# The number of automorphisms of models of $\aleph_1$-categorical theories

by

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**Abstract.** The number of automorphism of a model of a $\aleph_1$-categorical theory is computed in terms of a definability condition on the model. In addition the relationship between the cardinalities of the automorphism groups of various models of the same $\aleph_1$-categorical theory are considered.

Let $T$ be a complete first order theory having only infinite models. A theory $T$ is $\varkappa$-categorical if all models of $T$ with power $\varkappa$ are isomorphic. Morley ([6]) showed that every uncountable model of an $\aleph_1$-categorical theory $T$ is saturated. He further proved that if $T$ is not $\aleph_0$-categorical the countable models of $T$ can be arranged in an increasing sequence $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_\omega$ of order type $\omega+1$ such that $\mathcal{A}_i \precsim \mathcal{A}_{i+1}$ and $\mathcal{A}_\omega$ is saturated [5]. The members of this sequence are proved to be pairwise non-isomorphic in [1].

If $\mathcal{A}$ is a relational structure for a first order language $L$ we denote by $\mathrm{Aut}(\mathcal{A})$ the group of automorphisms of $\mathcal{A}$. In this paper we investigate the cardinality of $\mathrm{Aut}(\mathcal{A})$ ($|\mathrm{Aut}(\mathcal{A})|$) where $\mathcal{A}$ is a model of an $\aleph_1$-categorical theory. We show that if $\mathcal{A}$ is saturated and $|\mathcal{A}| = \varkappa$ then $|\mathrm{Aut}(\mathcal{A})| = 2^\varkappa$. If $\mathcal{A}$ is not saturated we show $|\mathrm{Aut}(\mathcal{A})| \leqslant \aleph_0$ or $|\mathrm{Aut}(\mathcal{A})| = 2^{\aleph_0}$. If $T$ is not $\aleph_0$-categorical we define a function $f$ from the natural numbers into the natural numbers union $\{\aleph_0, 2^{\aleph_0}\}$ by $f(n) = \alpha$ if $|\mathrm{Aut}(\mathcal{A}_n)| = \alpha$. We show this function is increasing and consider $\{n \mid f(n+1) > f(n)\}$. We assume familiarity with [1].

We have learned that Theorem 3 follows directly from a result of Kueker [4, Theorem 2.2 (i), (ii)]. It has also been obtained by purely algebraic means by K. K. Hickin. Both of these proofs do not involve any condition on the theory of the models involved. The definability conditions in Theorem 2 are the impact of assuming $T$ is $\aleph_1$-categorical.

With the following exceptions our notation follows [1]. The language of $T$ is $L$. We denote by $S_n(L)$ the set of formulas of $L$ which have at most $n$ free variables. For an arbitrary set $X$ we may form the language $L(X)$ by adjoining to $L$ a constant symbol $\underset{\sim}{x}$ for each member $x$ of $X$. If $X$ is a subset of the universe of an $L$-structure $\mathcal{A}$, then $(\mathcal{A}, x)_{x \in X}$ is the $L(X)$-structure where $\underset{\sim}{x}$ is the name of $x$. We denote by $S_n(X)$ the set of formulas of $L(X)$ with at most $n$ free variables. The Boolean algebra formed by identifying those members of $S_n(L)$ which are equivalent in $T$ is denoted $F_n(T)$. Let $X \subseteq |\mathcal{A}| = A$ (the universe of $\mathcal{A}$ and let $T' = \mathrm{Th}\big((\mathcal{A}, x)_{x \in X}\big)$. The Boolean algebra formed by identifying those members of $S_n(X)$ which are equivalent in $T'$ is denoted $F_n(X)$. We frequently identify a representative of an equivalence class with the class. A formula $\underset{\sim}{A} \in S_n(L)$ ($\underset{\sim}{A} \in S_n(X)$) is an atom if its equivalence class is an atom in the Boolean algebra $F_n(T)$ $F_n(X)$). $\mathcal{A}$ is an atomic model of $T$ if each $n$-tuple of elements of $|\mathcal{A}|$ satisfies an atom in $S_n(L)$.

We will frequently deal with finite or infinite sequences of elements from a model. We will not distinguish in our notation between the sequence and the set of elements in its range. For example, if $a \in A^n$ and $f$ is a function from $A$ to $A$, we write $f(a)$ for the sequence

$$\langle f(a_0), f(a_1), \ldots, f(a_{n-1})\rangle .$$

If $\mathcal{A}$ is an $L$-structure and $a, b$ are countable sequences of elements from $|\mathcal{A}|$, $a$ and $b$ realize the same type if $(\mathcal{A}, a)$ (that is $\big(\mathcal{A}, a(i)\big)_{i<\omega}$) is elementarily equivalent to $(\mathcal{A}, b)$. $\mathcal{A}$ is countably homogeneous if for each pair $a, b$ of countable sequences which realize the same type there is an automorphism $f$ of $\mathcal{A}$ such that $f(a) = b$.

THEOREM 1. *If $\mathcal{B}$ is a saturated model of an $\aleph_1$-categorical theory $T$ and $|\mathcal{B}| = \varkappa$ then $|\mathrm{Aut}(\mathcal{B})| = 2^{\varkappa}$.*

Proof. If $\varkappa > \aleph_0$, there is a model of $T$ with power $\varkappa$ having $2^{\varkappa}$ automorphisms by the Ehrenfeucht-Mostowski Theorem [8]. As $T$ is $\varkappa$-categorical the saturated model has $2^{\varkappa}$ automorphisms. By Lemma 9 of [1] there is a principal extension $T'$ of $T$ with a strongly minimal formula $\underset{\sim}{M}$. Let $\mathcal{B}'$ be the expansion of $\mathcal{B}$ to a model for $T'$. We first show the dimension of $\underset{\sim}{M}(\mathcal{B}')$ is $\aleph_0$. There is a countable model $\mathcal{C}$ of $T'$ with $\underset{\sim}{M}(\mathcal{C})$ having dimension $\aleph_0$. Thus, since $\mathcal{B}'$ is universal, the dimension of $\underset{\sim}{M}(\mathcal{B}')$ must be $\aleph_0$. By Lemma 8 of [1] $\mathcal{B}'$ is prime over a basis $X$ for $\underset{\sim}{M}(\mathcal{B}')$. Since $(\mathcal{B}', X)$ is a prime model of $\mathrm{Th}(\mathcal{B}', X)$ and $\mathrm{Th}(\mathcal{B}', X)$ is $\aleph_1$-categorical, $(\mathcal{B}', X)$ is a minimal model of $\mathrm{Th}(\mathcal{B}', X)$ so each elementary permutation of $X$ extends to an automorphism of $\mathcal{B}'$. Applying Lemma 4 of [1] each permutation of $X$ is an elementary permutation of $X$. Hence, $\mathcal{B}'$, and a fortiori, $\mathcal{B}$ has $2^{\aleph_0}$ automorphisms.

COROLLARY. *If $T$ is categorical in every infinite power, and $\mathcal{A}$ is a model of $T$ with power $\varkappa$, then $|\mathrm{Aut}(\mathcal{A})| = 2^{\varkappa}$.*

Proof. Each model of $T$ is saturated.

We will now deal with $\aleph_1$-but not $\aleph_0$-categorical theories. We first show that each non-saturated model has either countably many or $2^{\aleph_0}$ automorphisms.



Let $\mathcal{B}$ be a model of $T$ and $X \subseteq \mathcal{B}$. An element $b$ of $\mathcal{B}$ is determined or selected [3] by $X$ if there is a formula $\underset{\sim}{A} \in S_1(X)$ such that $\mathcal{B} \models \underset{\sim}{A}(b) \wedge \wedge \exists! \ \underset{\sim}{v}_0 A$. We denote by $D(X)$ the set of elements selected by $X$.

We will discuss the cardinality of $\mathrm{Aut}\mathcal{B}$ in terms of the following property.

$*$ For each finite $X \subseteq B$, $B - D(X)$ is infinite.

THEOREM 2. *Let $T$ be a complete first order theory having only infinite models. Let $\mathcal{B}$ be a countable, countably homogeneous, atomic model of $T$.*

1) *If $\mathcal{B}$ has $*$, $|\mathrm{Aut}(\mathcal{B})| = 2^{\aleph_0}$.*

2) *If $\mathcal{B}$ does not have $*$, $|\mathrm{Aut}(\mathcal{B})| \leqslant \aleph_0$.*

Proof. 1) Let $\sigma$ denote the constantly 0 map in $2^{\omega}$. For each $\tau \in 2^{\omega}$ we construct by induction a sequence $b_\tau$ of elements of $\mathcal{B}$ such that $(\mathcal{B}, b_\sigma) \equiv (\mathcal{B}, b_\tau)$ but $\tau_1 \neq \tau_2$ implies $b_{\tau_1} \neq b_{\tau_2}$.

Let $b_0 \in B - D(\emptyset)$ and let $\underset{\sim}{A}_0 \in S_1(L)$ be the atom realized by $b_0$. Since $b_0 \notin D(\emptyset)$, $\mathcal{B} \models \exists^{\geqslant 2} \ \underset{\sim}{v}_0 \underset{\sim}{A}_0$. Let $b_1$ be any other element such that $\mathcal{B} \models \underset{\sim}{A}(b_1)$.

If $\tau$ is a sequence of length $n$ let $\tau \frown i$ be the sequence of length $n+1$ extending $\tau$ with $(n+1)$-st coordinate $i$. Suppose for each $\tau \in 2^n$ a sequence $b_\tau \in B^n$ has been chosen such that $\tau_1 \neq \tau_2$ implies $b_{\tau_1} \neq b_{\tau_2}$ but for each $b_\tau$ $(\mathcal{B}, b_\tau) \equiv (\mathcal{B}, b_{\sigma\restriction n})$ where $\sigma\restriction n$ is the $n$-ary constantly zero sequence. Let

$$B_n = \bigcup_{\substack{\tau \in 2^n \\ i \leqslant n}} \{b_{\tau(i)}\} \quad \text{and} \quad B_n' = \bigcup_{i<n} \{b_{\sigma(i)}\} .$$

Since $B_n$ is finite there exists $x \in B - D(B_n)$. Let $x$ be $b_{\langle n, 0\rangle}$. Let $\underset{\sim}{A}_n \in S_{n+1}(L)$ be the atom in $F_{n+1}(T)$ realized by $b_{\sigma\restriction n+1}$. Then $\underset{\sim}{A}_n(b_{\sigma\restriction n}, v_0)$ is an atom in $F_1(B_n')$. Moreover $\mathcal{B} \models \exists^{\geqslant 2} \ \underset{\sim}{v}_0 \underset{\sim}{A}_n(b_{\sigma\restriction n}, v_0)$, Form $b_{\sigma\restriction n-1}$ by extending $b_{\sigma\restriction n}$ by any element satisfying $\underset{\sim}{A}_n(b_{\sigma\restriction n}, \underset{\sim}{v}_0)$ except $x$. Similarly for $\tau \in 2^n$ let $b_{[\tau-0](n+1)}$ and $b_{[\tau-1](n+1)}$ be distinct elements satisfying $\underset{\sim}{A}_n(b_\tau, v_0)$. There are two such elements because $(\mathcal{B}, b_\tau) \equiv \mathcal{B}, b_\sigma)$. By the atomicity of $\underset{\sim}{A}_n$, $(\mathcal{B}, b_{\sigma\restriction n+1}) \equiv (\mathcal{B}, b_{\tau \cap i})$ for each $\tau \in 2^n$ and $i = 0, 1$.

For $\tau \in 2^{\omega}$ let $b_\tau$ be the union of the finite sequences $b_{\tau_1}$ such that $\tau$ extends $\tau_1$. Then $(\mathcal{B}, b_\tau) \equiv (\mathcal{B}, b_\sigma)$. Since $\mathcal{B}$ is countably homogeneous, for each such $\tau$ there is an automorphism $f_\tau$ of $\mathcal{B}$ taking $b_{\sigma(i)}$ to $b_{\tau(i)}$. Since $\tau_1 \neq \tau_2$ implies for some $i$ $b_{\tau_1(i)} \neq b_{\tau_2(i)}$ these are $2^{\aleph_0}$ distinct automorphisms of $\mathcal{B}$.

2) If $*$ does not hold there is a finite subset $Z$ of $B$ with $D(Z) = B$. But then any automorphism of $\mathcal{B}$ is determined by the image of $Z$ and a finite set can have at most countably many images in a countable set.

A structure $\mathcal{A}'$ is an inessential expansion of $\mathcal{A}$ if it is obtained from $\mathcal{A}$ by naming a set of elements from the universe of $\mathcal{A}$. It is a finite inessential expansion if the set of elements named is finite.

LEMMA 1. *Let $\mathcal{A}'$ be a finite inessential expansion of $\mathcal{A}$. If $|\mathcal{A}| = \aleph_0$ and $|\mathrm{Aut}(\mathcal{A})| > \aleph_0$, then $|\mathrm{Aut}(\mathcal{A})| = |\mathrm{Aut}(\mathcal{A}')|$.*

Proof. Let $b$ be the finite sequence in $\mathcal{A}$ which is named in expanding $\mathcal{A}$ to $\mathcal{A}'$. There are at most countably many images for $b$ in $\mathcal{A}$ so if $|\mathrm{Aut}(\mathcal{A})| = \varkappa > \aleph_0$, there must be some $b'$ such that for each of a sequence $\langle f_a \rangle_{a<\varkappa}$ of automorphisms of $\mathcal{A}$, $f_a(b) = b'$. For each $a$ $f_0^{-1} \circ f_a$ is an automorphism of $\mathcal{A}$ which fixes $b$, i.e. an automorphism of $\mathcal{A}'$. But if $\alpha \neq \beta$ $f_0^{-1} \circ f_a \neq f_0^{-1} \circ f_\beta$ so $|\mathrm{Aut}(\mathcal{A}')| \geqslant \varkappa$. But each automorphism of $\mathcal{A}'$ is an automorphism of $\mathcal{A}$ so $|\mathrm{Aut}(\mathcal{A})| = |\mathrm{Aut}(\mathcal{A}')| = \varkappa$.

THEOREM 3. *If $\mathcal{A}$ is a non-saturated model of an $\aleph_1$-categorical theory $T$, then $|\mathrm{Aut}\,\mathcal{A}| \leqslant \aleph_0$ or $|\mathrm{Aut}\,\mathcal{A}| = 2^{\aleph_0}$.*

Proof. Since each uncountable model of $T$ is saturated, $\mathcal{A}$ is countable. Let $T'$ be a principal extension of $T$ with strongly minimal formula $\underset{\sim}{M}$ and let $\mathcal{A}'$ be an expansion of $\mathcal{A}$ to a model for $T'$. As it is an inessential expansion of $\mathcal{A}$, $\mathcal{A}'$ is not saturated. Hence, as remarked near the end of Section 3 in [1], $\mathcal{A}'$ does not contain an infinite set of indiscernibles. Thus there is a finite basis $x_1, \ldots, x_n$ for $\underset{\sim}{M}(\mathcal{A})$. Let $\mathcal{A}'' = (\mathcal{A}', x_1, \ldots, x_n)$ and $T'' = \mathrm{Th}(\mathcal{A}'')$. Then $\mathcal{A}''$ is a prime model of $T''$. Moreover, $\mathcal{A}$ ([1, Section 3]) and hence $\mathcal{A}''$ is countably homogeneous. Thus by Theorem 2 $|\mathrm{Aut}(\mathcal{A}'')| \leqslant \aleph_0$ or $|\mathrm{Aut}(\mathcal{A}'')| = 2^{\aleph_0}$. But then by Lemma 1 $|\mathrm{Aut}(\mathcal{A})| \leqslant \aleph_0$ or $|\mathrm{Aut}(\mathcal{A})| = 2^{\aleph_0}$.

Recall that by [7] and [1] the non-saturated models may be arranged in an elementary tower of order type $\omega$, $\mathcal{A}_0 < \mathcal{A}_1 < \ldots$ We will now consider the properties of a function $f$ mapping the natural numbers into a set consisting of the natural numbers, $\aleph_0$, and $2^{\aleph_0}$ defined by $f(n) = |\mathrm{Aut}(\mathcal{A}_n)|$.

LEMMA 2. *The function $f$ is increasing.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be non-saturated models of $T$, suppose $\mathcal{A} \precneqq \mathcal{B}$ and $g$ is an automorphism of $\mathcal{A}$. We show $g$ extends to an automorphism of $\mathcal{B}$. Let $\langle a_i \rangle_{i<\omega}$ be an enumeration of $\mathcal{A}$. Then $\langle a_i \rangle_{i<\omega}$ and $\langle g(a_i) \rangle_{i<\omega}$ realize the same type in $\mathcal{A}$ and hence in $\mathcal{B}$. Since $\mathcal{B}$ is countably homogeneous there is an automorphism of $\mathcal{B}$ taking $a_i$ to $g(a_i)$, that is extending $g$.

LEMMA 3. *For each $n$ $f(n) \geqslant n$.*

Proof. Let $T'$ be a principal extension of $T$ with strongly minimal formula $\underset{\sim}{M}$ and let $\mathcal{A}_n'$ be the expansion of $\mathcal{A}_n$ to a model of $T'$. Then by Lemma 7 and Lemma 4 of [1], $\mathcal{A}_n'$ is prime over a set of at least $n$ indiscernibles. Hence, there are at least $n!$ automorphisms of $\mathcal{A}_n'$ and hence of $\mathcal{A}_n$.

If $T$ is theory of equality in a language with $\aleph_0$ distinct constant symbols then $f(n) = n!$

We collect these lemmas in the following theorem.

THEOREM 4. *Let $T$ be an $\aleph_1$-but not $\aleph_0$-categorical theory. There exists an ordinal number $a$ $a < \omega+1$ such that for $0 \leqslant n < a$ $n! \leqslant f(n) \leqslant \aleph_0$ and if $a \leqslant n$ $f(n) = 2^{\aleph_0}$. If $a = \omega$, then all non-saturated models of $T$ have at most countably many automorphisms.*



If $T$ is the theory of torsion-free infinite divisible Abelian groups $f(n) = \aleph_0$ for each $n$. As was noticed in [2] $f(n) = 2^{\aleph_0}$ for each $n$ if $T$ is the theory of algebraically closed fields of characteristic zero.

If $T$ is any $\aleph_1$-but not $\aleph_0$-categorical theory extending $T$ to $T'$ by naming each element of the prime model does not change $f(n)$ for $n > 0$ but makes $f(0) = 1$.

To indicate how the behavior of $f$ may become more complex we now outline an example where $f(0) = 1$, $f(1) = \aleph_0$ and $f(2) = 2^{\aleph_0}$.

We first construct a structure $\mathcal{A}$ such that the required $T = \mathrm{Th}(\mathcal{A})$. The language $L$ contains a unary relation symbol $\underset{\sim}{R}$, a binary relation symbol $\underset{\sim}{S}$, a constant $\underset{\sim}{n}$ for each integer $n$, and ternary relation symbols, $\underset{\sim}{H}$ and for each integer $n$ $\underset{\sim}{G}_n$ and for each natural number $n$ $\underset{\sim}{F}_n$. The universe of $\mathcal{A}$ is the integers union the set of all pairs of the form $\langle\langle n, m\rangle, \langle m, n\rangle\rangle$ where $m$ and $n$ are integers. We define the relations on $\mathcal{A}$ as follows:

$$\begin{aligned}
\underset{\sim}{R}_{\mathcal{A}} &= \text{the integers},\ \underset{\sim}{n}_{\mathcal{A}} = n,\\
\neg\underset{\sim}{R}_{\mathcal{A}} &= \{\langle\langle n, m\rangle, \langle m, n\rangle\rangle \mid n, m \text{ integers}\},\\
\underset{\sim}{H}_{\mathcal{A}} &= \{\langle m, n, \langle\langle m, n\rangle, \langle n, m\rangle\rangle \mid m, n \text{ integers}\} \cup\\
&\quad \cup \{\langle m, n, \langle\langle n, m\rangle, \langle m, n\rangle\rangle\rangle \mid m, n \text{ integers}\},\\
(\underset{\sim}{F}_n)_{\mathcal{A}} &= \{\langle m, m+n, \langle\langle m+n, m\rangle, \langle m, m+n\rangle\rangle \mid m \text{ an integer}\},\\
(\underset{\sim}{G}_n)_{\mathcal{A}} &= \{\langle m, n, \langle\langle m, n\rangle, \langle n, m\rangle\rangle \mid m \text{ an integer}\},\\
\underset{\sim}{S}_{\mathcal{A}} &= \{\langle m, m+1\rangle \mid m \text{ an integer}\}\ \underset{\sim}{S}_{\mathcal{A}} \subseteq \underset{\sim}{R}_{\mathcal{A}} \times \underset{\sim}{R}_{\mathcal{A}}.
\end{aligned}$$

To show $T = \mathrm{Th}(\mathcal{A})$ is $\aleph_1$-categorical notice that $(\underset{\sim}{R}_{\mathcal{A}}, \underset{\sim}{S}_{\mathcal{A}})$ is a model of the theory of the integers under successor which is $\aleph_1$-categorical. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are models of $T$ with power $\aleph_1$ choose an isomorphism between $(\underset{\sim}{R}_{\mathcal{B}_1}, \underset{\sim}{S}_{\mathcal{B}_1})$ and $(\underset{\sim}{R}_{\mathcal{B}_2}, \underset{\sim}{S}_{\mathcal{B}_2})$ and extend it in the obvious manner to an isomorphism of $\mathcal{B}_1$ and $\mathcal{B}_2$.

The prime model of $T$, $\mathcal{A}$, has only the identity automorphism since every element of $|\mathcal{A}|$ is either named by some constant $\underset{\sim}{n}$, or is the unique element $c$ satisfying one of these two types of formulas

$$\text{(i)}\qquad \begin{aligned}
&\underset{\sim}{H}(\underset{\sim}{n}, \underset{\sim}{m}, \underset{\sim}{c}) \wedge \underset{\sim}{G}_m(\underset{\sim}{n}, \underset{\sim}{m}, \underset{\sim}{c}),\\
&\underset{\sim}{H}(\underset{\sim}{n}, \underset{\sim}{m}, \underset{\sim}{c}) \wedge \neg \underset{\sim}{G}_m(\underset{\sim}{n}, \underset{\sim}{m}, \underset{\sim}{c}).
\end{aligned}$$

In $\mathcal{A}_1$, there is an element $d \in \underset{\sim}{R}(\mathcal{A}_1) - \underset{\sim}{R}(\mathcal{A}_0)$ such that for each element $d_1 \in \underset{\sim}{R}(\mathcal{A}_1) - \underset{\sim}{R}(\mathcal{A}_2)$ there is integer $k$ that $d_1$ is the $k$th successor of $d$ (predecessor if $k$ is negative). The entire universe of $\mathcal{A}_1$ is selected by $d$. That is, if $c$ is an element of $\mathcal{A}_1$ it is either in $\underset{\sim}{R}(\mathcal{A}_1)$ or is the unique element satisfying a formula of type (i) or a formula of one of the following types

$$\text{(ii)}\qquad \begin{aligned}
&\underset{\sim}{H}(\underset{\sim}{S}^k(\underset{\sim}{d}), \underset{\sim}{n}, \underset{\sim}{c}) \wedge \underset{\sim}{G}_n(\underset{\sim}{S}^k(\underset{\sim}{d}), \underset{\sim}{n}, \underset{\sim}{c}),\\
&\underset{\sim}{H}(\underset{\sim}{S}^k(\underset{\sim}{d}), \underset{\sim}{n}, \underset{\sim}{c}) \wedge \neg \underset{\sim}{G}_n(\underset{\sim}{S}^k(\underset{\sim}{d}), \underset{\sim}{n}, \underset{\sim}{c}),
\end{aligned}$$

$$\text{(iii)}\qquad \begin{array}{ll} \underset{\sim}{H}(\underset{\sim}{S}^k(\underset{\sim}{d}), \underset{\sim}{S}^l(\underset{\sim}{d}), \underset{\sim}{c}) \wedge \quad \underset{\sim}{F}_{k,l}(\underset{\sim}{S}^k(\underset{\sim}{d}), \underset{\sim}{S}^l(\underset{\sim}{d}), \underset{\sim}{c}), & k > l, \\ \underset{\sim}{H}(\underset{\sim}{S}^k(\underset{\sim}{d}), \underset{\sim}{S}^l(\underset{\sim}{d}), \underset{\sim}{c}) \wedge \neg \underset{\sim}{F}_{k,l}(\underset{\sim}{S}^k(\underset{\sim}{d}), \underset{\sim}{S}^l(\underset{\sim}{d}), \underset{\sim}{c}), & k > l. \end{array}$$

In $R(\mathcal{A}_2) - R(\mathcal{A}_1)$ there exist elements $d_1$ and $d_2$ such that for no natural number $k$ is $d_1$ the $k$th successor or predecessor of $d_2$. The formulas $\underset{\sim}{H}(\underset{\sim}{S}^k(\underset{\sim}{d}_1), \underset{\sim}{S}^l(\underset{\sim}{d}_2), \underset{\sim}{v}_0)$ define principal types in $\mathrm{Th}(\mathcal{A}_2, d_1, d_2)$ which are realized by 2 points. That is, $*$ holds in $\mathcal{A}_2$ so $|\mathrm{Aut}(\mathcal{A}_2)| = 2^{\aleph_0}$.

Easy variations on this example will give theories $T$ with $f(0) = 1$, $f(n) = \aleph_0$ $0 < n < N$ and $f(n) = 2^{\aleph_0}$ for $n \geqslant N$, for any choice of $N$. Is there a theory $T$ such that for some natural number $N > 1$, $f(n)$ is finite for $n < N$, $f(n) = \aleph_0$ for some segment beginning with $N$, say $N \leqslant n \leqslant M$ and $f(n) = 2^{\aleph_0}$ for $n > M$? That is, is it possible in other than the trivial case when all elements of the prime model are named for the value of $f$ to jump from finite to countable to uncountable?

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# On certain homological properties of finite-dimensional compacta. Carriers, minimal carriers and bubbles (*)

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**Abstract.** A compact metric space $X$ is $q$-cyclic ($q = 1, 2, \ldots$) provided that there exists a coefficient group $G$ such that the Vietoris-Čech homology group $H_q(X, G)$ of $X$ is non-trivial. An irreducibly $q$-cyclic compact metric space is called a $q$-dimensional closed Cantorian manifold, or a $q$-bubble. The following question, asked by P. S. Alexandroff, is considered in the paper: Given a $q$-dimensional $q$-cyclic compact metric space $X$, does $X$ contain a $q$-bubble? As is known, the answer is not always positive, but by adding some assumptions on $X$ a positive result is obtained. A class of spaces, the so called $\mathrm{WSC}_q$-compacta, is exhibited in the paper and it is proved that each $q$-dimensional $q$-cyclic $\mathrm{WSC}_q$-compactum contains at least one and at most countably many $q$-bubbles. Furthermore, some other properties of $\mathrm{WSC}_q$-compacta are studied.


**1. Introduction.** By a compactum is meant a compact metric space. As is well known, the Čech and the Vietoris homology theories are equivalent in the algebraic sense (see for instance [13], p. 273). The $q$-dimensional Vietoris-Čech homology group of a compactum $X$ with coefficients in an Abelian group $G$ will be denoted by $H_q(X, G)$. This group will be sometimes represented as the limit of the inverse system of the (simplicial) $q$th homology groups of the nerves of all finite open coverings of $X$, with $G$ as the coefficient group (the well-known Čech construction) and sometimes as the group of homology classes of $q$-dimensional true cycles in $X$ with coefficients in $G$ (the Vietoris construction). In the first case, the notations, symbols and terminology from [9], chap. IX will be applied here, in the second case, we shall base ourselves on the construction described in [4], chap. II, sec. 3. In particular, the concepts of infinite chains and infinite cycles are very useful. By means of these concepts the following characterization of the dimension of compacta has been established:

(*) This is the present writer's doctoral thesis (in a modified form), defended at the University of Warsaw, Poland, in February 1969. The original title of the thesis was "Dimensional properties of ANR-spaces".