in V(x)\subset U(x)$, and let h: $V(x)\times I\to U(x)$ be a homotopy such that the identity on $V(x)$ is homotopic to a point $z\in U(x)$. We assume W.L.O.G. that $U(x)=\langle U_{\alpha_1},\dots,U_{\alpha_n}\rangle$ and $V(x)=\langle V_{\alpha_1},\dots,V_{\alpha_n}\rangle$. Define i: $X_\alpha\times I\to\prod_{\alpha\in\mathcal{A}} X_\alpha\times I$ by $i(s,t)=\big(i_\alpha(s),t\big)$ for each $\alpha\neq\alpha_1,\dots,\alpha_n$. Now define h_α: $X_\alpha\times I\to X_\alpha$ by $h_\alpha=\pi_\alpha hi$, for $\alpha\neq\alpha_1,\dots,\alpha_n$. The identity on X_α is homotopic to the point $z_\alpha\in X_\alpha$ by h_α.

Sufficiency. Let $x\in\prod_{\alpha\in\mathcal{A}} X_\alpha$ and let $x\in U(x)$, an open set in $\prod_{\alpha\in\mathcal{A}} X_\alpha$. We assume W.L.O.G. that $U(x)=\langle U_{\alpha_1},\dots,U_{\alpha_n}\rangle$ and that X_α is contractible for all $\alpha\neq\alpha_1,\dots,\alpha_n$. Let h_α: $X_\alpha\times I\to X_\alpha$ such that the identity on X_α is homotopic to a point $z_\alpha\in X$ by h_α, $\alpha\neq\alpha_1,\dots,\alpha_n$, and let h_{α_j}: $V_{\alpha_j}\times I\to U_{\alpha_j}$ such that the identity on V_{α_j} is homotopic to $z_{\alpha_j}\in U_{\alpha_j}$ by h_{α_j}, where $x_{\alpha_j}\in V_{\alpha_j}\subset U_{\alpha_j}$. Now $x\in V(x)=\langle V_{\alpha_1},\dots,V_{\alpha_n}\rangle\subset U(x)$. Define h: $V(x)\times I\to U(x)$ by $h(y,t)=\{h_\alpha(y,t)\}_{\alpha\in\mathcal{A}}$. The identity on $V(x)$ is homotopic to the point $\{z_\alpha\}_{\alpha\in\mathcal{A}}\in\prod_{\alpha\in\mathcal{A}} X_\alpha$ by h. Q.E.D.

We note that if X is a normal space and A is a neighborhood retract of X and is contractible, then A is a retract of X. A known consequence of this (see [5]) is

PROPOSITION 14. *If $X\in \mathrm{ANR}(\mathcal{N})$ and X is contractible, then $X\in \mathrm{AR}(\mathcal{N})$.*

It is not true that $\prod_{\alpha\in\mathcal{A}} X_\alpha\in \mathrm{AR}(\mathcal{N})$ if and only if each $X_\alpha\in \mathrm{AR}(\mathcal{N})$, since products of normal spaces need not be normal. However, if $\prod_{\alpha\in\mathcal{A}} X_\alpha$ is normal, the above is true (see [2]) and with this same restriction, we obtain a similar theorem for the neighborhood retract case.

THEOREM 15. *Let $\prod_{\alpha\in\mathcal{A}} X_\alpha$ be normal, where X_α is a topological space for each $\alpha\in\mathcal{A}$. If $\prod_{\alpha\in\mathcal{A}} X_\alpha\in \mathrm{ANR}(\mathcal{T})$, then $X_\alpha\in \mathrm{ANR}(\mathcal{N})$ for each $\alpha\in\mathcal{A}$, and $X_\alpha\in \mathrm{AR}(\mathcal{N})$ for all but finitely many $\alpha\in\mathcal{A}$. Conversely, if this condition holds, then $\prod_{\alpha\in\mathcal{A}} X_\alpha\in \mathrm{ANR}(\mathcal{N})$.*

Proof. For the first part of the theorem, we observe X_α is a normal space and, letting $\beta\in\mathcal{A}$ and $z\in\prod_{\alpha\neq\beta} X_\alpha$, $X_\beta\times\{z\}$ is a retract of $\prod_{\alpha\in\mathcal{A}} X_\alpha$. By Theorem 11, $X_\beta\times\{z\}\in \mathrm{ANR}(\mathcal{N})$. But then $X_\beta\in \mathrm{ANR}(\mathcal{N})$. To see that all but finitely many of X_α are $\mathrm{AR}(\mathcal{N})$, observe that $\prod_{\alpha\in\mathcal{A}} X_\alpha$ is locally contractible by Theorem 9. So by Lemma 13, X_α is contractible for all but finitely α. Thus by Proposition 14, we have $X_\alpha\in \mathrm{AR}(\mathcal{N})$ for all but finitely many α. The converse follows from 4.1 page 39, 3.1 page 83, and 3.2 page 84 of [5], observing that $\mathcal{N}$ is a weakly hereditary class. Q.E.D.

By Corollary 3 and the results of this section we have

COROLLARY 16. (1) *Let* $X \in \mathrm{ANR}(\mathcal{CH})$. *If* A *is a neighborhood retract of* X, *then* $A \in \mathrm{ANR}(\mathcal{CH})$.

(2) *Let* $X \in \mathrm{ANR}(\mathcal{CH})$ $[X \in \mathrm{AR}(\mathcal{CH})]$. *If* A *is a retract of* X, *then* $A \in \mathrm{ANR}(\mathcal{CH})$ [resp., $A \in \mathrm{AR}(\mathcal{CH})$].

(3) $\prod_{a \in \mathcal{A}} X_a \in \mathrm{ANR}(\mathcal{CH})$ $\left[\prod_{a \in \mathcal{A}} X_a \in \mathrm{AR}(\mathcal{CH})\right]$ *if and only if* $X_a \in \mathrm{ANR}(\mathcal{CH})$ *for each* $a \in \mathcal{A}$ *and* $X_a \in \mathrm{AR}(\mathcal{CH})$ *for all but finitely many* a [resp., $X_a \in \mathrm{AR}(\mathcal{CH})$ *for each* $a \in \mathcal{A}$].

5. Locally compact neighborhood retracts. In this section we prove that for locally compact Tychonoff spaces, the neighborhood retract concept is equivalent to a much stronger condition. Let $\mathcal{C}$ be a class of topological spaces. A space X is said to be a *strong absolute neighborhood retract for the class* $\mathcal{C}$ (written $X \in S\text{—}\mathrm{ANR}(\mathcal{C})$) providing condition (2) of our original condition can be replaced by the following weaker statement: if whenever $h\colon X \to Y$ is an embedding and $Y \in \mathcal{C}$, then $h[X]$ is a retract of some open set in Y (see [4] and [6] for the separable metric case). Note that for Hausdorff spaces, the "strong absolute retract" concept would be no different from the $\mathrm{AR}(\mathcal{C})$ condition.

The proof of the following lemma is analogous to that in [6].

LEMMA 17. *Let* X *be a Tychonoff space. Then* $X \in S\text{-}\mathrm{ANR}(\mathcal{T})$ *if and only if for each Tychonoff cube* T *and each embedding* $h\colon X \to T$, *there is an open set* U *in* T *and a retraction* $r\colon U \to h[X]$.

THEOREM 18. *Let* X *be a locally compact Tychonoff space. Then* $X \in \mathrm{ANR}(\mathcal{T})$ *if and only if* $X \in S\text{-}\mathrm{ANR}(\mathcal{T})$.

Proof. Sufficiency is obvious. For necessity, suppose $X \in \mathrm{ANR}(\mathcal{T})$ and let $h\colon X \to T$ be an embedding of X into some Tychonoff cube T. We create an open set U in T and a retraction $r\colon U \to h[X]$. From Theorem 1, there is an open set V in $(h[X] \times \{O\}) \cup (T \times (0,1])$ containing $h[X] \times \{O\}$ and a retraction $r'\colon V \to h[X] \times \{O\}$. Let $\Gamma = \overline{h[X]} - h[X] \subset T$. Since X is locally compact, Γ is closed in T (see [6]). Now if $\overline{h[X]} = T$, then $h[X]$ is open in T and $U = h[X]$ is the desired open set. Hence we may assume $\overline{h[X]} \neq T$. Let y be a point of $T - \overline{h[X]}$ and let $\varrho\colon T \to I$ be a map such that $\varrho(y) = 1$ and $\varrho\left[\overline{h[X]}\right] = 0$. Define $F\colon T - \Gamma \to (h[X] \times \{O\}) \cup (T \times (0,1])$ by $F(x) = \big(x, \varrho(x)\big)$ and let $U = F^{-1}[V]$. Then U is open in $T - \Gamma$, and hence in T, and $r = \pi r' F | U\colon \to h[X]$ is a retraction, where $\pi\colon h[X] \times \{O\} \to h[X]$ is the projection. By appeal to Lemma 17, $X \in S\text{-}\mathrm{ANR}(\mathcal{T})$. Q.E.D.

Lemma 17 and Theorem 18 yield a corollary that is somewhat similar to Theorem 0 stated in section 1 and in compact Hausdorff spaces reduces to Corollary 3.

COROLLARY 19. *Let* X *be a locally compact Tychonoff space. Then* $X \in \mathrm{ANR}(\mathcal{T})$ *if and only if for each Tychonoff cube* T *and each embedding* $h\colon X \to T$, $h[X]$ *is a retract of some open set in* T.

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Reçu par la Rédaction le 11. 8. 1971