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where h is canonical isomorphism and $g(s^{-1}r\otimes a)=r\otimes a$. Hence $\ker f_s=\ker(hgf_s)=\{a\in A|\ sa=0\}\subseteq T(A),\ \text{and}\ \text{evidently,}\ \bigcup_{s\in S}\ker f_s=T(A).$ Thus

$$T(A) = \lim(\ker f_s) \cong \ker f \cong \operatorname{Tor}_1(K, A)$$
.

3.4. Proposition. For every module A there exists an S-module.

Proof. Write $A_s = R' \otimes A$ and consider the map $\varphi \colon A \to A_s$ ($\varphi(a) = 1 \otimes a$, $a \in A$). Since A_s is S-divisible, therefore $A' = A_s/T(A_s)$ is S-torsion free and S-divisible.

If ψ_A is the combined map $A \stackrel{\varphi}{\to} A_s \stackrel{g}{\to} A'$, where g is the natural homomorphism, then $\psi_A(a) = 0$ implies that $\varphi(a) \in T(A_s)$, i.e., $\sigma \varphi(a) = 0$ for some $\sigma \in S$. Hence by 3.3 $\sigma a \in T(A)$, and it is clear that $\ker \psi_A = T(A)$.

Now every element $x \in A'$ can be written as $g(s^{-1} \otimes a)$, so that $sx = \psi_A(a)$. This completes the proof.

It follows that, if A is S-torsion module, then its S-module is zero.

3.5. Proposition. If $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is an exact sequence of modules, then the sequence

$$0 \rightarrow A' \rightarrow B' \rightarrow C' \rightarrow 0$$

is exact, where (ψ_A, A') , (ψ_B, B') , and (ψ_C, C') are S-modules of A, B, and C respectively.

Proof. The exactness of the given sequence implies the exactness of

$$0 \to A_s \stackrel{f^*}{\to} B_s \stackrel{g^*}{\to} C_s \to 0$$
 $(f^* = \mathbf{1}_{R'} \otimes f, \text{ and } g^* \text{ similarly}).$

Since A_s is S-divisible, therefore by 2.1 we obtain the exact sequence $0 \to A' \xrightarrow{\overline{f^*}} B' \xrightarrow{\overline{f^*}} C' \to 0$.

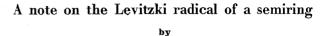
In view of 3.5 and the remark at the end of 3.4 it follows that $A' \cong (A/T(A))'$ for any module A. Furthermore, it is evident that A' is a covariant exact functor of A.

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Abstract. In this paper the authors prove that the Levitzki radical of an arbitrary semiring S is necessarily a k-ideal of S. A preliminary lemma states that if I is a locally nilpotent ideal of a semiring S then the closure of I is also a locally nilpotent ideal of S. These results strengthen certain of those obtained by E. Barbut [1].

- 1. A set S with two binary operations + and \cdot is called a *semiring* if (R, +) is a commutative semigroup with zero, (R, \cdot) is a semigroup, and both the left and right distributive laws hold for multiplication over addition. It is also required that $0 \cdot x = x \cdot 0 = 0$ for all $x \in S$. A nonempty subsemiring I is called a *right ideal of* S if for all $x \in I$ and $r \in S$, $xr \in I$. Left ideals and (two-sided) ideals are defined in a similar manner. An ideal I of S is called a k-ideal of S [5] if $x + y \in I$ and $y \in I$ implies $x \in I$ for each $x, y \in S$.
- E. Barbut [1] defined the Levitzki radical of a semiring and could prove many results concerning this radical providing the Levitzki radical is a k-ideal. In this note we prove that this radical is necessarily a k-ideal which strengthens many of Barbut's results.
- 2. If I is an ideal of the semiring S, the quotient semiring S/I is the one defined by S. Bourne [2] where for $a, b \in S$,

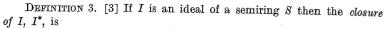
 $a \equiv b \pmod{I}$ iff there exists $i_1, i_2 \in I$ such that $a + i_1 = b + i_2$.

DEFINITION 1. A semiring S is called *locally nilpotent* if every finite subset F of S generates a nilpotent subsemiring of S, or equivalently, if for each finite subset F of S there exists a positive integer N_F such that every product of N_F elements from F is zero.

DEFINITION 2. [1] The Levitzki radical L(S) of a semiring S is the sum of all locally nilpotent ideals of S.

E. Barbut [1] has shown that L(S) is a locally nilpotent ideal of S which contains every locally nilpotent right or left ideal of S.

^(*) This paper is part of Dwight M. Olson's Ph. D. dissertation prepared under the direction of Professor Terry L. Jenkins at the University of Wyoming, Laramie.



$$I^* = \{x \in S: x + i \in I \text{ for some } i \in I\}.$$

It is known that if I is an ideal of S then I^* is an ideal of S. One can also see from the definition that $I \subseteq I^*$, $(I^*)^* = I^*$ and that I is a k-ideal of S if and only if $I = I^*$.

LEMMA 1. If I is a locally nilpotent ideal of the semiring S then I^* is also a locally nilpotent ideal of S.

Proof. We must show that for every finite subset F of I^* we can find a positive integer p such that any product of p elements from F is zero. We do so by induction on the number, n, of elements in F which are in I^*-I .

Any finite subset F of I^* which has n=0 elements from I^*-I is a subset of I which is locally nilpotent. Now assume for induction that for any finite subset of I^* with n=k elements from I^*-I that there exists a positive integer p such that any product of p elements from this subset is zero.

Let F be any finite subset of I^* with n=k+1 elements from I^*-I . Choose $x_1 \in F$ such that $x_1 \in I^*-I$ so that by definition of I^* $x_1+a=b$ for some $a, b \in I$. Then $F_1 = \{F - \{x_1\}\} \cup \{a, b\}$ is a finite subset of I^* with only n=k elements from I^*-I . By induction, there exists a positive integer p_0 such that any product of p_0 elements from F_1 is zero.

Let $F_2 = F_1 \cup \{x_1\} = F \cup \{a, b\}$. We show that any product of p_0 elements from F_2 is zero by induction on the number, m, of times x_1 occurs in the product.

Consider any product of p_0 elements from F_2 . If x_1 occurs m=0 times the product is entirely of elements from F_1 and thus must be zero. Now assume that any product of p_0 elements from F_2 in which x_1 occurs m=k times is zero. Any product of p_0 elements from F_2 in which x_1 occurs m=k+1 times can be written $A \cdot x_1 \cdot B$ where A and B are products of elements from F_2 . But $x_1+a=b$ for some $a,b \in I$ so

$$A \cdot x_1 \cdot B + A \cdot a \cdot B = A \cdot b \cdot B$$
.

However, $A \cdot a \cdot B$ and $A \cdot b \cdot B$ are products of p_0 elements from F_2 in which x_1 occurs m = k times and as a result must be zero. Consequently, $A \cdot x_1 \cdot B$ is zero.

By induction, then, any product of p_0 elements from F_2 is zero and since $F \subseteq F_2$, any product of p_0 elements from F is zero. Thus we have completed our original induction and we conclude that I^* is locally nilpotent.

THEOREM 1. For any semiring S, the Levitzki radical L(S) of S is a k-ideal of S.



Proof. L(S) is a locally nilpotent ideal of S ([4], p. 26) so by Lemma 1, $L(S)^*$ is locally nilpotent. However L(S) contains all locally nilpotent ideals of S. Thus $L(S)^* \subset L(S) \subset L(S)^*$ and L(S) is a k-ideal of S.

Theorem 1 renders E. Barbut's Lemma 11 [1] unnecessary and we may improve his Lemma 6, Theorem, and Corollary all in [1] as follows.

LEMMA. In a semiring R, L(R/L(R)) = 0.

Theorem. If R is a semiring which satisfies the ascending chain condition on left and right annihilators then any nil subsemiring of R is nilpotent.

COROLLARY. If R is a semiring satisfying the ascending chain condition on left and right k-ideals then any nil subsemiring of R is nilpotent.

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